The Stone-Čech remainder of $\omega^* \setminus \{x\}$

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Joint work with Rolf Suabedissen

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Motivation

What is the Stone-Čech compactification of $\omega^* \setminus \{x\}$?

- Fine and Gillman '60: CH implies that for every point $x \in \omega^*$ there are continuous bounded real-valued functions on $\omega^* \setminus \{x\}$ that cannot be continuously extended to ω^*.
- van Douwen, Kunen and van Mill '89: It is consistent with $\mathfrak{c} = \aleph_2$ that for every point $x \in \omega^*$ all continuous real-valued functions on $\omega^* \setminus \{x\}$ can be continuously extended to ω^*.

In other words, under CH, the space $\omega^* \setminus \{x\}$ has a non-trivial Stone-Čech remainder.

- Question for this talk: Under CH, how do the Stone-Čech remainders of $\omega^* \setminus \{x\}$ look for different x?
- Frolík '67: There are $2^{\mathfrak{c}}$ different spaces of the form $\omega^* \setminus \{x\}$.

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The Stone-Čech remainder ω^* of the integers

A topological characterisation of ω^* requiring the Continuum Hypothesis

The Parovičenko properties of ω^*:

- Compactness, zero-dimensionality, no isolated points;
- Disjoint open F_σ-sets have disjoint closures;
  ($\Leftrightarrow$ F-space: open F_σ-sets are C^*-embedded)
- Non-empty G_δ-sets have non-empty interior.

Theorem (Parovičenko '63 and van Douwen/van Mill '78)

The Continuum Hypothesis is equivalent to the assertion that every Parovičenko space of weight $\mathfrak{c}$ is homeomorphic to ω^*.

The κ-Parovičenko spaces of weight κ

A common generalisation of the Cantor space and ω^* to higher cardinals

The κ-Parovičenko properties:

- Compactness, zero-dimensionality, no isolated points;
- Disjoint open $F_{<\kappa}$-sets have disjoint closures;
- Non-empty $G_{<\kappa}$-sets have non-empty interior.

Brouwer 1910: C

There is a unique 0-dim. compact space of weight ω without isolated points.

Parovičenko '63: ω^*

Under CH there is a unique Parovičenko space of weight $\mathfrak{c} = \omega_1$.

Negrepontis '69: S_κ

Under the assumption $\kappa = \kappa^{<\kappa}$ there is a unique κ-Parovičenko space of weight κ.

- S_ω is the Cantor space C and S_{ω_1} equals ω^* under CH.

The Stone-Čech remainder of $\omega^* \setminus \{x\}$

Parovičenko properties improve when taking Stone-Čech remainders

Theorem (Main Result)

Assuming CH, for every x in ω^*, the remainder of $\omega^* \setminus \{x\}$ is an ω_2-Parovičenko space of weight 2^{ω_1}.

Hence, under the cardinal assumption $2^{\mathfrak{c}} = \omega_2$ we get:

- For every point x the space $(\omega^* \setminus \{x\})^*$ is homeomorphic to S_{w_2}, the unique ω_2-Parovičenko space of weight ω_2.
- In particular: the remainders of $\omega^* \setminus \{x\}$ and $\omega^* \setminus \{y\}$ are homeomorphic for all points x and y of ω^*.

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Overview of proof strategy

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Proof ingredients:

1. The remainder of $\omega^* \setminus \{x\}$ is compact, 0-dim, crowded;
2. Dow '85: The theorem holds for $\omega^* \setminus \{p\}$ where p is a point with a nested neighbourhood base (a so-called P-*point*);
3. Locally, apart from at most one rogue point $\star$, $(\omega^* \setminus \{x\})^*$ looks like $(\omega^* \setminus \{p\})^*$;
4. This point $\star$ in the remainder of $\omega^* \setminus \{x\}$ has an ω_1-complete neighbourhood base, so is quite nice after all;
5. Putting it all together.

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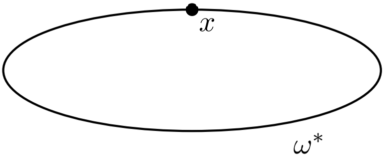
3a Finding the rogue point $\star$ in $(\omega^* \setminus \{x\})^*$

A comic-strip proof



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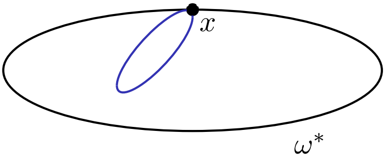
- Pick an open F_σ-set U such that $x \in \partial U$.



