# Random graphs from minor-closed classes

Colin McDiarmid

Oxford University

Birmingham, August 2013

# OR: 'Random planar graphs and beyond'

thanks to

This is an outgrowth of work on random planar graphs started with Angelika Steger and Dominic Welsh. and with Manuel Bodirsky, Chris Dowden, Stefanie Gerke, Mihyun Kang, Valentas Kurauskas, Mike Loeffler, Marc Noy, Bruce Reed and Andreas Weißl.

## Question

Let $\mathcal{A}$ be a class of (simple) graphs closed under isomorphism, eg the class $\mathcal{P}$ of planar graphs.

$\mathcal{A}_n$ is the set of graphs in $\mathcal{A}$ on vertices $1, \ldots, n$.

$R_n \in_u \mathcal{A}$ means that $R_n$ is picked uniformly at random from $\mathcal{A}_n$.

What are typical properties of $R_n$?

usually a giant component? probability of being connected? many vertices of degree 1? size of the 2-core?

# Minors

$H$ is a minor of $G$ if $H$ can be obtained from a subgraph of $G$ by edge-contractions.

$\mathcal{A}$ is minor-closed if

$$G \in \mathcal{A},\ H \text{ a minor of } G \quad \Rightarrow\ H \in \mathcal{A}$$

Examples:

forests, series-parallel graphs, and more generally graphs of treewidth $\leq k$;

outerplanar graphs, planar graphs, and more generally graphs embeddable on a given surface;

graphs with at most $k$ (vertex) disjoint cycles.

# Minors

$Ex(\mathcal{H})$ is the class of graphs with no minor a graph in $\mathcal{H}$. For example, the class of planar graphs is $Ex(\{K_5, K_{3,3}\})$.

Easy to see that: $\mathcal{A}$ is minor-closed iff $\mathcal{A} = Ex(\mathcal{H})$ for some class $\mathcal{H}$.

Robertson and Seymour's graph minors theorem (once Wagner's conjecture) is that if $\mathcal{A}$ is minor-closed then $\mathcal{A} = Ex(\mathcal{H})$ for some **finite** class $\mathcal{H}$.

The unique minimal such $\mathcal{H}$ consists of the excluded minors for $\mathcal{A}$.

Mostly we shall assume that $\mathcal{A}$ is minor-closed.

# Two related problems

## typical properties and counting

- finding typical properties of $R_n \in_u \mathcal{A}_n$
- counting - that is, estimating $|\mathcal{A}_n|$

(We will not discuss here a third related problem: how to generate $R_n$.)

We want to talk about typical properties but must start with counting.

# Estimating $|\mathcal{A}_n|$

two levels

Two natural levels at which to work:

(a) combinatorial / probabilistic

(b) generating functions / singularity analysis

Typically (a) is approximate, and (b) is asymptotically exact – if you can do it!.

We shall focus on (a) but first we say something brief on (b).

For a class $\mathcal{A}$ of graphs, let $A(x)$ denote the exponential generating function (egf) $\sum_n |\mathcal{A}_n| x^n / n!$.

# Asymptotically exact counting

For suitable classes of graphs, we can relate the egfs (or two variable versions) of all graphs, connected graphs, 2-connected graphs and 3-connected graphs.

If we know enough about the 3-connected graphs (as we do for planar graphs, thanks to Tutte and others) then we may be able to extend to all graphs.

Gimenéz and Noy (2009) extended work of Bender, Gao and Wormald (2002) to handle planar graphs.

Chapuy, Fusy, Gimenéz, Mohar and Noy (2011) and Bender and Gao (2011) went on to handle $\mathcal{G}^S$.

You have heard or will hear more on this topic this week!

# Decomposable families

If a graph is in $\mathcal{A}$ if and only if each component is, then we call $\mathcal{A}$ decomposable.

For example the class of planar graph is decomposable but the class of graphs embeddable on the torus is not.

A minor-closed class is decomposable iff each excluded minor is connected.

