Shadowing in Dynamical Systems

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Outline

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Dynamical Systems

- Let X be a compact space with metric d and let $f : X \to X$ be a continuous function.

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- For $\epsilon > 0$, an ϵ-pseudo-orbit is a sequence $\langle x_i \rangle_{i \in \mathbb{N}}$ satisfying $d(f(x_i), x_{i+1}) < \epsilon$ for all $i \in \mathbb{N}$.

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- An *ϵ-chain from x to y* is a finite sequence $x_0, x_1, \ldots x_n$ with $x_0 = x$ and $x_n = y$ satisfying $d(f(x_i), x_{i+1}) < \epsilon$ for $0 \leq i < n$.

Shadowing

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A map $f : X \to X$ has *shadowing* provided that for all $\epsilon > 0$ there exists a $\delta > 0$ such that for every δ-pseudo-orbit $\langle x_i \rangle_{i \in \mathbb{N}}$ there exists an orbit $\langle f^i(z) \rangle_{i \in \mathbb{N}}$ satisfying $d(x_i, f^i(z)) < \epsilon$.

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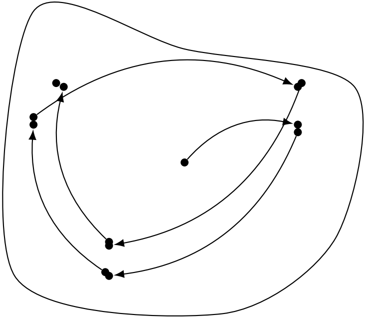
- We say that the orbit $\langle f^i(z) \rangle_{i \in \mathbb{N}}$ ϵ-shadows the δ-pseudo-orbit $\langle x_i \rangle_{i \in \mathbb{N}}$.

Shadowing



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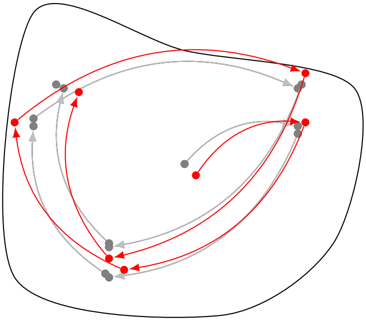

Perturbation and Pseudo-orbits

An Observation

Let f, g be maps on X with $d(f(x), g(x)) < \delta$ for all $x \in X$. Then $\langle g^i(x) \rangle_{i \in \mathbb{N}}$ is a δ-pseudo-orbit for f.

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- Anosov (1969) and Bowen (1975) used this observation to address stability of orbits under perturbation in hyperbolic systems

Perturbation and Shadowing

Lemma

Let f be a map with shadowing. Then for all $\epsilon > 0$ there exists $\delta > 0$ such that for any map g with $d(f(x)), g(x)) < \delta$, and any point $x \in X$, there exists a point $z \in X$ such that $d(f^i(z), g^i(x)) < \epsilon$ for all $i \in \mathbb{N}$.

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Lemma

Let f be a map with shadowing. Then for all $\epsilon > 0$ there exists $\delta > 0$ such that for any map g with $d(f(x)), g(x)) < \delta$, and any point $x \in X$, there exists a point $z \in X$ such that $d(f^i(z), g^i(x)) < \epsilon$ for all $i \in \mathbb{N}$.

- In this sense maps with shadowing exhibit stability of orbits under perturbation.

Finite Precision and Pseudo-orbits

Another Observation

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Another Observation

Since computers have only finite precision, any computed orbit (or orbit segment) is necessarily a pseudo-orbit.

- In a chaotic system, a computed orbit diverges rapidly from a true orbit.
- In a chaotic system with shadowing, the computed orbit is still representative of the true orbit of a (possibly) different point.

Outline

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- For $a = \langle a_i \rangle_{i \in \mathbb{N}} \in \Sigma^{\mathbb{N}}$ and $N \in \mathbb{N}$, let $a\restriction_N = \langle a_0, a_1, a_2, \ldots a_N \rangle$

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- Let for $a, b \in \Sigma^{\mathbb{N}}$, defined $d(a, b) = 2^{-N}$ where N is maximal so that $a\restriction_N = b\restriction_N$ (or zero if $a = b$).

