Dynamical properties of the doubling map with holes

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Overview

1. Open dynamical systems
   - The doubling map with holes
   - Symbolic dynamics

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  and $f_U = f \mid_{X_U}$.
- We call (X_U, f_U) an *open dynamical system* or *a map with a hole*. (Pianigiani, Yorke 1979)

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Let $(a, b) \subset S^1$. Consider

$$X_{(a,b)} = \{x \in S^1 \mid f^n(x) \notin (a, b) \text{ for every } n \geq 0\}.$$

Introducing holes and restricting the problem

Lemma (Glendinning - Sidorov 2013)

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i) *If* $0 < a < \frac{1}{4}$ *and* $\frac{1}{2} < b < 1$, *or* $0 < a < \frac{1}{2}$ *and* $\frac{3}{4} < b < 1$ *then* $X_{(a,b)} = \{0\}$.

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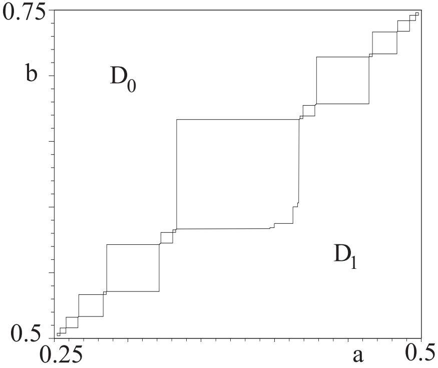
We will just consider $a \in (\frac{1}{4}, \frac{1}{2})$ and $b \in (\frac{1}{2}, \frac{3}{4})$ with $b \leq \chi(a)$.



0.75
b
D_0
0.5
0.25
a
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Motivation: Is $(\Lambda_{(a,b)}, f_b^a)$ intrinsically ergodic?

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Observe that π is a semi-conjugacy between $2x \mod 1$ and the one sided shift σ, where $\sigma : \Sigma_2 \to \Sigma_2$ $\sigma((x_i)_{i=1}^{\infty}) = (x_{i+1})_{i=1}^{\infty}$.

Lexicographic subshifts

Given $x, y \in \Sigma_2$ we say that x is *lexicographically less that* y, $x \prec y$ if there exists $k \in \mathbb{N}$ such that $x_j = y_j$ for $i < k$ and $x_k < y_k$.

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Moreover, $(\Sigma_{(a,b)}, \sigma_{(a,b)})$ *is a subshift and* $(\Sigma_{(a,b)}, \sigma_{(a,b)})$ *is conjugated to* $(\Lambda_{(a,b)}, f_b^a)$.

For every $n \in \mathbb{N}$, *the set of admissible words of length* n *of* $\Sigma_{\mathcal{F}}$ is given by:

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- has *the specification property* if there exist $m \in \mathbb{N}$ such that for every $u = u_1 \dots u_{\ell(u)}$ and $v = v_1 \dots v_{\ell(v)}$ there exist $w = w_1 \dots w_n$ such that uwv is admissible and $n \leq m$.

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- is *coded* if $A = \overline{\bigcup_{n=1}^{\infty} A_n}$ where (A_n, σ_{A_n}) is a transitive subshift of finite type.

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We say that $(\Sigma_{(a,b)}, \sigma_{(a,b)})$ is *symmetric* if for every $x \in \Sigma_{(a,b)}$, $\bar{x} \in \Sigma_{(a,b)}$.

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Theorem (Glendinning-Sidorov 2001)

For $a \in [0, \frac{1}{2}]$, then:

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2. $\Sigma_{(a,1-a)} = \{(01)^\infty, (10)^\infty\}$ *if* $a \in [\frac{1}{3}, \frac{13}{32})$;
3. $\Sigma_{(a,1-a)}$ *is countable if* $a \in [\frac{13}{32}, a^*)$;
4. $\Sigma_{(a,1-a)}$ *is uncountable and* $\dim_H(\Sigma_a) > 0$ *if* $a \in (a^*, \frac{1}{2}]$.

where $a^* = \pi(t)$ and t is the Thue-Morse sequence.

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Then, we considered $a \in (a^*, \frac{1}{2})$

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(Let $N = \{x \in \Sigma_2 \mid x_1 = 1\} \setminus P$. If $\pi^{-1}(2a) \in N$ then Σ_a is a SFT (Bundfuss, Krüger, Troubetzkoy 2011).)

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If $\pi^{-1}(2a)$ is a finite (periodic) sequence then $\Sigma_{(a,1-a)}$ is a subshift of finite type. Moreover, if $\pi^{-1}(2a)$ is irreducible then $\Sigma_{(a,1-a)}$ is a transitive subshift of finite type.