Let $\mathcal{A}$ be a decomposable class of graphs; and let $\mathcal{C}$ consist of the connected graphs in $\mathcal{A}$, with egf $C(x)$. The exponential formula says that $A(x) = e^{C(x)}$.

# Growth constant

## definition

$\mathcal{A}$ has a growth constant $\gamma$ if

$$(|\mathcal{A}_n|/n!)^{1/n} \to \gamma \quad \text{as } n \to \infty,$$

that is, if

$$|\mathcal{A}_n| = (\gamma + o(1))^n \; n!.$$

$\mathcal{A}$ has growth constant $\gamma \implies A(x)$ has radius of convergence $\rho = 1/\gamma$.

If $\mathcal{A}$ is decomposable, then the exponential formula shows that $\mathcal{A}$ and $\mathcal{C}$ have the same radius of convergence.

# When is there a growth constant?

small

Call $\mathcal{A}$ small if $\exists c$ such that $|\mathcal{A}_n| \leq c^n n!$

Norine, Seymour, Thomas and Wollan (2006); Dvorák and Norine (2010):

**Lemma**

*Each (proper) minor-closed graph class $\mathcal{A}$ is small.*

We can prove this using Mader's result that each graph in $\mathcal{A}$ has average degree at most $b$.

# When is there a growth constant?

## bridge-addable and addable

$\mathcal{A}$ is bridge-addable if whenever $G \in \mathcal{A}$ and $u$ and $v$ are in different components of $G$ then $G + uv \in \mathcal{A}$.

$\mathcal{A}$ is addable if it is decomposable and bridge-addable.

A minor-closed class $\mathcal{A}$ is addable iff each excluded minor is 2-connected.

$\mathcal{G}^S$ is bridge-addable but not decomposable (and so not addable) except in the planar case.

## When is there a growth constant?

bridge-addable and being connected

From McDiarmid, Steger and Welsh (2005):

**Lemma**

*If $\mathcal{A}$ is bridge-addable and $R_n \in_u \mathcal{A}$ then*

$$\mathbb{P}(R_n \textit{ is connected}) \geq 1/e.$$

For trees $\mathcal{T}$ and forests $\mathcal{F}$, $|\mathcal{T}_n| = n^{n-2}$ and $|\mathcal{F}_n| \sim e^{\frac{1}{2}} n^{n-2}$. Thus for $R_n \in_u \mathcal{F}$,

$$\mathbb{P}(R_n \text{ is connected}) \sim e^{-\frac{1}{2}}.$$

# Aside: bridge-addable and being connected

McDiarmid, Steger and Welsh (2006) conjectured:

**Conjecture**

*If $\mathcal{A}$ is bridge-addable then $\mathbb{P}(R_n$ is connected$) \geq e^{-\frac{1}{2}+o(1)}$.*

Balister, Bollobás and Gerke (2008) give an asymptotic lower bound of $e^{-0.7983}$.

Addario-Berry, McDiarmid and Reed (2013?), and Kang and Panagiotou (2013?) proved the conjecture **if** $\mathcal{A}$ is also closed under deleting bridges.

Further recent work by Norine, but the full conjecture is still open.

# When is there a growth constant?

small and addable

## Lemma

*$\mathcal{A}$ small and addable $\Rightarrow$ $\exists$ growth constant $\gamma(\mathcal{A})$*

Proof. Since $\mathcal{A}$ is bridge-addable, $\mathbb{P}(R_n$ is connected$) \geq 1/e$.

