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Exceptional Family of Elements for a Variational Inequality Problem and its Applications

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Abstract. This paper introduces a new concept of exceptional family of elements (abbreviated, exceptional family) for a finite-dimensional nonlinear variational inequality problem. By using this new concept, we establish a general sufficient condition for the existence of a solution to the problem. Such a condition is used to develop several new existence theorems. Among other things, a sufficient and necessary condition for the solvability of pseudo-monotone variational inequality problem is proved. The notion of coercivity of a function and related classical existence theorems for variational inequality are also generalized. Finally, a solution condition for a class of nonlinear complementarity problems with so-called P_*-mappings is also obtained.

Key words: Complementarity problem, Exceptional family of elements, Generalized monotonicity, Nonlinear variational inequality, P_*-mapping

1. Introduction

Let K be a closed convex subset of R^n. Assume that it is given as follows

$$K = \{x \in R^n : g(x) \leq 0, h(x) = 0\}$$

where $g(x) = (g_1(x), ..., g_m(x))^T, h(x) = (h_1(x), ..., h_l(x))^T$, and $g_i(x)(i = 1, ..., m)$ and $h_j(x)(j = 1, ..., l)$ are convex and linear real-valued continuously differentiable functions from R^n into R^1, respectively. Specially, if $g(x) = -x$, then $K = \{x \in R^n : g(x) \leq 0\} = R^n_+$ (nonnegative orthant).

Let f be a mapping from R^n into R^n. The finite-dimensional variational inequality problem, denoted by $VI(K, f)$, is to find a vector $x^* \in K$ such that

$$(x - x^*)^T f(x^*) \geq 0, \text{ for all } x \in K.$$

When $K = R^n_+$, the above problem reduces to the following nonlinear complementarity problem, denoted by $NCP(f)$,

$$x \geq 0, f(x) \geq 0, x^T f(x) = 0.$$

The variational inequality problem has had many successful practical applications in the last three decades. It has been used to formulate and investigate equilibrium models arising in economics, transportation, regional science and operations research (see [3, 5]). The development of solution conditions for this problem has played a very important role in theory, algorithms and applications of the problem. So far, a large number of existence conditions have been established by many researchers, including Eaves [1, 2], Karamardian [8–11], Kojima [13], Moré [18, 19], Smith [26], and Pang and Yao [21]. It should be noted that most of the previous existence results on $VI(K, f)$ are developed in general by means of fixed point arguments and minimax theory (see [1, 2, 8–11, 13, 18, 19]). The topological degree theory are also used to develop solution conditions for variational inequality and its various generalizations (see [21], for example)

Our analysis in this paper is motivated by the papers [26, 7, 28]. Smith [26] introduced a concept of exceptional sequence for a continuous function and used it to investigate the solution conditions of complementarity problems. Isac et al. [7] using the topological degree generalized Simth's concept and introduced the notion of exceptional family of elements for continuous functions and applied it to several kinds of complementarity problems. The applications of Smith's results in spatial price equilibrium problem and network equilibrium problem are given in [26] and [4], respectively. Both [26] and [7] have shown that the nonexistence of exceptional family is a sufficient condition for the solvability of complementarity problem. However, under what conditions can one guarantee the nonexistence of exceptional family? This problem is discussed a little both in [26] and [7] except for the case of coercive functions. Recently, Zhao [28] extends the concepts and results in [26] and [7] to variational inequality problems over polyhedral sets.

The present paper is intended to introduce the notion of a general exceptional family of elements for nonlinear variational inequality problems. This concept is so general that it includes as a special case the notions in [26] and [7]. Our notion of exceptional family of elements for variational inequalities provides a unified analysis for the investigation of the solvability of $VI(K, f)$ from the point of view that many well-known existence theorems for $VI(K, f)$, which were proved by different ways in the literature can be either generalized or reobtained by using our analysis. In this paper, we focus our main attention on developing new existence theorems for $VI(K, f)$ by using the proposed concept of exceptional family. Among other things, a necessary and sufficient condition for the solvability of pseudo-monotone $VI(K, f)$ is proved, and an existence result for quasi-monotone problem is also established. The property of coercivity of a function and related existence results for $VI(K, f)$ are extended to so-called p-order coercive functions. We also establish a solution condition for a class of complementarity problems, where the functions are nonlinear P_*-mappings which include several important classes of functions as the special cases.