Definitions

- A *shift space* is a compact subset of $\Sigma^{\mathbb{N}}$ which is invariant under the shift map σ

$$\langle a_i \rangle_{i\in\mathbb{N}} \mapsto \langle a_{i+1} \rangle_{i\in\mathbb{N}}$$

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- A shift of finite type is a shift space X characterized by a finite set $\mathcal{F}$ of 'forbidden words' where $a \in X$ if and only if for all $i, N \in \mathbb{N}$, $\sigma^i(a)\restriction_N$ does not belong to $\mathcal{F}$.

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- Without loss of generality, each element of $\mathcal{F}$ has the same length.

Pseudo-orbits in Shift Spaces

Observation

Let $\sigma : X \to X$ be a shift space. Then $\langle a_i \rangle_{i \in \mathbb{N}}$ is a 2^{-N}-pseudo-orbit if and only if $\sigma(a_i)\restriction_N = a_{i+1}\restriction_N$.

```
0 1 1 0 0 1 1 1 1 0 0 0 0 0 0 ...
  1 1 0 0 0 1 0 0 1 1 0 0 1 0 0 ...
    1 0 0 0 1 1 1 0 0 1 0 0 0 1 0 ...
      0 0 0 1 1 1 1 0 1 0 0 1 0 0 1 ...
        0 0 1 1 0 0 0 0 0 0 0 1 0 0 0 ...
          0 1 1 0 0 0 1 0 1 1 1 0 0 0 0 ...
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$$
\begin{array}{l}
0\ \overbrace{1\ 1\ 0\ 0}^{N}\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ \ldots \\
1\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ \ldots \\
1\ 0\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ \ldots \\
0\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ \ldots \\
0\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ \ldots \\
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\boxed{1\ 1\ 0\ 0}\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ \ldots \\
1\ 0\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ \ldots \\
0\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ \ldots \\
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\boxed{1\ 0\ 0\ 0}\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ \ldots \\
0\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ \ldots \\
0\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ \ldots \\
0\ 1\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ \ldots \\
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$$1\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ \ldots$$

$$1\ \boxed{0\ 0\ 0\ 1}\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ \ldots$$

$$\boxed{0\ 0\ 0\ 1}\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ \ldots$$

$$0\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ \ldots$$

$$0\ 1\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ \ldots$$

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$$
\begin{array}{llllllllllllllllllllll}
0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\
 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & \dots \\
 & & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & \dots \\
 & & & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & \dots \\
 & & & & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots \\
 & & & & & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \dots \\
 & & & & & & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots
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1 1 0 0 0 1 0 0 1 1 0 0 1 0 0 ...
1 0 0 0 1 1 1 0 0 1 0 0 0 1 0 ...
0 0 0 1 1 1 1 0 1 0 0 1 0 0 1 ...
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\begin{array}{llllllllllllllllllllll}
0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & & & & & & \\
 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & \dots & & & & & \\
 & & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & \dots & & & & \\
 & & & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & \dots & & & \\
 & & & & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots & & \\
 & & & & & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \dots & \\
 & & & & & & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots
\end{array}
$$

$$c = \; 0\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ \dots$$

Shadowing in Shift Spaces

Obesrvation

In fact this is the *unique* element c of $\Sigma^{\mathbb{N}}$ which could *possibly* ϵ-shadow the pseudo-orbit for *any* $\epsilon < 1$.

$$
\begin{array}{cccccccccccccccccccccc}
0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & & & & & & \\
& 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & \dots & & & & & \\
& & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & \dots & & & & \\
& & & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & \dots & & & \\
& & & & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \dots & & \\
& & & & & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \dots & \\
& & & & & & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \dots
\end{array}
$$

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Shadowing in Shift Spaces

Question

Does c belong to the shift space X?

$$
\begin{array}{l}
0\ 1\ 1\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ \ldots \\
\quad 1\ 1\ 0\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ \ldots \\
\quad\quad 1\ 0\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ \ldots \\
\quad\quad\quad 0\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ \ldots \\
\quad\quad\quad\quad 0\ 0\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ \ldots \\
\quad\quad\quad\quad\quad 0\ 1\ 1\ 0\ 0\ 0\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ \ldots \\
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Shadowing and Shifts of Finite Type

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- Suppose $\sigma : X \to X$ is a shift of finite type and let $N \in \mathbb{N}$ be the length of the elements of $\mathcal{F}$.
- Let $\langle a_i \rangle_{i \in \mathbb{N}}$ be a 2^{-N}-pseudo-orbit in X.
- Construct $c \in \Sigma^{\mathbb{N}}$ as above.
- Observe that for all $i \in \mathbb{N}$, $\sigma^i(c)\restriction_N = a_i\restriction_N \notin \mathcal{F}$ and therefore $c \in X$.