Since also $\mathcal{A}$ is decomposable

$$|\mathcal{A}_{a+b}| \geq \binom{a+b}{a} \frac{|\mathcal{A}_a|}{e} \frac{|\mathcal{A}_b|}{e} \frac{1}{2}$$

and so $f(n) = \frac{|\mathcal{A}_n|}{2e^2 n!}$ satisfies $f(a+b) \geq f(a) \cdot f(b)$; that is, $f$ is supermultiplicative. Now use 'Fekete's lemma' to show that

$$f(n)^{1/n} \to \sup_k f(k)^{1/k} < \infty.$$

# When is there a growth constant?

## minor-closed and addable – and $\mathcal{G}^S$

**Theorem**

*Each addable proper minor-closed class $\mathcal{A}$ has a growth constant $\gamma(\mathcal{A})$.*

Thus $\mathcal{P}$ has a growth constant.

For any surface $S$ other than the plane, $\mathcal{G}^S$ is bridge-addable but not addable. However, we can show that $\mathcal{G}^S$ has a growth constant since it is 'not much bigger' than $\mathcal{P}$, and in fact has the same growth constant as $\mathcal{P}$.

As we noted, we now know **much** more, indeed asymptotic formulae.

# Having a growth constant yields ..

## Pendant copies theorem - introduction

$G$ has a pendant copy of $H$ if $G$ has a bridge $e$ with $H$ at one end, and $e$ incident with the root of $H$.

A connected graph $H$ with root vertex $r$ is freely attachable to $\mathcal{A}$ if whenever we have a graph $G$ in $\mathcal{A}$ and a disjoint copy of $H$, and we add an edge between $r$ and a vertex in $G$, then the resulting graph must be in $\mathcal{A}$.

For an addable minor-closed class $\mathcal{A}$, the class of graphs freely attachable to $\mathcal{A}$ is the class of connected graphs in $\mathcal{A}$.

For $\mathcal{G}^S$, the class of freely attachable graphs is the class of connected planar graphs.

# Pendant copies theorem

**Theorem**

*Let $\mathcal{A}$ have a finite positive growth constant, and let the connected graph $H$ be freely attachable to $\mathcal{A}$. Let $R_n \in_u \mathcal{A}$. Then there exists $\alpha > 0$ such that*

$$\Pr(R_n \textit{ has } < \alpha n \textit{ pendant copies of } H) = e^{-\Omega(n)}.$$

For $R_n \in_u \mathcal{P}$, whp $\omega(R_n) = 4$ and so $\chi(R_n) = 4$.

For $R_n \in_u \mathcal{G}^S$, whp $\omega(R_n) = 4$ and $\chi(R_n) \in \{4, 5\}$ (and $\chi_{list}(R_n) = 5$).

Hadwiger's Conjecture being false says that for some $k$, there is a graph $G \in Ex(K_k)$ with $\chi(G) \geq k$. But then for $R_n \in_u Ex(K_k)$, whp $\chi(R_n) \geq k$.

## Big component

The big component $\mathrm{Big}(G)$ of a graph $G$ is the (lex first) component with most vertices.

The fragment 'left over', $\mathrm{Frag}(G)$, is the subgraph induced on the vertices not in the big component.

Write $\mathrm{frag}(G)$ for $v(\mathrm{Frag}(G))$.

**Theorem**

*If $\mathcal{A}$ is bridge-addable then $\mathbb{E}[\mathrm{frag}(R_n)] < 2$.*

Thus $\mathrm{Big}(R_n)$ is giant!

# Smoothness

Let $\mathcal{A}$ be any small class of graphs.

Call $\mathcal{A}$ smooth if $\frac{|\mathcal{A}_n|}{n|\mathcal{A}_{n-1}|} \to$ a limit as $n \to \infty$.

In this case the limit must be the growth constant $\gamma$.

All the classes for which we know an asymptotic counting formula are smooth, for example series-parallel graphs, $\mathcal{P}$, $\mathcal{G}^S$.

Showing smoothness is an important step in proving results about $R_n \in_u \mathcal{A}$.

## When is $\mathcal{A}$ smooth?

Bender, Canfield and Richmond (2008) show:

**Theorem**

*$\mathcal{G}^S$ is smooth for any surface $S$.*

The proof did not involve an asymptotic counting formula (and indeed none was then known). The method can be adapted to show more. The key idea is to consider the core (2-core).

