Dynamics of Generalized Inverse Limits on Intervals

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Outline

Shift on Generalized Inverse Limits

Suppose $f : I \to 2^I$ is a surjective, upper semicontinuous bonding map. Let $M = \varprojlim f$.

Even though f is not even a function in the usual sense, it induces a continuous function σ from M onto M.

The function σ is called the *shift map* on M, since for $\mathbf{x} = (x_0, x_1, ...) \in M$, $\sigma(\mathbf{x}) = \sigma(x_0, x_1, ...) = (x_1, x_2, ...)$.

Beginning Study of Dynamics

We begin an investigation of the dynamical behavior of the shift map σ. We would like to know which properties of the bonding map f imply that the map σ has certain dynamical properties.

There are many, many possible dynamical properties that could be studied. We list a few:

1. Does the shift map σ admit fixed points? How about periodic points of larger periods?
2. Is the set of periodic points dense in M?
3. Is σ topologically transitive?
4. Does σ have sensitive dependence on initial conditions?
5. Is σ chaotic? And then, chaotic in what sense?
6. Does σ have positive topological entropy?

Starting Easy - An Example

Def. Suppose $f : I \to 2^I$ is surjective and upper semicontinuos, and that the graph G of f is connected. Then we say that f is *full*.

Proposition. Suppose that $f : I \to 2^I$ is full, the points $(0,0)$ and $(1,1)$ are in G, and the graph G of f lies under the diagonal $\Delta = \{(x,x) : x \in I\}$ except for the points $(0,0)$ and $(1,1)$. Then the points $\mathbf{0} := (0,0,0,...)$ and $\mathbf{1} := (1,1,1,...)$ are fixed under σ, and the point $\mathbf{0}$ attracts all points of M except $\mathbf{1}$; i.e., if $\mathbf{x} \in M$, $\mathbf{x} \neq \mathbf{1}$, then $\lim_{n\to\infty} \sigma^n(\mathbf{x}) = \mathbf{0}$.

Proof.

It is obvious that $\mathbf{0}$ and $\mathbf{1}$ are fixed under σ. Further, suppose $\mathbf{x} = (x_0, x_1, \dots) \in M$ and $\mathbf{x} \neq \mathbf{0}$, $\mathbf{x} \neq \mathbf{1}$. Suppose $i \geq 0$. If $x_i \neq 0$, $x_i \neq 1$, then because G lies below the diagonal except for $(0,0),(1,1)$, it must be the case that $x_{i+1} < x_i$. If $x_i = 0$, then $x_{i+1} = 0$. If $x_i = 1$, then $x_{i+1} = 1$ or $x_{i+1} < 1$. Since $\mathbf{x} \neq \mathbf{1}$, there is some N such that $x_N < 1$. Hence, for all cases, $x_i \geq x_{i+1}$, eventually $x_i < 1$, and if there is J such that $x_J = 0$, then $x_j = 0$ for $j \geq J$.

We claim that $x_0, x_1, \dots$ is a nonincreasing sequence of points in I that converges to 0: If not, the sequence converges to a number $p > 0$. But then $(x_1, x_0), (x_2, x_1), \dots$ is a sequence of points in G, and it must converge to the point (p, p), and, since G is closed, the point (p, p) must be in G - a contradiction to our assumptions about G. So the sequence $x_0, x_1, \dots$ converges to 0, and therefore so does the sequence $\mathbf{x}, \sigma(\mathbf{x}), \dots$.

Similarly, we get

Proposition. Suppose that $f : I \to 2^I$ is full, the points $(0,0)$ and $(1,1)$ are in G, and the graph G of f lies above the diagonal $\Delta = \{(x,x) : x \in I\}$ except for the points $(0,0)$ and $(1,1)$. Then the points $\mathbf{0} := (0,0,0,...)$ and $\mathbf{1} := (1,1,1,...)$ are fixed under σ, and the point $\mathbf{1}$ attracts all points of M except $\mathbf{0}$; i.e., if $\mathbf{x} \in M$, $\mathbf{x} \neq \mathbf{0}$, then $\lim_{n\to\infty} \sigma^n(\mathbf{x}) = \mathbf{1}$.

topological transitivity

Def. Suppose X is a metric space and $g : X \to X$ has the property that for every pair of open sets U, V, there is a positive integer k such that $g^{-k}(U) \cap V \neq \emptyset$. Then g is *topologically transitive*.

