Random Graphs from Restricted Classes

Lecture Notes for the Summer School "Random Graphs, Geometry and Asymptotic Structure"

Lectures 4–5

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August 23, 2013

1 Analysis Background

Let us consider a class of graphs $\mathcal{C}$ that is *stable* and *nice*. As we saw in the previous lecture, in this case there are positive constants α, β, c, b and $0 < \rho_B, \rho_C < 1$ such that

$$c_n^\bullet \sim c n^{-\alpha} \rho_C^{-n} n!$$
$$b_n^\bullet \sim b n^{-\beta} \rho_B^{-n} n!$$

where $c_n^\bullet$ is the n-th coefficient of the generating function $C^\bullet(x)$ and $b_n^\bullet$ is the n-th coefficient of the generating function $B^\bullet(x)$.

Recall from Lecture 3 that, denoting by $b(s, C_n)$ the number of blocks with s vertices in C_n (i.e., a random graph on n vertices), then for all $\varepsilon > 0$, with high probability we have

$$b(s, C_n) = (1 \pm \varepsilon) \frac{|\mathcal{B}_s|}{(s-1)!} \tau^{s-1} n, \tag{1}$$

where $\tau := C^\bullet(\rho_C)$.

Now, since every vertex of C_n is contained in some block, by summing this quantity over s we should obtain the number of non-root vertices. Hence we must check that

$$\sum_s b(s, C_n)(s-1) = n - 1.$$

On the other hand, by (1) we get that

$$\sum_s b(s, C_n)(s-1) \sim \sum_s (s-1) \frac{|\mathcal{B}_s|}{(s-1)!} \tau^{s-1} n = \sum_s \frac{|\mathcal{B}_s|}{(s-2)!} \tau^{s-1} n = n \tau B''(\tau),$$

where the last equality follows from the definition of the exponential generating function $B(x) = \sum_s \frac{|\mathcal{B}_s|}{s!} x^s$.

Question. *Is it always the case that $\tau B''(\tau) = 1$?*

There is a purely analytic condition that tells us whether such quantity is smaller or equal than 1.

Lemma 1.1. *Let $\xi := \rho_B B''(\rho_B) \in (0, \infty]$. Then,*

(i) if $\xi > 1$, then $\tau B''(\tau) = 1$.

(ii) If $\xi < 1$, then $\tau B''(\tau) < 1$.

Sketch of the Proof. Recall from the previous lecture that

$$C^\bullet(x) = x \exp\left\{B'(C^\bullet(x))\right\}.$$

Now we introduce the inverse function of $C^\bullet(x)$, and denote it by

$$\psi(u) = ue^{-B'(u)}.$$

It is easy to verify that $\psi(u)$ is indeed the sought inverse function, in fact for all x such that $C^\bullet(x) \leq \rho_B$ we have

$$\psi(C^\bullet(x)) = C^\bullet(x)e^{-B'(C^\bullet(x))} = (xe^{B'(C^\bullet(x))})e^{-B'(C^\bullet(x))} = x.$$

By differentiation we get

$$\psi'(u) = e^{-B'(u)}(1 - uB''(u)).$$

Hence, if $\rho_B B''(\rho_B) > 1$, then there is a (unique) value $0 < \tau < \rho_B$ such that

$$1 = \tau B''(\tau).$$

Otherwise, $\psi'(u) \neq 0$ for all $0 < u < \rho_B$.

The value τ must be given by $C^\bullet(\rho_c)$, otherwise we have $C^\bullet(\rho_c) = \rho_B$. $\square$

Remark 1.2. *Actually, if $\xi > 1$ we always have $c_n^\bullet \sim cn^{-3/2}\rho^{-n}n!$.*

Theorem 1.3. *Let $\mathcal{C}$ be a nice and stable class, and let C_n be a random graph from $\mathcal{C}_n$. Denote by*

$$M(C_n) := \#\{\text{vertices in the largest block}\}.$$

Then we have:

(i) If $\xi = \rho_B B''(\rho_B) > 1$, then there is a constant $c > 0$ s.t.

$$\mathbb{P}\big(M(C_n) \leq c\ln(n)\big) \geq 1 - n^{-3}.$$

(ii) If $\xi < 1$, then

$$M(C_n) = \big(1 - \tau B''(\tau)\big)n + o(n) \text{ w.h.p.}$$

Furthermore, all other blocks contain $o(n)$ vertices.