The core of $G$, $\mathrm{core}(G)$, is the unique maximal subgraph such that the minimum degree $\delta(G) \geq 2$.

We start again, in greater generality.

## General model

### Binomial random graph $G_{n,p}$

In the classical binomial random graph $G_{n,p}$ on the vertex set $[n]$, the $\binom{n}{2}$ possible edges appear independently with probability $p$, $0 < p < 1$.

For each $H \in \mathcal{A}_n$

$$\mathbb{P}(G_{n,p} = H | G_{n,p} \in \mathcal{A}) = \frac{p^{e(H)}(1-p)^{\binom{n}{2}-e(H)}}{\sum_{G \in \mathcal{A}_n} p^{e(G)}(1-p)^{\binom{n}{2}-e(G)}} = \frac{\lambda^{e(H)}}{\sum_{G \in \mathcal{A}_n} \lambda^{e(G)}}$$

where $\lambda = p/(1-p)$.

Here we assume that $\mathcal{A}_n \neq \emptyset$, and $e(G)$ denotes the number of edges in $G$.

# General model

## Random cluster model

In the more general random-cluster model, we are also given $\nu > 0$; and the random graph $R_n$ ranges over the graphs $H$ on $[n]$, with

$$\mathbb{P}(R_n = H) \propto p^{e(H)}(1-p)^{\binom{n}{2}-e(H)} \cdot \nu^{\kappa(H)}.$$

Here $\kappa(H)$ denotes the number of components of $H$. For each $H \in \mathcal{A}_n$ we have

$$\mathbb{P}(R_n = H \mid R_n \in \mathcal{A}) = \frac{\lambda^{e(H)}\nu^{\kappa(H)}}{\sum_{G \in \mathcal{A}_n} \lambda^{e(G)}\nu^{\kappa(G)}}.$$

# General model

## New model

The distribution of our random graphs in $\mathcal{A}$ is as follows.

Given edge-parameter $\lambda > 0$ and component-parameter $\nu > 0$, we let the weighting $\tau$ be the pair $(\lambda, \nu)$. For each graph $G$ we let $\tau(G) = \lambda^{e(G)} \nu^{\kappa(G)}$; and we denote $\sum_{G \in \mathcal{A}_n} \tau(G)$ by $\tau(\mathcal{A}_n)$.

$R_n \in_\tau \mathcal{A}$ means that $R_n$ is a random graph which takes values in $\mathcal{A}_n$ with

$$\mathbb{P}(R_n = H) = \frac{\tau(H)}{\tau(\mathcal{A}_n)}.$$

We call $R_n$ a $\tau$-*weighted random graph from* $\mathcal{A}$.

When $\lambda = \nu = 1$ we are back to random graphs sampled uniformly.

# Well-behaved graph classes

Results below will involve a well-behaved weighted class of graphs $(\mathcal{A}, \tau)$. We require that $\mathcal{A}$ is proper, minor-closed and bridge-addable, and satisfies certain further conditions.

The following classes of graphs are all well-behaved, with any weighting $\tau$:

any proper, minor-closed, addable class (for example the class of forests, or series-parallel graphs or planar graphs);

the class $\mathcal{G}^S$ of graphs embeddable on any given surface $S$;

the class of all graphs which contain at most $k$ vertex-disjoint cycles.

The definition of 'well-behaved' requires $\mathcal{A}$ also to be 'freely-addable-or-limited'.

# Well-behaved graph classes

## Freely-addable-or-limited classes of graphs

$H \in \mathcal{A}$ is freely addable to $\mathcal{A}$ if the disjoint union $G \cup H \in \mathcal{A}$ whenever $G \in \mathcal{A}$.

$H \in \mathcal{A}$ is limited in $\mathcal{A}$ if $kH$ is not in $\mathcal{A}$ for some positive integer $k$.