Def. Suppose $f : I \to I$ is full and has the property that for every nonempty open set $u \subset I$, there is a positive integer N such that $f^n(u) = I$ if $n \geq N$. Then we say that f is *locally eventually onto*.

locally eventually onto $\Rightarrow$ topological transitivity

Proposition. Suppose that $f : I \to I$ is full and locally eventually onto, and $M = \varprojlim f$. Then σ is topologically transitive on M.

Proof Suppose $U = (u_0 \times \ldots \times u_n \times I^\infty) \cap M$, and $V = (v_0 \times \ldots \times v_m \times I^\infty) \cap M$ are nonempty basic open sets in M. Then $\sigma^{-1}(U) = M \cap (f(u_0) \times u_0 \times \ldots \times u_n \times I^\infty)$. Since u_0 is a nonempty open set in I, there is a positive integer N such that if $l \geq N$, then $f^l(u_0) = I$. Then for $l \geq N$, $\sigma^{-l}(U) = M \cap (f^l(u_0) \times f^{l-1}(u_0) \times \ldots \times f(u_0) \times u_0 \times \ldots \times u_n \times I^\infty)$.

Suppose $\mathbf{x} = (x_0, x_1, \ldots) \in M$. Since $f^I(u_0) = I$ for $I \geq N$, there is a point $\mathbf{y_i} = (y_{i0}, y_{i1}, \ldots) \in \sigma^{N+i-1}(U)$ such that $(y_{i0}, \ldots, y_{i-1}) = (x_0, \ldots x_{i-1})$. Hence the sequence $\mathbf{y_1}, \mathbf{y_2}, \ldots$ converges to the point $\mathbf{x}$. So $\cup_{i \geq N} \sigma^{-i}(U)$ is a dense open set in M and there is some integer K such that $V \cap \sigma^{-K}(U)$ is not empty. It follows that σ is topologically transitive. $\square$

Some Examples

Finding full bonding maps that are locally eventually onto is not hard:

1. Let T denote the usual tent map, i.e., $T(x) = 2x$ for $0 \leq x \leq 1/2$, and $T(x) = 2 - 2x$ for $1/2 \leq x \leq 1$.
2. Let f be the full bonding map whose graph consists of the union of three line segments: L_1 is the line segment from $(0, 0)$ to $(1/2, 1)$; L_2 is the vertical line segment from $(1/2, 0)$ to $(1/2, 1)$; L_3 is the line segment from $(1/2, 0)$ to $(1, 1)$.

Dense set of periodic points?

Periodic points are relatively easy to spot as the next proposition demonstrates.

Proposition. Suppose that $\mathbf{x} = (x_0, x_1, \ldots)$ is a periodic point of period N under $\sigma : M \to M$. Then $x_0 \in f^N(x_0)$. Conversely, if $x_0 \in f^N(x_0)$, then there is a periodic point $\mathbf{x}$ of period N in M such that $\pi_0(\mathbf{x}) = x_0$.

One might hope, or expect, that if the full bonding map f is locally eventually onto, then the induced map σ on M would admit a dense set of periodic points. At this time we cannot show that, but we can prove that every open set in M contains a periodic closed set.

"Dense" set of closed sets

Proposition. Suppose that for each open set u in I there is a positive integer N such that for $n \geq N$, $f^n(u) = I$ (so f is locally eventually onto). Then if U is a nonempty open set in M, U contains a periodic closed set.