Symbolics in Continua

- Continuum dynamics are often studied by assigning itineraries to points and then working in the space of itineraries

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- Continuum dynamics are often studied by assigning itineraries to points and then working in the space of itineraries
- A significant issue with this approach is that itinerary spaces are naturally totally disconnected.

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- The topology of a dendrite is compatible with a *taxicab metric* d, i.e.
- Given two points x, y and a third point z on the arc connecting x and z, we have

$$d(x, y) = d(x, z) + d(z, y)$$

Baldwin's symbolics

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- Let X be a dendrite and $f : X \to X$ such that f has a single turning point t and f is expanding by a factor $\lambda > 1$ on components of $X \setminus \{t\}$.
- Furthermore, suppose that f is *self-similar* in the sense that for each component M of $X \setminus \{t\}$, $f(M \cup \{t\}) = X$.

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- Furthermore, suppose that f is *self-similar* in the sense that for each component M of $X \setminus \{t\}$, $f(M \cup \{t\}) = X$.
- Then $f : X \to X$ is conjugate to a map in the collection we will now describe.

Baldwin's symbolics

Itinerary Space

Give $\{0, 1, \ldots, n, *\}$ with topology generated by the basis

$$\{\{0\}, \{1\}, \ldots, \{n\}, \{0, 1, \ldots, n, *\}\}.$$

Let $\Lambda = \{0, 1, \ldots, n, *\}^{\mathbb{N}}$ with the induced product topology.

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Let $\Lambda = \{0, 1, \ldots, n, *\}^{\mathbb{N}}$ with the induced product topology.

- The topology on Λ is not Hausdorff.
- There are many shift invariant Hausdorff subspaces.

Baldwin's symbolics

- A sequence $\tau = \langle \tau_n \rangle \in \Lambda$ is called *acceptable* if
  - $\tau_n = *$ if and only if $\sigma^{n+1}(\tau) = \tau$
  - If $\sigma^n(\tau) \neq \tau$, then $\sigma^n(\tau)$ and τ are distinguishable in Λ

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  - If $\sigma^n(\tau) \neq \tau$, then $\sigma^n(\tau)$ and τ are distinguishable in Λ
- A sequence $\alpha \in \Lambda$ is τ-*consistent* provided that if $\alpha_n = *$, then $\sigma^{n+1}(\alpha) = \tau$.

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  - If $\sigma^n(\tau) \neq \tau$, then $\sigma^n(\tau)$ and τ are distinguishable in Λ
- A sequence $\alpha \in \Lambda$ is τ*-consistent* provided that if $\alpha_n = *$, then $\sigma^{n+1}(\alpha) = \tau$.
- A τ-consistent sequence α is τ*-admissible* provided that if $\sigma^n(\alpha) \neq *\tau$, then $\sigma^n(\alpha)$ and $*\tau$ are distinguishable in Λ

Baldwin's symbolics

The dendrite D_τ

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- D_τ is a dendrite
- $\sigma(D_\tau) = D_\tau$
- $*\tau$ is the only turning point of $\sigma|D_\tau$.