If $\mathcal{A}$ is $\mathcal{G}^S$ then the freely addable graphs are the planar graphs, and the limited graphs are the non-planar graphs.

$\mathcal{A}$ is freely-addable-or-limited if each graph in $\mathcal{A}$ is either freely addable or limited (it cannot be both).

Decomposable classes, $\mathcal{G}^S$ and $Ex(kC_3)$ are all freely-addable-or-limited.

$Ex(C_3 \cup C_4)$ is **not** freely-addable-or-limited.

# Boltzmann Poisson random graph

For a weighted graph class $(\mathcal{A}, \tau)$ the egf is

$$A(x, \tau) = \sum_{n \geq 0} \tau(\mathcal{A}_n) \frac{x^n}{n!}.$$

We use $\mathcal{U}\mathcal{A}$ to denote the unlabelled graphs in $\mathcal{A}$.

Fix $\rho > 0$ such that $A(\rho, \tau)$ is finite; and let

$$\mu(H) = \frac{\rho^{v(H)} \tau(H)}{\mathrm{aut}(H)} \text{ for each } H \in \mathcal{U}\mathcal{A} \tag{1}$$

Easy manipulations give $A(\rho, \tau) = \sum_{H \in \mathcal{U}\mathcal{A}} \mu(H)$.

# Boltzmann Poisson random graph

Let $\mathcal{A}$ be decomposable. The Boltzmann Poisson random graph $R = R(\mathcal{A}, \rho, \tau)$ takes values in $\mathcal{UA}$, with

$$\mathbb{P}[R = H] = \frac{\mu(H)}{A(\rho, \tau)} \quad \text{for each } H \in \mathcal{UA}.$$

Let $\mathcal{C}$ denote the class of connected graphs in $\mathcal{A}$. For each $H \in \mathcal{UC}$ let $\kappa(G, H)$ be the number of components of $G$ isomorphic to $H$.

**Theorem**

*The random variables $\kappa(R, H)$ for $H \in \mathcal{UC}$ are independent, with $\kappa(R, H) \sim \mathrm{Po}(\mu(H))$.*

# Fragments theorem

## Theorem

*Let* $(\mathcal{A}, \tau)$ *be well-behaved, and let* $\rho$ *be the radius of convergence of* $A(x, \tau)$*. Let* $\mathcal{F}_{\mathcal{A}}$ *be the class of graphs freely addable to* $\mathcal{A}$*, with egf* $F_{\mathcal{A}}$*.*

*Then* $0 < \rho < \infty$ *and* $F_{\mathcal{A}}(\rho, \tau)$ *is finite;*

*and for* $R_n \in_\tau \mathcal{A}$*,* $F_n = \mathcal{U}\mathrm{Frag}(R_n)$ *satisfies*

$$F_n \to_d R$$

*where* $R$ *is the Boltzmann Poisson random graph* $R(\mathcal{F}_{\mathcal{A}}, \rho, \tau)$*.*

# Corollaries on Fragments and connectivity

## Corollary

*Let $\mathcal{D}$ be the class of connected graphs in $\mathcal{F}_{\mathcal{A}}$. Given distinct graphs $H_1, \ldots, H_k$ in $\mathcal{UD}$ the $k$ random variables $\kappa(F_n, H_i)$ are asymptotically independent with distribution $\mathrm{Po}(\mu(H_i))$.*

- $\kappa(F_n) \to_d \mathrm{Po}(D(\rho, \tau))$
- $\mathbb{P}[R_n \text{ is connected }] \to e^{-D(\rho,\tau)} = F_{\mathcal{A}}(\rho, \tau)^{-1}$
- $\mathbb{E}[\kappa(F_n)] \to D(\rho, \tau)$
- variance of $\kappa(F_n) \to D(\rho, \tau)$
- $\mathrm{frag}(R_n) = v(F_n) \to_d v(R)$

# Example: forests and trees

Let us illustrate these results for the classes $\mathcal{A} = \mathcal{F}$ of forests and $\mathcal{T}$ of trees. Observe that $\mathcal{F}_{\mathcal{A}}$ is $\mathcal{F}$ and $\mathcal{D}$ is $\mathcal{T}$.