Proof Without loss of generality, suppose $U = (u_0 \times ... \times u_m \times I^\infty) \cap M$ is a nonempty basic open set in M. Then choose, for $0 \leq i \leq m$, a nonempty open interval v_i such that $\overline{v_i} \subset u_i$ and $(v_0 \times \cdots \times v_m \times I^\infty) \cap M \neq \emptyset$. Let $V_0 = (v_0 \times \cdots \times v_m \times I^\infty) \cap M$. Then $\emptyset \neq \overline{V_0} \subset U$. Suppose $l \geq N + m + 1$. Then for $N \leq i \leq N + m + 1$, $f^i(v_0) = I$, and, it follows that $(v_0 \times \cdots \times v_m \times I^N \times v_0 \times \cdots \times v_m \times I^\infty) \cap M \neq \emptyset$. Let $V_1 = (v_0 \times \cdots \times v_m \times I^N \times v_0 \times \cdots \times v_m \times I^\infty) \cap M$.

Then $V_2 := (v_0 \times \cdots \times v_m \times I^N \times v_0 \times \cdots \times v_m \times I^N \times v_0 \times \cdots \times v_m \times I^\infty) \cap M$ is nonempty, and we can continue. Construct for each l the set V_l in a similar fashion, obtaining a nonempty basic open set in M. Also, $\overline{V_l} \subset \overline{V_{l-1}} \subset U$ for each positive integer l. Hence, $\cap_{l=0}^{\infty} \overline{V_l} \neq \emptyset$. Let P denote $\cap_{l=0}^{\infty} \overline{V_l}$. Note that $\sigma^{N+m+1}(P) = P$ and $\emptyset \neq P \subset U$. Thus, P is the desired periodic closed set in U. $\square$

Van's Lemma

Lemma Suppose $h : I \to 2^I$ is full and $M = \varprojlim f$. Also, suppose A and B are subcontinua of M such that $\{0,1\} \subset \pi_0(A)$, then for each n there is a continuum W in M such that $d_H(W,B) \leq 2^{-n}$ and $\sigma^{n-1}(W) \subset A$. Moreover, for each n,

1. if there exists $b \in B$ such that $\pi_n(b) \in \{0,1\}$ and $\sigma^{n-1}(b) \in A$, then there is a continuum $W \in M$ such that $b \in W$, $d_H(W,B) \leq 2^{-n}$, and $\sigma^{n-1}(W) \subset A$, and
2. if there exist $b_0, b_1 \in B$ such that $\pi_n(b_i) = i$ and $\sigma^{n-1}(b_i) \in A$ for each i, then there is a continuum W in M such that $\{b_0, b_1\} \subset W$, $d_H(W,B) \leq 2^{-n}$, and $\sigma^{n-1}(W) = A$.

Dynamical implications

Van has used this lemma, to great advantage, to prove that many continua *cannot* be inverse limits when there is only one bonding map, but it is really a dynamical result.

Def. Suppose X, Y, Z are compact metric spaces, $A \subset X \times Y$, $B \subset Y \times Z$. Define the *Mahavier product* $A \star B$ to be $\{(x, y, z) \in X \times Y \times Z : (x, y) \in A, (y, z) \in B\}$.

Notation Suppose for each i, X_i is a compact metric space ($i \geq 0$), and $A_i \subset X_{i-1} \times X_i$ ($i \geq 1$). Then $\star_{i=1}^{\infty} A_i$ denotes $A_1 \star A_2 \star \cdots$, and if n is a positive integer, then $\star_{i=1}^{n} A_i$ denotes $A_1 \star A_2 \star \cdots \star A_n$.

shift on 2 symbols

Let Σ_2 denote the set $\{\mathbf{s} = (s_0, s_1, \ldots) : s_i \in \{0,1\}, i \geq 0\}$. Then Σ_2 is a closed subset of the Hilbert cube, and admits a natural map $\overline{\sigma}$ which is continuous and surjective, and is defined by $\overline{\sigma}(s_0, s_1, s_2, \ldots) = (s_1, s_2, \ldots)$. $\overline{\sigma}$ is called the *shift on 2 symbols*.