In what follows, Section 2 develops the notion of exceptional family of elements for $VI(K, f)$ and establishes a main theorem. Section 3 establishes several new solution conditions for $VI(K, f)$. Final remarks is given in Section 4.

2. Definition and main theorem

In this paper, $\|\cdot\|$ denotes the Euclidean-norm, R^n_+ denotes the nonnegative orthant, and $P_K(\cdot)$ denotes the projection operator on the convex set K with Euclidean-norm, that is, for any $z \in R^n$, the projection $P_K(z)$ of z on the set K is the unique solution to the following problem

$$\min_y\{\|y - z\| : y \in K\}.$$

Let D be an open bounded set of R^n , we denote by $\overline{D}$ and $\partial(D)$ the closure and boundary of D, respectively. Let $C(\overline{D})$ denote the linear space of continuous functions from $\overline{D} \to R^n$. If $f \in C(\overline{D})$ and $y \in R^n$ such that $y \notin f(\partial(D))$, the notation $\deg(f, D, y)$ is the topological degree associated with f, D and y. See [15,20]. To show our main result, we will make use of the following lemmas.

LEMMA 2.1 [3]. *x^* is a solution of $VI(K, f)$ if and only if x^* is the solution of the following equation*

$$x - P_K(x - f(x)) = 0.$$

The following two results can be found in [15] and [20].

LEMMA 2.2 [Poincaré–Bohl]. *Let $D \subset R^n$ be an open bounded set and $F, G \in C(\overline{D})$ be two continuous functions. The homotopy $H(x, t)$ is defined as follows*

$$H(x, t) = tG(x) + (1 - t)F(x), 0 \leq t \leq 1.$$

Let y be an arbitrary point in R^n. If y satisfies the following condition

$$y \notin \{H(x, t) : x \in \partial D \text{ and } t \in [0, 1]\},$$

then

$$\deg(G, D, y) = \deg(F, D, y).$$

LEMMA 2.3 [Kronecker]. *Let D and F be given as in Lemma 2.2, if $y \notin F(\partial D)$ and* $\deg(F, D, y) \neq 0$, *then the system of equations $F(x) = y$ has a solution in D.*

It is well-known (see [6, 10, 19]) that the $VI(K, f)$ is solvable if f is a continuous mapping on the nonempty compact convex set K. To study the existence of a solution to $VI(K, f)$, there is nothing to do when K is bounded. Thus, throughout the remainder of this paper, we consider only the case that K is an unbounded convex set. We now introduce the general concept of *exceptional family of elements for variational inequality problems.*

DEFINITION 2.1. Let $\hat{x}$ be an arbitrary feasible point, i.e., $\hat{x} \in K$. A sequence $\{x^r\}_{r\to+\infty} \subset K$ is said to be an exceptional family of elements (abbreviated, exceptional family) for variational inequality problem $VI(K, f)$, if the sequence satisfies the following two conditions:

1. $\|x^r\| \to +\infty$ as $r \to +\infty$.
2. There exists two vector sequences $\{\lambda_r\} \subset R^m_+, \{\mu_r\} \subset R^l$ and a positive scalar sequence $\{\alpha_r\}$, where $\alpha_r > 1$ for all r, such that the sequence $\{\alpha_r x^r + (1-\alpha_r)\hat{x}\} \subset K$, and the following equations hold for all r

$$f(x^r) = -(\alpha_r - 1)(x^r - \hat{x}) - \frac{1}{2}[\nabla g(\alpha_r x^r + (1-\alpha_r)\hat{x})^T \lambda_r + \nabla h(\alpha_r x^r + (1-\alpha_r)\hat{x})^T \mu_r], \tag{1}$$

$$(\lambda_r)^T g(\alpha_r x^r + (1-\alpha_r)\hat{x}) = 0. \tag{2}$$

Where $\nabla g(\cdot)$ and $\nabla h(\cdot)$ denote the Jacobian matrix of g and h, respectively.