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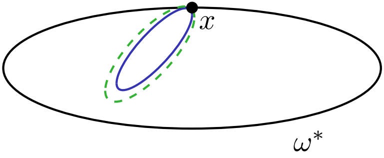
- $\overline{U} = \beta U$ by the F-space property.



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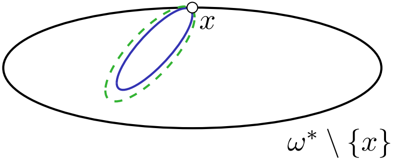

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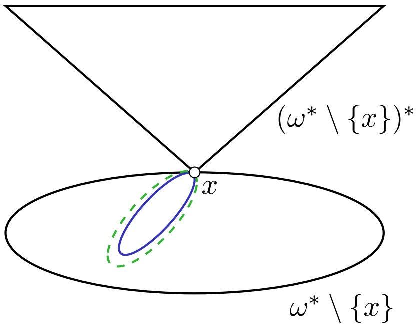

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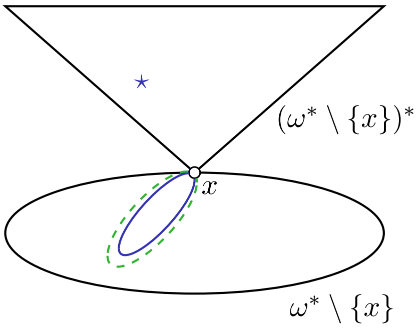
- $\overline{U} \setminus \{x\}$ has a 1-pt Stone-Čech compactification.
- U limits only onto $\star$ in the remainder.



3b Apart from point $\star$, $(\omega^* \setminus \{x\})^*$ looks like $(\omega^* \setminus \{p\})^*$

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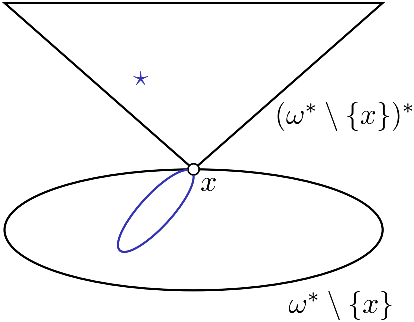

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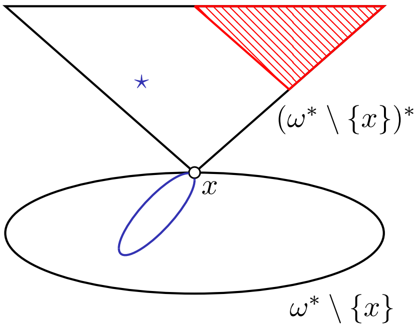

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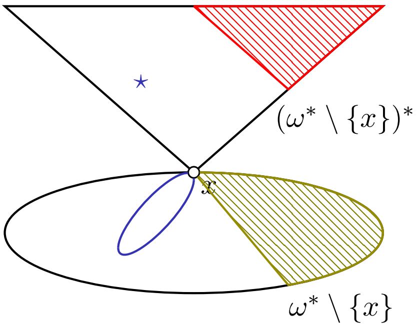

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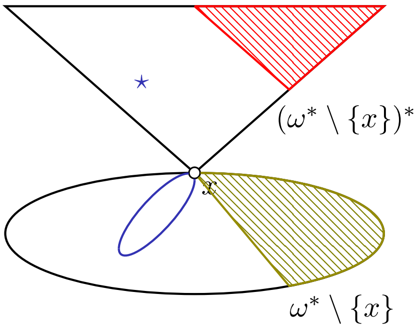

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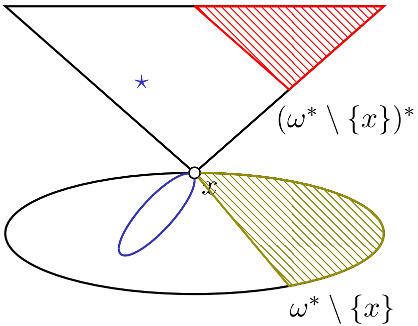
- So C^* looks like $(\omega^* \setminus \{p\})^*$.



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The Stone-Čech remainder of $\omega^* \setminus \{x\}$

An open question

In this talk: It is consistent with CH that the remainders of $\omega^* \setminus \{x\}$ are all homeomorphic, regardless of the choice of x.

Question

Is it consistent with CH that for a P-point p and a non-P-point x the remainders of $\omega^* \setminus \{p\}$ and $\omega^* \setminus \{x\}$ are non-homeomorphic?