- $\sigma|D_\tau$ is self-similar in the earlier sense.

Baldwin's symbolics

Theorem (Baldwin)

Let X be a dendrite and let $f : X \to X$ be a self-similar piecwise expanding dendrite map with a single turning point(as described earlie).Then there exists $n \in \mathbb{N}$ and $\tau \in \{1, 2, \ldots, n, *\}^{\mathbb{N}}$ such that f is conjugate to the shift map restricted to D_τ.

Distance in D_τ

Definition

Let $x, y \in D_\tau$ and let $N \in \mathbb{N}$. We say $x\restriction_N \simeq y\restriction_N$ provided that there exists $z \in D_\tau$ for which $z\restriction_N$ is indistinguishable from both $x\restriction_N$ and $y\restriction_N$ (in $\{0, 1, *\}^N$).

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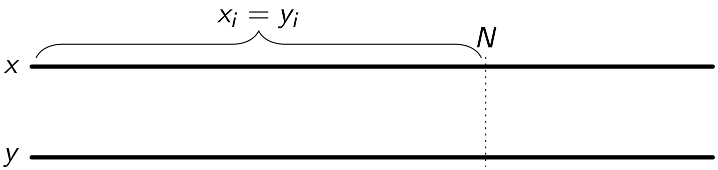
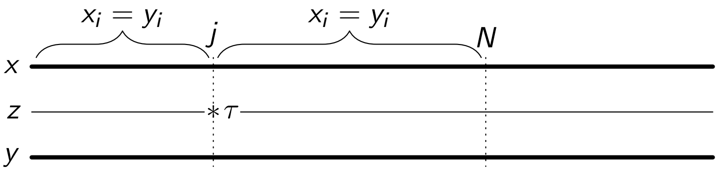
- $x\restriction_N \simeq y\restriction_N$ provided that
  - $x_i = y_i$ for all $i \leq N$, or
  - there exists $z = z_1 z_2 \ldots z_j * \tau$ with $j \leq N$ such that for all $i \leq N$ either $x_i = y_i = z_i$ or $z_i = *$

Distance in $D_\mathcal{T}$



Distance in D_τ



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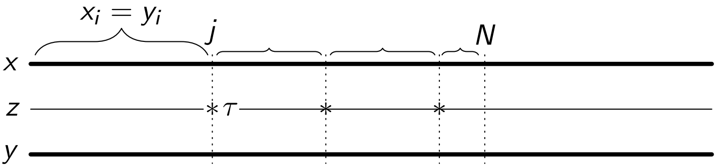

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- *For each $N \in \mathbb{N}$ there exists $\delta_N > 0$ such that for $x, y \in D_\tau$, $d(x, y) < \delta_N$ implies $x\restriction_N \simeq y\restriction_N$.*

Pseudo-orbits in D_τ

Observation

Let $\sigma : D_\tau \to D_\tau$. Then $\langle a_i \rangle_{i \in \mathbb{N}}$ is a δ-pseudo-orbit only if $\sigma(a_i)\restriction_{N_\delta} \simeq a_{i+1}\restriction_{N_\delta}$.

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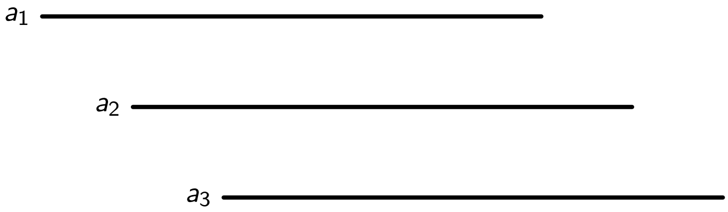

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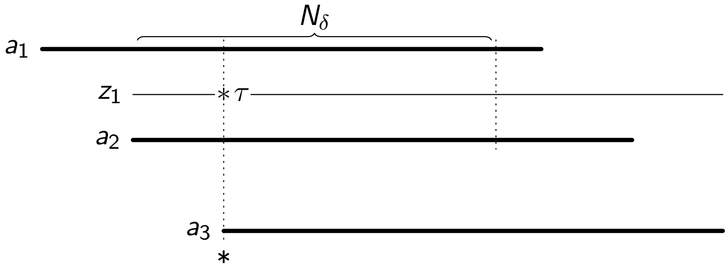

Pseudo-orbits in $D_\mathcal{T}$

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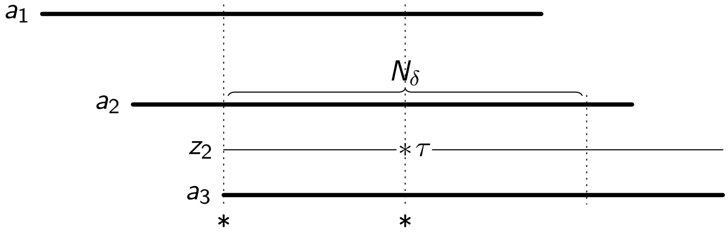

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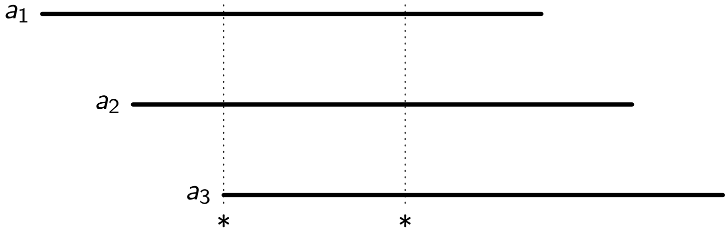

Shadowing in D_τ

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- The only obstacle to using the same construction as in shifts of finite type is those columns in which we have a disagreement of symbols.
- In particular, we might run into trouble when two such columns are within N_ϵ of one another.

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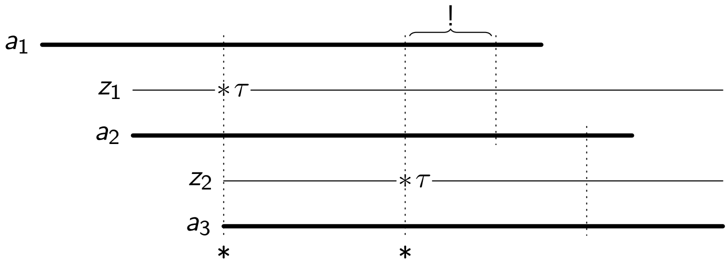

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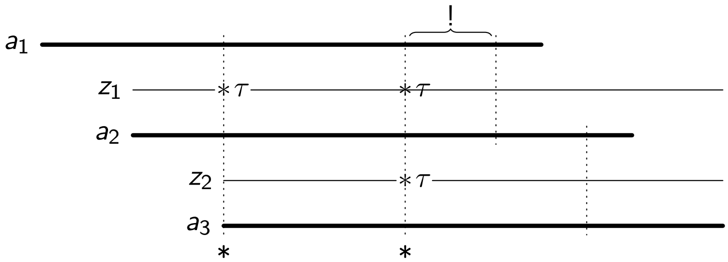

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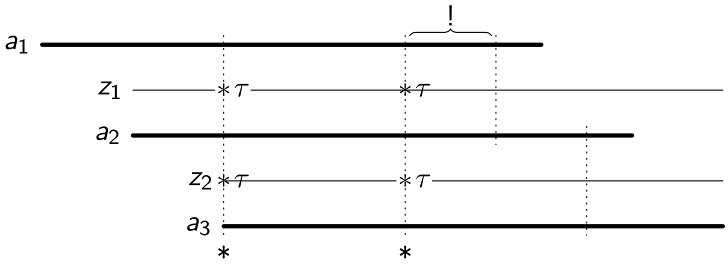

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- Construct c as in shifts of finite type, with the exception that if the i-th column has a disagreement within the first N many symbols, choose an arbitrary symbol.
- Any N_ϵ length piece of the constructed itinerary either
  - misses all such columns, or