From now on we will distinguish between **simple** (all blocks are of small size) and **complex** graphs (one block has linearly many vertices and the other blocks are small).

A few examples of simple graphs are:

- outerplanar graphs,
- series-parallel graphs,
- cactus graphs (where $\mathcal{B}$ is the class of cycles),
- clique graphs (where $\mathcal{B}$ is the class of cliques).

A few examples of complex graphs are

- planar graphs (for which $1 - \tau B''(\tau) \simeq 0.96$),
- $K_{3,3}$-minor free (for which $1 - \tau B''(\tau) \simeq 0.98$).

Sketch of the Proof of Theorem 1.3. We start by proving (i).

Let us choose $s = c\ln(n)$, for some $c > 0$. Then,

$$\begin{aligned}\mathbb{P}(\exists \text{ a block of size} \geq s) = \mathbb{P}(M(C_n) \geq s) &= \mathbb{P}(M(\Gamma C_n^\bullet) \geq s \mid |\Gamma C_n^\bullet| = n) \\ &= \frac{\mathbb{P}(M(\Gamma C_n^\bullet) \geq s,\, |\Gamma C_n^\bullet| = n)}{\mathbb{P}(|\Gamma C_n^\bullet| = n)} \\ &= \Theta(1) n^\alpha \mathbb{P}(\exists i,\ 1 \leq i \leq n \ :\ |B_i'| \geq s).\end{aligned}$$

The last equality follows from Lemma 1.4 from the previous lecture.

Now we compute the probability that at least one graph of the class $\mathcal{B}'$ has size at least s.

From what we have seen so far, we can deduce that

$$\mathbb{P}(|B_i'| = s) = \frac{|\mathcal{B}_s|}{(s-1)!}\frac{\tau^{s-1}}{B'(\tau)} \sim \frac{b\, s^{-\beta}\rho_B^{-s} s!}{(s-1)! B'(\tau)}\tau^{s-1} = \Theta(1) s^{1-\beta}\left(\frac{\tau}{\rho_B}\right)^s.$$

By assumption, we know that $\tau < \rho_B$, hence such probability is exponentially small. Therefore,

$$\mathbb{P}(|B_i'| \geq s) \lesssim \mathbb{P}(|B_i'| = s) = \Theta(1) s^{1-\beta}\gamma^{-s},$$

where $\gamma = \rho_B/\tau$.

At this point, by Markov's inequality we get

$$\mathbb{P}(\exists i,\ 1 \leq i \leq n \ :\ |B_i'| \geq s) \geq n\mathbb{P}(|B_i'| \geq s) \leq \Theta(1) n s^{1-\beta}\gamma^{-s}.$$

Therefore we have

$$\mathbb{P}(\exists \text{ a block of size} \geq s) = \mathbb{P}(M(C_n) \geq s) \leq \Theta(1) n^{\alpha+1} s^{1-\beta}\gamma^{-s} \leq n^{-3},$$

where the last inequality holds for a suitable choice of c in $s = c\ln(n)$.

The proof of (ii) relies on a counting argument.

Let us set $N := \big(1 - \tau B''(\tau)\big)n + o(n)$. Our aim is to compare directly the number of graphs in $\mathcal{C}_n$ having exactly *one* block of size N, with the number of graphs having *two or more* such blocks.

In order to perform this comparison, one starts by considering a graph with one component of size N together with a graph with two components of size $N/2$. In particular, consider each subgraph C_i of the first graph, whose size is smaller than N. Now attach a copy of C_i to a vertex of the second graph, chosen uniformly at random.