The above general definition of exceptional family for $VI(K, f)$ generalizes the notions of exceptional sequence and exceptional family of elements for continuous functions introduced in [26] and [7], respectively. To see this, let

$$K = \{x \in R^n : g(x) = -x \leq 0\} = R^n_+,$$

then (1) and (2) is reduced to

$$f(x^r) = -(\alpha_r - 1)(x^r - \hat{x}) + \frac{1}{2}\lambda_r,$$

$$(\lambda_r)^T x^r = 0.$$

Setting $\hat{x} = 0$, then the above equations can be further simplified as follows

$$f_i(x^r) = -(\alpha_r - 1)x_i^r, \text{ if } x_i^r > 0, \tag{3}$$

$$f_i(x^r) = \frac{1}{2}(\lambda_r)_i \geq 0, \text{ if } x_i^r = 0, \tag{4}$$

which is just the definition of *exceptional family of elements* for the continuous functions introduced in [7]. Moreover, if $\|x^r\| = r$ for all $r > 0$, the sequence $\{x^r\}$ satisfying (3) and (4) is just the concept of *exceptional sequence* introduced in [26], which is also discussed by G. Isac under the name of *opposite radial sequence*.

THEOREM 2.1. *Let K be nonempty. If $f : R^n \to R^n$ is a continuous function, then there exists either a solution or an exceptional family for $VI(K, f)$.*

Proof. By Lemma 2.1, the solvability of $VI(K, f)$ is equivalent to the existence of a solution to the equation $\phi(x) = 0$, where

$$\phi(x) = x - P_K(x - f(x)).$$

It follows from the continuity of the mapping f and the property of the projection operator that $\phi(x)$ is continuous . Let $\hat{x}$ be an arbitrary feasible point, i.e., $\hat{x} \in K$. We consider the homotopy between the mapping $x - \hat{x}$ and $\phi(x)$, that is

$$H(x, t) = t(x - \hat{x}) + (1 - t)\phi(x), \quad 0 \leq t \leq 1.$$

Let $M \in R^{n\times n}$ be an arbitrary positive-definite matrix and d be a vector in R^n. Consider the following convex quadratic function in variable x.

$$c(x) = x^T Mx + x^T d.$$

The family of bounded open sets, denoted by $\{D_r\}_{r\to+\infty}$, is defined as follows

$$D_r = \{x \in R^n : c(x) < r\}.$$

Hence the boundary $\partial D_r = \{x \in R^n : c(x) = r\}$. Without loss of generality, we assume that $r > c(\hat{x})$. Two cases are possible.

Case 1. There exists a $r > c(\hat{x})$ such that

$$0 \notin \{H(x, t) : x \in \partial D_r, t \in [0, 1]\}.$$

Then by Lemma 2.2

$$\deg(\phi, D_r, 0) = \deg(x - \hat{x}, D_r, 0)$$

It is clear that $|\deg(x - \hat{x}, D_r, 0)| = 1$, hence from the above and Lemma 2.3, the equation $\phi(x) = 0$ has at least one solution.

Case 2. For each $r > c(\hat{x})$ there exists a point $x^r \in \partial D_r$ and a number $t_r \in [0, 1]$ such that

$$\begin{aligned} 0 = H(x^r, t_r) &= t_r(x^r - \hat{x}) + (1 - t_r)[x^r - P_K(x^r - f(x^r))] \\ &= x^r - t_r\hat{x} - (1 - t_r)P_K(x^r - f(x^r)). \end{aligned} \tag{5}$$

If $t_r = 0$, by Lemma 2.1, (5) implies that x^r is a solution of $VI(K, f)$. On the other hand, since $c(x^r) = r > c(\hat{x})$, $x^r \neq \hat{x}$. Thus it follows from (5) that $t_r \neq 1$. Hence, in the rest of the proof, we consider only the case that $t_r \in (0, 1)$ for all $r > c(\hat{x})$. From (5) we have

$$\frac{1}{1 - t_r}x^r - \frac{t_r}{1 - t_r}\hat{x} = P_K(x^r - f(x^r)) \in K. \tag{6}$$