Theorem

Theorem. Suppose $G \subset I \times I$ and J, K are disjoint continua in G such that $\pi_0(J) = \pi_0(K) = [0,1]$. Then $\star_{i=1}^{\infty} G$ contains an uncountable collection $\mathcal{A}$ of mutually disjoint continua such that

1. For each $\mathbf{s}$ in Σ_2, there is an element $A_{\mathbf{s}}$ of $\mathcal{A}$. In fact, there is a one-to-one onto mapping p of Σ_2 onto $\mathcal{A}$. (So $p(\mathbf{s}) = A_{\mathbf{s}}$.)
2. For each $\mathbf{s}$ in Σ_2, $\pi_0(A_{\mathbf{s}}) = [0,1]$.
3. $\mathcal{A}$ is invariant under σ. Moreover, $p \circ \overline{\sigma}(\mathbf{s}) = \sigma \circ p(\mathbf{s})$ for $\mathbf{s}$ in Σ_2, or, $\overline{\sigma}$ factors over σ.
4. For each continuum $W \subset \star_{i=1}^{\infty} G$ and positive integer n and $\mathbf{s} \in \Sigma_2$, there is a continuum $W_{n,\mathbf{s}} \subset \star_{i=1}^{\infty} G$ such that $\pi_{[0,n]}(W_{n,\mathbf{s}}) = \pi_{[0,n]}(W)$ and $\sigma^{n+1}(W_{n,\mathbf{s}}) \subset A_{\mathbf{s}}$.

topological entropy

Suppose $\alpha = \{\alpha_1, ..., \alpha_n\}$ is a minimal open cover of $[0,1]$ by (nonempty) open intervals.

Let G be a closed subset of $[0,1] \times [0,1]$.

Let V_1 denote a minimal (in terms of cardinality) subcover of G in $W_1 := \{\alpha_i \times \alpha_j : 1 \leq i, j \leq n, G \cap (\alpha_i \times \alpha_j) \neq \emptyset\}$.

Let $N(V_1)$ denote the cardinality of V_1. Note that $N(V_1) \leq n^2$.

Continue for each positive integer m: Let V_m denote a minimal subcover (in terms of cardinality) of $\star_{i=1}^m G$ in
$W_m := \{\alpha_{i_1} \times \cdots \times \alpha_{i_m} : (\alpha_{i_1} \times \cdots \times \alpha_{i_m}) \cap \star_{i=1}^n G \neq \emptyset\}$

Note that $N(V_n) \leq n^m$.

(work with Goran Erceg)

Finally, define the entropy of G with respect to α:
$ent(G, \alpha) = \lim_{m\to\infty} log(N(V_m))/m$

Theorem. The limit above always exists. (Has the standard proof from P. Walters.)

Finally, define $ent(G) = \sup_\alpha ent(G, \alpha)$.

The standard properties of topological entropy hold (except for one). For example:

1. $ent(G) = ent(G^{-1})$
2. $ent(G) = ent(\sigma)$
3. $ent(\star_{i=1}^m G) = m\, ent(G)$

One standard property doesn't hold

For continuous maps $f : [0,1] \to [0,1]$, $ent(f^n) = n\, ent(f)$.

If G is the graph of $f : I \to 2^I$, let $ent(f) = ent(G)$.

In general, $ent(f^n) \neq n\, ent(f)$.

Counterexample

Let g be the "diamond" bonding map, that is, the map whose graph consists of the union of four line segments -

1. L_1 is the line segment from $(0, 1/2)$ to $(1/2, 1)$
2. L_2 is the line segment from $(1/2, 1)$ to $(1, 1/2)$
3. L_3 is the line segment from $(1/2, 0)$ to $(1, 1/2)$
4. L_4 is the line segment from $(0, 1/2)$ to $(1/2, 0)$

Let f be the bonding map defined by $f(x) = \{x, 1 - x\}$ for $x \in I$.

Then $g^2(x) = g \circ g(x) = f(x)$. (These examples are in the Ingram-Mahavier text.)

However, $ent(f) = 0 = ent(g^2)$ and $ent(g) = log(2)$, so $ent(g^2) \neq 2\, ent(g)$.

good news or bad news?

We actually take this as additional evidence that, for set-valued mappings, taking "composition" in the usual way is not the best way.

Taking compositions in the usual way, loses information.

On the other hand, taking the Mahavier product does NOT lose information, and it also behaves nicely with respect to topological entropy.

Thank you so much.