By Cayley's formula $\tau(\mathcal{T}_n) = n^{n-2}\lambda^{n-1}\nu$, so the egf $T$ is:

$$T(x,\tau) = \nu \sum_{n\geq 1} n^{n-2}\lambda^{n-1}x^n/n!$$

The radius of convergence is $\rho = (e\lambda)^{-1}$; and

$$T(\rho,\tau) = \nu \sum_{n\geq 1} n^{n-2}\lambda^{n-1}(e\lambda)^{-n}/n! = \frac{\nu}{\lambda}\sum_{n\geq 1}\frac{n^{n-2}}{e^n n!} = \frac{\nu}{2\lambda}.$$

# Example: random forests

Consider $R_n \in_\tau \mathcal{F}$.
Since $T(\rho, \tau) = \frac{\nu}{2\lambda}$, $\kappa(R_n) \to_d 1 + \mathrm{Po}(\frac{\nu}{2\lambda})$; and

$$\mathbb{P}(R_n \text{ is connected }) = \frac{\tau(\mathcal{T}_n)}{\tau(\mathcal{F}_n)} \to e^{-\frac{\nu}{2\lambda}}.$$

Thus

$$\tau(\mathcal{F}_n) \sim \nu e^{\frac{\nu}{2\lambda}} \; n^{n-2} \lambda^{n-1}.$$

Also, $F_n = \mathcal{U}\mathrm{Frag}(R_n)$ satisfies $F_n \to_d R$, and $\mathrm{frag}(R_n) \to_d v(R)$, and indeed (with a little more work)

$$\mathbb{E}[\mathrm{frag}(R_n)] \to \mathbb{E}[v(R)] = \rho T'(\rho, \tau) = \frac{\nu}{\lambda}.$$

# Smoothness and $\mathrm{core}(R_n)$ theorem

## Theorem

*Let $(\mathcal{A}, \tau)$ be well-behaved, with growth constant $\gamma$; let $\mathcal{C}$ denote the class of connected graphs in $\mathcal{A}$; and let $R_n \in_\tau \mathcal{A}$. Then*

*(a) Both $(\mathcal{A}, \tau)$ and $(\mathcal{C}, \tau)$ are smooth with growth constant $\gamma$, and $\gamma \geq \lambda e$.*

*(b) Let $\mathcal{C}^{\delta \geq 2}$ denote the class of graphs in $\mathcal{C}$ with minimum degree at least 2. If $\gamma > \lambda e$ then $(\mathcal{C}^{\delta \geq 2}, \tau)$ has growth constant $\beta$ where $\beta$ is the unique root $> \lambda$ to $\beta e^{\lambda/\beta} = \gamma$; and if $\gamma \leq \lambda e$ then $\rho(\mathcal{C}^{\delta \geq 2}, \tau) \geq \lambda^{-1}$.*

more ..

# Smoothness and $\mathrm{core}(R_n)$ theorem continued

## Theorem

*(c) If $\gamma > \lambda e$ let $\alpha = 1 - x$ where $x$ is the unique root $< 1$ to $xe^{-x} = \lambda/\gamma$; and otherwise let $\alpha = 0$. Then for each $\epsilon > 0$*

$$\mathbb{P}(|v(\mathrm{core}(R_n)) - \alpha n| > \epsilon n) = e^{-\Omega(n)}. \tag{2}$$

*(d) Let $\mathcal{D}$ denote the class of connected graphs freely addable to $\mathcal{A}$, with egf $D$. Let $\rho = 1/\gamma$, and suppose that $\gamma > \lambda e$. Then $T(\rho,\tau) < D(\rho,\tau) < \infty$, and the probability that $\mathrm{core}(R_n)$ is connected tends to $e^{T(\rho,\tau) - D(\rho,\tau)}$ as $n \to \infty$.*

# Example: graphs on surfaces

Uniform case, $R_n \in_u \mathcal{G}^S$, $\mathrm{core}(R_n)$

Solving $\beta e^{1/\beta} = \gamma$ gives $\beta = \beta_0 \approx 26.207554$. This is the growth constant of the class of (connected) graphs in $\mathcal{G}^S$ with minimum degree at least 2.