By the property of projection operator, $\frac{1}{1-t_r}x^r - \frac{t_r}{1-t_r}\hat{x}$ is the unique solution to the following problem

$$\begin{aligned} &\text{minimize } T(y) = \|y - (x^r - f(x^r))\|^2 \\ &\text{s.t.} \qquad y \in K = \{x \in R^n : g(x) \leq 0, h(x) = 0\}. \end{aligned}$$

Since the problem is a convex quadratic program, the Karush–Kuhn–Tucker conditions completely characterize the solution of the convex program. Therefore, there exist two vectors $\lambda_r \in R^m_+$ and $\mu_r \in R^l$ such that

$$\nabla T\left(\frac{x^r}{1-t_r}-\frac{t_r}{1-t_r}\hat{x}\right)+\nabla g\left(\frac{x^r}{1-t_r}-\frac{t_r}{1-t_r}\hat{x}\right)^T\lambda_r + \nabla h\left(\frac{x^r}{1-t_r}-\frac{t_r}{1-t_r}\hat{x}\right)^T\mu_r = 0,$$

$$(\lambda_r)^T g\left(\frac{x^r}{1-t_r}-\frac{t_r}{1-t_r}\hat{x}\right) = 0.$$

Denote $\alpha_r = 1/1-t_r > 1$ and note that $\nabla T(y) = 2[y-(x^r - f(x^r))]$. The above two equations can be written as follows

$$f(x^r) = -(\alpha_r - 1)(x^r - \hat{x}) - \frac{1}{2}[\nabla g(\alpha_r x^r + (1-\alpha_r)\hat{x})^T\lambda_r + \nabla h(\alpha_r x^r + (1-\alpha_r)\hat{x})^T\mu_r],$$

$$(\lambda_r)^T g(\alpha_r x^r + (1-\alpha_r)\hat{x}) = 0.$$

Hence, to show the sequence $\{x^r\}_{r>c(\hat{x})}$ is an exceptional family, it suffices to show $\|x^r\| \to +\infty$ as $r \to \infty$ and $\{x^r\} \subset K$. It follows from $c(x^r) = r$ that $\|x^r\| \to \infty$ as $r \to \infty$. Note that K is convex, from (6) and $\hat{x} \in K$, we have

$$x^r = t_r\hat{x} + (1-t_r)\left[\frac{x^r}{1-t_r}-\frac{t_r}{1-t_r}\hat{x}\right] \in K.$$

Hence $\{x^r\} \subset K$. The proof is complete.

COROLLARY 2.1. *If the problem $VI(K, f)$ has no exceptional family, then it has at least a solution.*

3. Applications

Theorem 2.1 in last section establishes a new sufficient condition of the existence of a solution to $VI(K, f)$. We will see that this sufficient condition is weaker than several known solution conditions existing in literature. We will show that such a sufficient condition is so weak that it is also necessary for the solvability of pseudo-monotone variational inequality problem. The conclusion of Theorem 2.1 makes it possible for us to develop new solution conditions for $VI(K, f)$ via investigating the conditions of nonexistence of the exceptional family for $VI(K, f)$. Particularly, an existence result for a class of nonlinear functions called P_*-mappings is established. P_*-mappings include as the special cases the monotone functions and a subclass of uniform P-functions. Recently, the linear P_*-mapping has obtained special attention in the field of interior-point algorithm for linear complementarity problems (see [14, 17, 22–25, 27] for example).

THEOREM 3.1. *Let $f : R^n \to R^n$ be a continuous function, if there is a feasible point $\hat{x} \in K$ such that for each sequence $\{x^r\} \subset K$ with $\|x^r\| \to \infty$, the following inequality*

$$(x^r - \hat{x})^T f(x^r) \geq 0 \tag{7}$$

holds for a point x^r with $\|x^r\| \neq \|\hat{x}\|$. Then $VI(K, f)$ has no exceptional family, and hence $VI(K, f)$ is solvable.