Hence, one can see that the number of the second type of graph is

$$\left[b\left(\frac{N}{2}\right)^{-\beta}\rho_B^{-N/2}\,(N/2)!\right]^2\binom{N}{N/2}\times N\times N. \tag{2}$$

This value comes from the following reasoning: having two components of size $N/2$ gives rise to the first term squared. Furthermore, we have exactly $\binom{N}{N/2}$ ways to label the two components of size $N/2$, which leads to the second term. Finally, we have N possible ways to choose the graph that acts as a "bridge" between the two large components, and we have $\Theta(N)$ vertices where we can attach it.

At this point, we simplify (2), obtaining

$$\Theta(1)N^{-2\beta+2}\rho_B^{-N}N!,$$

which we compare with the number of graphs with only one component of size N, namely

$$\Theta(1)N^{-\beta}\rho_B^{-N}N!.$$

By direct comparison, we find that if $\beta > 2$, then with high probability we have only one block of size N.

But in fact, from the condition

$$\xi = \rho_B B''(\rho_B) < 1 \quad \Leftrightarrow \quad \sum_n n^{-\beta+2} < 1,$$

it follows that $\beta > 3$, hence the statement. $\square$

2 Simple vs. Complex Case

2.1 Simple Graphs: Degree Sequence

For any graph G, let $T(G) = \big(V(T(G)), E(T(G))\big)$ be its **block tree**. In other words, denoting by $B(G)$ the set of blocks of G, and $V(G)$ its set of vertices, we have

$$\begin{aligned} V(T(G)) &:= \big(B(G) \cup V(G)\big) \\ E(T(G)) &:= \big\{\mathcal{B}_v \ : \ v \in B(G)\big\}. \end{aligned}$$

Now we can construct an equivalence relation using the concept of block-tree. In fact, we declare two graphs G and G' to be equivalent (we write $G \sim G'$) as follows:

$$G \sim G' \quad \Leftrightarrow \quad T(G) = T(G').$$

We denote by $[C_n]$ the equivalence class of such relation, for every C_n random graph from the class $\mathcal{C}_n$.

Choose a representative E from the equivalence class $[C_n]$. Then the random graph $C_n \mid E$ can be constructed as follows.

Choose independently a block b_1 from the class $\mathcal{B}_{s_1}$, a block b_2 from the class $\mathcal{B}_{s_2}$ and so on. Then we get the following result:

Theorem 2.1. *Let $\mathcal{C}$ be a nice and stable class, and consider $C_n \in \mathcal{C}_n$. Denote by*

$$d_{k,n} := \#\{\text{degree-}k \text{ vertices in } C_n\}, \qquad \text{and} \qquad \mu_{k,n} := \mathbb{E}(d_{k,n}).$$

Then, for some positive constant C we have:

$$\mathbb{P}\big(|d_{k,n} - \mu_{k,n}| \geq \varepsilon \mu_{k,n}\big) \leq C \frac{\ln^2(n)}{\varepsilon^2 \mu_{k,n}}$$

Proof. We start by applying Chebyshev's inequality, yielding that

$$\mathbb{P}\big(|d_{k,n} - \mu_{k,n}| \geq \varepsilon \mu_{k,n}\big) \leq \frac{\mathrm{Var}(d_{k,n})}{\varepsilon^2 \mu_{k,n}}.$$

To simplify the notation, define the following set

$$\mathcal{E}_n := \{E \in [C_n] \ : \ E \text{ contains blocks with at least } c \ln(n) \text{ vertices}\}.$$

Now we directly compute $\mathrm{Var}(d_{k,n})$ as follows.

$$\begin{aligned} \mathrm{Var}(d_{k,n}) &= \sum_{E \in [C_n]} \mathrm{Var}(d_{k,n}) \mathbb{P}(E) \\ &\leq \sum_{\mathcal{E}_n} \mathrm{Var}(d_{k,n}) \mathbb{P}(E) + \sum_{[C_n] \setminus \mathcal{E}_n} \mathrm{Var}(d_{k,n}) \mathbb{P}(E) \\ &\leq n^{-3} n^2 + \sum_{[C_n] \setminus \mathcal{E}_n} \mathrm{Var}(d_{k,n}) \mathbb{P}(E). \end{aligned} \tag{3}$$

The last inequality follows from Theorem 1.3, part (i).