$\beta_0$ is only slightly larger than the growth constant $\approx 26.18412$ for 2-connected graphs in $\mathcal{G}^S$, from Bender, Gao and Wormald (2002).

Solving $\alpha = 1 - 1/\beta_0$ gives $\alpha = \alpha_0 \approx 0.961843$; and for $R_n \in_u \mathcal{G}^S$

$$v(\mathrm{core}(R_n)) \sim \alpha_0 n \text{ whp}$$

$\alpha_0 n$ is only slightly larger than the number of vertices in the largest block of $R_n$, which is about $0.95982n$, from Giménez, Noy and Rué (2007).

# Example: graphs on surfaces

$R_n \in_u \mathcal{G}^S$, connectivity of $\mathrm{core}(R_n)$

The class $\mathcal{D}$ of connected freely addable graphs is the class of connected planar graphs. From Giménez and Noy (2009), $e^{-D(\rho)} \approx 0.963253$ where $\rho = 1/\gamma$.

Further $e^{T(\rho)} \approx 1.038138$, so by part (d) of the last Theorem, the probability that $\mathrm{core}(R_n)$ is connected $\approx 0.999990$.

Thus

$$\mathbb{P}(\mathrm{core}(R_n) \text{ not connected }) \approx 10^{-5}.$$

For comparison

$$\mathbb{P}(\mathrm{Frag}(R_n) = C_3) \sim e^{-D(\rho)}\rho^3/6 \approx 8 \cdot 10^{-6}.$$

# Other minor-closed classes

## Connected excluded minors

Let us return to the uniform case. What behaviour can we see with other minor-closed classes, not well-behaved?

*Example*

Path forests, ie $Ex(C_3, K_{1,3})$.

Not bridge-addable.

Smooth with growth constant 1 (and asymptotic formula).

$\kappa(R_n)$ asymptotically normal, mean $\sim \sqrt{n}$.

Largest component size $\sim \sqrt{n} \log n$.

More examples in:

M. Bousquet-Mélou and K. Weller, Asymptotic properties of some minor-closed classes of graphs, *CPC* to appear.

# Other minor-closed classes

## Disconnected excluded minors

*Example*

At most $k$ disjoint cycles, ie $Ex((k+1)C_3)$.

Not decomposable.

Looks like a forest with $k$ additional 'free' vertices.

Smooth with growth constant $\gamma_k = 2^k e$ (and asymptotic formula).

$\mathbb{P}(R_n)$ is connected$) \to p_k$, where $p_k = e^{-T(1/\gamma_k)}$ ($p_0 = e^{-\frac{1}{2}}$).

Similar behaviour for $Ex((k+1)C_t)$, $Ex((k+1)K_{1,t})$,.. (not for $Ex(2K_4)$).

Kurauskas and McDiarmid (2011, 2012)

# All minor-closed classes

Bernardi, Noy and Welsh (2010) investigate minor-closed graph classes with small growth constants.

They ask:

does every proper minor-closed class of graphs have a growth constant?

We may ask further:

are they always smooth?

# Some recent references

E. Bender and Z. Gao, Asymptotic enumeration of labelled graphs with a given genus, *Electron J Combin* **18** (2011).

G. Chapuy, E. Fusy, O. Giménez, B. Mohar and M. Noy. Asymptotic enumeration and limit laws for graphs of fixed genus. *JCT A* **118** (2011) 748 – 777.

C. McDiarmid, Random graphs from a weighted minor-closed class, *Electron J Combin* **20** (2) (2013) P52.