  - all corresponding pieces of the a_i are indistinguishable from the apropriate shifts of eachother (and hence from c).

Shadowing in D_τ

Theorem

For each acceptable τ *in* $\{0, 1, \ldots, n, *\}^{\mathbb{N}}$, $\sigma : D_\tau \to D_\tau$ *has shadowing.*

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For each acceptable τ in $\{0, 1, \ldots, n, *\}^{\mathbb{N}}$, $\sigma : D_\tau \to D_\tau$ has shadowing.

- In particular, unimodal, self-similar dendrite maps have shadowing
- As a corollary it follows (with some work) that Julia sets of quadratic polynomials that are dendrites all have shadowing.

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- Define a topology on Γ by taking as basis the collection

$$\{\{\alpha\restriction_N, s_0(\alpha)\restriction_N, s_1(\alpha)\restriction_N\} : \alpha \in \Gamma, N \in \mathbb{N}\}$$

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- For an acceptable sequence τ, the space E_τ of τ-admissible sequences is well-structured and exhibits shadowing.

Shadowing in Quadratic Julia sets

Theorem

Let $c \in \mathbb{C}$ and suppose that f_c defined by $z \mapsto z^2 + c$ has an attracting or parabolic periodic point. If the associated kneading sequence τ is not an n-tupling, then τ is an acceptable sequence in Γ and f_c restricted to its Julia set is conjugate to σ on E_τ.

Shadowing in Quadratic Julia sets

Corollary

Let $c \in \mathbb{C}$ and suppose that f_c defined by $z \mapsto z^2 + c$ has an attracting or parabolic periodic point. If the associated kneading sequence τ is not an n-tupling, then f_c restricted to its Julia set has shadowing.

Asymptotic Shadowing

- In all of these settings, once we fix $\epsilon > 0$ and find the $\delta > 0$ such that every δ-pseudo-orbit $\langle a_i \rangle$ is ϵ-shadowed by some $x \in X$, we can actually say a bit more.

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- If the sequence $\langle a_i \rangle$ also has the property that for every $\eta > 0$ there is an $N \in \mathbb{N}$ such that $\langle a_i \rangle_{i \geq N}$ is an η-pseudo-orbit, then the constructed shadowing point will have the property that for all $\gamma > 0$ there exists $M \in \mathbb{N}$ such that $f^M(x)$ γ shadows $\langle a_i \rangle_{i \geq M}$.

Asymptotic Shadowing

Theorem

Let X be a shift of finite type or either D_τ or E_τ for an acceptable τ. Then $\sigma : X \to X$ has asymptotic shadowing.

Questions for Further Research

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- Can these techniques be extended to handle higher degree polynomial Julia sets?
- Can shadowing be classified in the category of dendrite maps?

Outline

ω-limit sets

- For a map $f : X \to X$, the ω-limit set of a point $x \in X$ is the set
$$\omega(x) = \bigcap_{n \in \mathbb{N}} \overline{\{f^i(x) : i \geq n\}}$$

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- Bowen used shadowing to characterize ω-limit sets for Axiom A diffeormorphisms.