Proof. Suppose that there exists an exceptional family $\{x^r\}$ for the problem $VI(K, f)$. By Definition 2.1, we have $\{x^r\} \subset K$ and $\|x^r\| \to \infty$ as $r \to \infty$ and there exist two vector sequences $\{\lambda_r\} \subset R^m_+$ and $\{\mu_r\} \subset R^l$ and a scalar sequence $\{\alpha_r\}$, where each $\alpha_r > 1$, such that $\{\alpha_r x^r + (1 - \alpha_r)\hat{x}\} \subset K$ and the equations (1) and (2) hold. Note that $g_i(x)(i = 1, ..., m)$ is convex and $h_j(x)(j = 1, ..., l)$ is linear, we have that

$$g(\hat{x}) \geq g(\alpha_r x^r + (1 - \alpha_r)\hat{x}) + \alpha_r \nabla g(\alpha_r x^r + (1 - \alpha_r)\hat{x})(\hat{x} - x^r), \tag{8}$$

$$h(\hat{x}) = h(\alpha_r x^r + (1 - \alpha_r)\hat{x}) + \alpha_r \nabla h(\alpha_r x^r + (1 - \alpha_r)\hat{x})(\hat{x} - x^r). \tag{9}$$

By (2), (8) and the feasibility of $\hat{x}$, we have

$$(\lambda_r)^T \nabla g(\alpha_r x^r + (1 - \alpha_r)\hat{x})(\hat{x} - x^r) \leq \frac{1}{\alpha_r}(\lambda_r)^T g(\hat{x}) \leq 0. \tag{10}$$

Since $\alpha_r x^r + (1 - \alpha_r)\hat{x}$ is feasible, (9) implies that

$$\nabla h(\alpha_r x^r + (1 - \alpha_r)\hat{x})(\hat{x} - x^r) = 0. \tag{11}$$

Therefore by (1), (10) and (11), we have

$$\begin{aligned}(x^r - \hat{x})^T f(x^r) &= -(x^r - \hat{x})^T [(\alpha_r - 1)(x^r - \hat{x}) \\ &\quad + \frac{1}{2}\nabla g(\alpha_r x^r + (1 - \alpha_r)\hat{x})^T \lambda_r + \nabla h(\alpha_r x^r + (1 - \alpha_r)\hat{x})^T \mu_r)] \\ &\leq -(\alpha_r - 1)\|x^r - \hat{x}\|^2\end{aligned} \tag{12}$$

which implies that

$$(x^r - \hat{x})^T f(x^r) < 0, \text{ for all } \|x^r\| \neq \|\hat{x}\|.$$

This is in contradiction with the hypothesis condition (7). Hence, $VI(K, f)$ has no exceptional family, and hence $VI(K, f)$ has a solution by Theorem 2.1.

Harker and Pang (see Theorem 3.3 in [3]) show that the condition "there exists a vector $\hat{x} \in K$ such that the set $K(\hat{x}) = \{x \in K : (x - \hat{x})^T f(x) < 0\}$ is bounded (possibly empty)" implies that $VI(K, f)$ has a solution. It should be noted that the above Harker and Pang condition implies our condition in Theorem 3.1. Actually, for each sequence $\{x^r\} \subset K$ with $\|x^r\| \to \infty$, if the set $K(\hat{x})$ is bounded (possibly empty), we have that $x^r \notin K(\hat{x})$ for sufficiently large r , i.e.,

$$(x^r - \hat{x})^T f(x^r) \geq 0 \text{ for all sufficiently large } r$$

which implies the condition of Theorem 3.1.

From the proof of the above Theorem 3.1, one can actually prove the following consequence.

THEOREM 3.1′. *Let $f : R^n \to R^n$ be a continuous function and $\hat{x}$ be a point in K, if $VI(K, f)$ has an exceptional family, then the set*

$$K(\hat{x}) = \{x \in K : (x - \hat{x})^T f(x) < 0\}$$

must be nonempty and unbounded.

Harker and Pang's result can be also viewed as an immediate consequence of the above results. The next definition extends the coercivity of a function to so-called p-order coercivity.