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In general $\Sigma_{(a,1-a)}$ is not transitive.

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Lemma

If $a > \frac{5}{12}$ satisfies that $\pi^{-1}(2a) \in \mathcal{E}$ there exist a sequence ω_n^- of irreducible sequences such that;

1. $\omega_n^- \prec \omega_{n+1}^-$ *and* $\omega_n^- \longrightarrow \pi^{-1}(2a)$. *Then* $\Sigma_{(a,1-a)}$ *is coded, i.e*

$$\Sigma_{(a,1-a)} = \overline{\bigcup_{n=1}^{\infty} \Sigma_{\omega_n^-}}.$$

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3. *If* 0^n *occurs infinitely many times in* $\pi^{-1}(2a)$ *and* $\pi^{-1}(2a)$ *satisfies a not nice technical theorem.*

Technical Theorem

Theorem (A.B. 2014)

Let $n \geq 2$ and $\omega \in \mathcal{E} \cap (1^n, 1^{n+1})$. If for every $r \in \mathbb{N}$, ω_r^+ satisfies that

$$\frac{1}{2^{2\ell(\omega_r^+)}} < d({\omega_{r-1}^+}', \omega_r^+) \leq \frac{1}{2^{\ell(\omega_r^+)+n}},$$

then $\lim_{n\to\infty} s_{\omega_n^+} < \infty$. Here $\omega' = \omega\bar{\omega}_1 \ldots \omega_{\ell(\bar{\omega})-1} 1$.

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Intuitively, the technical theorem says that $s_{\omega_r^+}$ does not increase exponentially fast.

Symmetric subshifts with no specification

Theorem (A.B. 2014)

Let $n \geq 2$ fixed. Let $\pi^{-1}(2a) \in \mathcal{E}$ such that $\pi^{-1}(2a) \in (1^n, 1^{n+1})_{\prec}$, $a > \frac{5}{12}$, 0^n occurs in $\pi^{-1}(2a)$ infinitely many times. If there exists an increasing sequence $\{r_i\}_{i=1}^{\infty} \subset \mathbb{N}$ and $R \in \mathbb{N}$ such that for every $r_i \geq R$ $\omega_{r_i}^-$ satisfies

$$\ell(\omega_{r_{i-1}}^-(\overline{\omega_{r_{i-1}1}^-} \ldots \overline{\omega_{r_{i-1}\ell(\omega_{r_{i-1}}^-)-1}^+}1)^{k_{r_i}}) \leq \ell(\omega_{r_i}^-)$$

and

$$\frac{1}{2^{(k_{r_i}+1)\ell(\omega_{r_i}^-)+n}} \leq d(\omega_{r_i}^{-\prime\prime\prime}, \omega) \leq \frac{1}{2^{(k_{r_i}+1)\ell(\omega_{r_i}^-)}}$$

for some $k_{r_i} \geq 1$ then $(\Sigma_{(a,1-a)}, \sigma_{(a,1-a)}$ does not have specification.

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$$\pi^{-1}(a) = \omega\nu^{n_1^\nu}\omega^{n_1^\omega}\nu^{n_2^\nu}\omega^{n_2^\omega}\nu^{n_3^\nu}\dots$$

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If $\ell(\omega) + \ell(\nu) = 3$ we call (a, b) trivially renormalizable.
If ω or ν can be infinite. In this case, we say that (a, b) is *renormalizable by an infinite sequence*.

Renormalization and Transitivity

Theorem (A.B. 2014)

1. *If $(a, b) \in \mathcal{LW}$ is renormalizable by ω and ν and $\ell(\omega) + \ell(\nu) > 4$ then $(\Sigma_{(a,b)}, \sigma_{(a,b)})$ is not transitive.*

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2. *If* $(a, b) \in \mathcal{LW}$ *is renormalizable by an infinite sequence then* $(\Sigma_{(a,b)}, \sigma_{(a,b)})$ *is not transitive.*
3. *If* (a, b) *is not renormalizable and* $(\Sigma_{(a,b)}, \sigma_{(a,b)})$ *is a subshift of finite type, then* $(\Sigma_{(a,b)}, \sigma_{(a,b)})$ *is transitive.*

Specification

Given a sequence $a \in \Sigma_2$, consider
$0_a = \max\{n \in \mathbb{N} \mid 0^n \text{ is a factor of } a\}$, and
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Theorem (A.B. 2014)

If $0_a + 2 \leq 0_b$ and $1_a > 1_b$ then $(\Sigma_{(a,b)}, \sigma_{(a,b)})$ has specification.