- In particular, for an Axiom A diffeomorphism f, the ω-limit sets of f are precisely those sets which are 'abstract ω-limit sets.'

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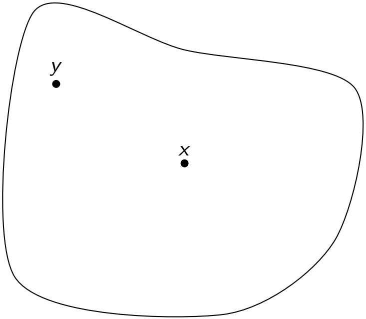

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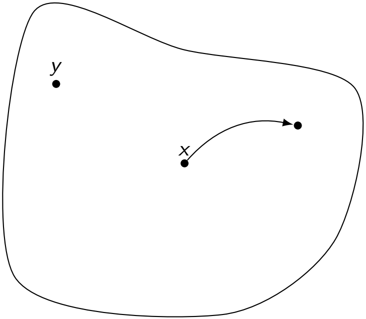

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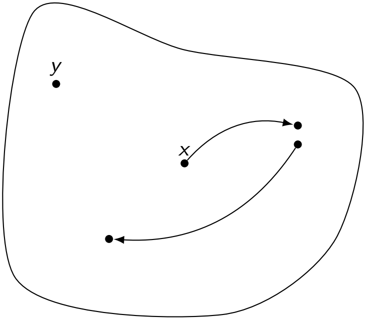

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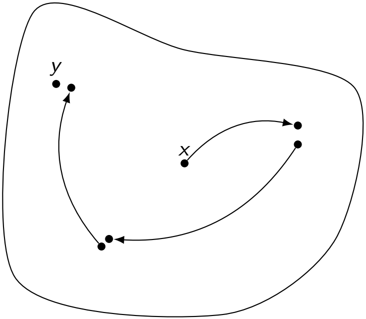

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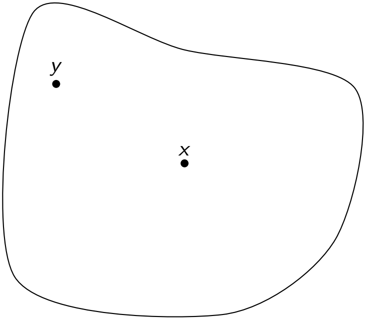

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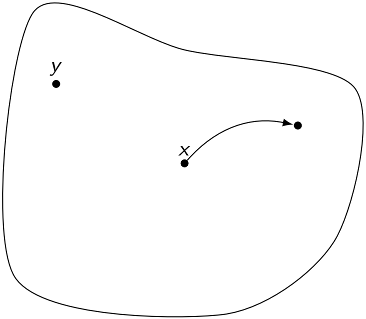

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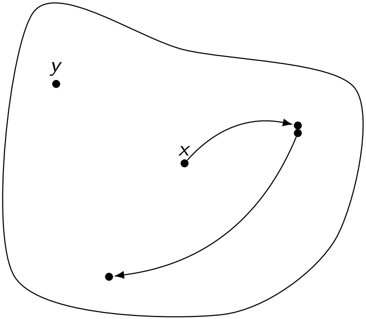

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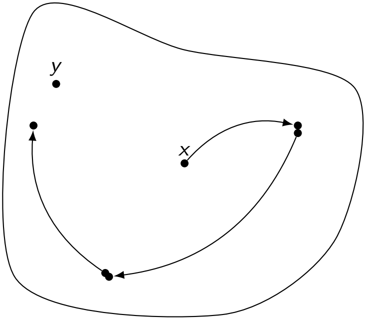

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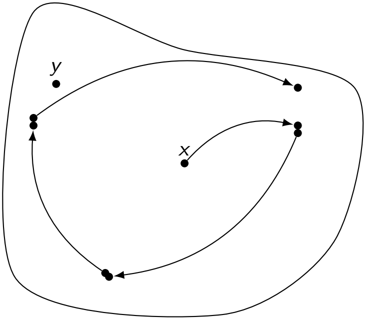

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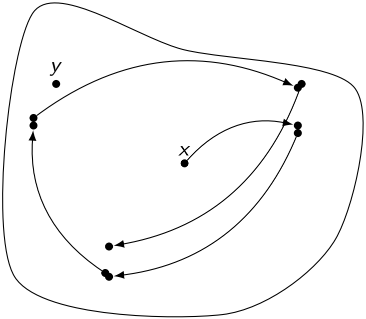

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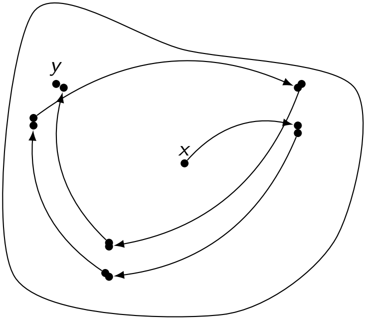

Asymptotic Pseudo-Orbits

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Theorem (Barwell, Good, Oprocha, Raines)

A nonempty closed set $A \subseteq X$ is internally chain transitive if and only if it is the ω-limit of some asymptotic pseudo-orbit $\langle x_i \rangle_{i\in\mathbb{N}}$ in X.

Hausdorff metric

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The metric d on X induces a metric d_H on 2^X given by:

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- 2^X is compact with respect to the topology generated by d_H.

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- $\omega(f) \subseteq ICT(f) \subseteq 2^X$.

Internal chain transitivity and ω-limit sets

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Question

Can we characterize those systems for which the collection of nonempty closed internally chain transitive sets is equal to the collection of ω-limit sets?

A conjecture

- Many of the known examples of systems in which internal chain transitivity characterizes ω-limit sets exhibit shadowing.

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- Recently, a counterexample was discovered (Puljiz 2013).
- However, there still seemed to be a strong connection to shadowing.

Main Theorem

Theorem

If $f : X \to X$ has shadowing, then $\omega(f) = ICT(f)$ if and only if $\omega(f)$ is closed with respect to the Hausdorff metric.

Outline of Proof

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Outline of Proof

Corollary

If $\omega(f) = ICT(f)$, then $\omega(f)$ is closed.

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If $f : X \to X$ has shadowing and $\omega(f)$ is closed, then $\omega(f) = ICT(f)$.

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- Let $a, b \in C$ and fix $\epsilon > 0$.