Now, denoting by $d(v)$ the degree of vertex v, we have

$$d_{k,n} = \sum_v \mathbf{1}_{d(v)=k}.$$

This, using the independence of the vertices belonging to different blocks, implies

$$\begin{aligned}\mathrm{Var}(d_{k,n} \mid E) &= \sum_{v,v'} \left[\mathbb{E}(\mathbf{1}_{d(v)=k}\mathbf{1}_{d(v')=k}) - \mathbb{E}(\mathbf{1}_{d(v)=k})\mathbb{E}(\mathbf{1}_{d(v')=k})\right] \\ &\leq \sum_{v,v'\in \text{ same block}} \mathbb{P}(d(v)=k)\mathbb{P}(d(v')=k \mid d(v)=k) \\ &\leq \sum_{v,v'\in \text{ same block}} \mathbb{P}(d(v)=k) \leq C^2 \ln^2(n)\mathbb{E}(d_{k,n} \mid E).\end{aligned}$$

Now, by substituting this result back into (3), and applying Chernoff bounds we obtain the claim. □

Remark 2.2. *The value of $\mu_{k,n}$ can be computed explicitly, obtaining exponential bounds on the tails. The right approach comes from expressing the degree of a vertex v as the sum of two quantities. We write*

$$d(v) = d_B(v) + d_2(v),$$

where $d_B(v)$ is the degree of v inside the block (we say that v "is born" with this degree), while $d_2(v)$ is due to each call to the algorithm constructing $\Gamma C^\bullet$.

Hence, one obtains that

$$\mathbb{P}(d(v)=k) = \sum_{l=1}^{k} \mathbb{P}(d_B(v)=l)\mathbb{P}(d_2(v)=k-l).$$

Example: in the case of outerplanar graphs we have:

$$\mathbb{P}(d(v)=k) \sim ck^{-1/2}\rho^{-k}\mu^{-\sqrt{k}},$$

for a suitable positive constant μ.

The complex case cannot be handled in a similar fashion, because one cannot exploit the independence of the vertices in different blocks.

2.2 Complex Graphs

A possible approach for this type of random graphs, is to look for a **decomposition** (first results go back to Tutte).

In this case one can replace the class of 2-connected graphs by the class of 3-connected ones. Any 2-connected graph *rooted at an edge* can be obtained by taking one element from the class of:

(i) cycle;

(ii) bond;

(iii) 3-connected graph;

and replace each edge by a rooted 2-connected graph.

The next result expresses a criterion that allows us to distinguish when a graph is *simple* or *complex*.

Theorem 2.3. *There is a critical condition that determines whether a random* 2*-connected graph is, with high probability,*

(i) "simple": if all 3*-connected components are of size at most* $\ln(n)$*;*

(ii) "complex": if there is a 3*-connected component that has linear size.*

A class $\mathcal{C}$ can be subdivided into two categories according to the 2-connected graph classification: hence they can be *simple* or *complex*. The latter can still be subdivided into two further categories, according to a 3-connected graph classification: we shall be saying *complex-simple* or *complex-complex*.

An example of *complex-complex* graphs are the planar graphs, for which some results are known, e.g. the law of the degree sequence, as well as the law of the maximum degree. Further studies are needed to understand the typical structure of 3-connected random graphs.

Open Questions

Several problems are still open in this matter, here we quote a few examples.

- The study of "global" parameters, for example the diameter of the graph, for which a behavior of order $\Theta(\sqrt{n})$ is conjectured for the simple case.

- Treat the complex-complex case in a systematical way.

- Proceed with the study of the unlabeled case.

Exercises

Let $\mathcal{C}$ be a nice stable class, C_n a graph from $\mathcal{C}_n$, drawn uniformly at random, and denote by Δ the maximum degree of C_n. Show that with high probability

$$\Delta = \Omega\big(\log(n)/\log\log(n)\big).$$

Hint: Relate the maximum degree to the Z_i's.

If in addition if $\mathcal{C}$ is simple, argue that with high probability we have

$$\Delta = O\big(\log^2(n)/\log\log(n)\big).$$