- By unif. cont., let $\delta > 0$ such that if $d(p,q) < \delta$, then $d(f(p), f(q)) < \epsilon/3$. WLOG, $\delta < \epsilon/3$.

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- Choose k such that $d_H(C_k, C) < \delta$.

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- Then $\langle x_i \rangle$ is an ϵ-chain in C from a to b:

$$d(f(x_i), x_{i+1}) \leq d(f(x_i), f(x_i')) + d(f(x_i'), x_{i+1}') + d(x_{i+1}', x_{i+1})$$

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- Thus $C \in ICT(f)$ and $ICT(f)$ is closed.

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- Let $C \in ICT(f)$. By [BGOR], there exists an asymptotic pseudo-orbit $\langle x_i \rangle_{i\in\mathbb{N}}$ with $\omega(\langle x_i \rangle) = C$.
- For all $\delta > 0$ there exists $M_\delta \in \mathbb{N}$ such that $\langle x_{i+M_\delta} \rangle$ is a δ-pseudo-orbit.

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- Since f has shadowing, choose $\delta > 0$ such that every δ-pseudo-orbit is $\epsilon/2$-shadowed.
- In particular, choose $z \in X$ such that $\langle f^i(z) \rangle$ $\epsilon/2$-shadows $\langle x_{i+M_\delta} \rangle$.

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- WLOG, the sequence $\langle x_{n_i+M} \rangle$ converges to some $b \in C$.
- Then $d(a,b) = \lim d(f^{n_i}(z), x_{n_i+M}) \leq \epsilon/2$ and so

$$\sup_{a \in \omega(z)} \inf_{b \in C} d(a,b) \leq \epsilon/2 < \epsilon.$$

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- Additionally, for all $b \in C$, there exists a sequence $\langle n_i \rangle$ of natural numbers greater than M_δ with $x_{n_i} \to b$.

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- Then $d(a, b) = \lim d(f^{n_i - M_\delta}(z), x_{n_i}) \leq \epsilon/2$ and so

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- In particular, $d_H(\omega(z), C) < \epsilon$.
- This holds for all $\epsilon > 0$, and so $C \in \overline{\omega(f)}$.

Interval Maps

- Interval maps are known to satisfy $\omega(f)$ being closed [Blokh, Bruckner, Humke, Smítal].

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Corollary

If $f : I \to I$ has shadowing, then $\omega(f) = ICT(f)$.

Shifts of finite type

- Shifts of finite type are known to exhibit both shadowing and $\omega(f) = ICT(f)$ [Barwell, Good, Knight, Raines].

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In shifts of finite type, $\omega(\sigma)$ is closed.

Quadratic Julia sets

- The complex map $f_c(z) = z^2 + c$ restricted to its Julia set exhibits both shadowing and $\omega(f) = ICT(f)$ for certain parameters c [Barwell, M, Raines]

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Corollary

For parameters c such that either J_c is a dendrite, or f_c has an attracting or parabolic periodic point, and kneading sequence τ which is not an n-tupling, $\omega(f_c|J_c)$ is closed.

Thank you

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