DEFINITION 3.1. A function $f : R^n \to R^n$ is said to be $p-$order coercive with respect to K if there exists some $p \in (-\infty, 1)$ and $\hat{x} \in K$ such that

$$\lim_{x \in K, \|x\| \to \infty} \frac{f(x)^T (x - \hat{x})}{\|x\|^p} = +\infty. \tag{13}$$

When $p = 1$ (13) reduces to the standard definition of coercivity of a function in literature (see [19,3]). It is evident that the standard coercive function must be p-order coercive, but the converse is not true in general. For example, we consider the one-variable real function $f(t) = t^\alpha/(1 + t^\alpha)$, where the scalar $\alpha \geq 0$. Let $K = [1, +\infty)$ for any $p \in (-\infty, 1)$ and any $t^0 \in [1, +\infty)$, we have

$$\lim_{t \geq 1, t \to +\infty} \frac{f(t)(t - t^0)}{t^p} = +\infty$$

which implies that $f(t)$ is p-order coercive with respect to K, but $f(t)$ is not coercive, since that

$$\lim_{t \geq 1, t \to +\infty} \frac{f(t)(t - t^0)}{t} = 1.$$

Coercivity property has played very important role in the existence theory of $VI(K, f)$ (see [3, 6, 7, 19, 26, 28]). Hartman and Stampacchia [6] and Moré [19] show that the standard coercivity implies that $VI(K, f)$ has a solution (see also Theorem 3.3 in [3]). This existence result is generalized to the case of p-order coercivity in the following consequence.

COROLLARY 3.1. *Let $f : R^n \to R^n$ be a continuous function, if there exists a scalar p such that f is p-order coercive, where $-\infty < p < 1$, then there exists no exceptional family for $VI(K, f)$, and hence the problem $VI(K, f)$ has a solution.*

Proof. It is easy to see that p-order coercivity condition implies the condition of Theorem 3.1. Actually, if $0 \le p < 1$, then from (13), we have

$$\lim_{x \in K, \|x\| \to \infty} f(x)^T (x - \hat{x}) = +\infty. \tag{14}$$

If $-\infty < p < 0$, for any sequence $\{x^r\} \subset K$ with $\|x^r\| \to +\infty$, (13) implies that

$$f(x^r)(x^r - \hat{x}) > 0 \text{ for sufficiently large } r. \tag{15}$$

Both (14) and (15) imply that the condition of Theorem 3.1 holds. Thus the desired result is an immediate consequence of Theorem 3.1.

In summary, the existence Theorem 3.3 in [3] (due to Harker and Pang) and Theorem 3.2 in [3](due to Moré, Hartman and Stampacchia) can be viewed as the special cases of Theorem 3.1. It is worthwhile to mention that the aforementioned conclusion "p-order coercivity implies nonexistence of exceptional family for $VI(K, f)$" is also a generalized version of Proposition 4.7 in [26] and Proposition 4 in [7] for $NCP(f)$ under the standard coercivity assumption.

In what follows, we develop the sufficient and necessary condition for pseudo-monotone type $VI(K, f)$. Recall that a mapping $f : R^n \to R^n$ is monotone on K if $(f(x) - f(y))^T (x - y) \ge 0$ holds for all distinct pair $x, y \in K$. A mapping f is said to be pseudo- monotone on K if for every pair of distinct points $x, y \in K$, the condition $(y - x)^T f(x) \ge 0$ implies that $(y - x)^T f(y) \ge 0$. Denote by K^* the dual cone of the convex set K, i.e.,

$$K^* = \{y \in R^n : y^T x \ge 0, \forall x \in K\}.$$

It is well-known that for nonlinear pseudo-monotone functions the feasible condition "there exists a vector $\hat{x} \in K$ such that $f(x) \in K^*$" is not a sufficient condition for the solvability of $VI(K, f)$. Megiddo [16] gave an example to show that this feasibility can not guarantee the existence of the solution to $VI(K, f)$ even for nonlinear monotone mapping f. If the feasibility condition is replaced by strictly feasibility condition, i.e., there exists a vector $\hat{x} \in K$ such that $f(\hat{x}) \in \text{int}(K^*)$, where $\text{int}(\cdot)$ denotes the interior of a set, Karamardian [11] and Moré [19] showed that there exists a solution to the nonlinear complementarity problem $NCP(f)$. Their results remain valid for pseudo-monotone $VI(K, f)$ (see Theorem 3.4 in [3] due to Harker and Pang). It should be noted that the strictly feasibility is sufficient, however, it is not necessary for the solvability of $VI(K, f)$ (see Corollary 3.2). The question how to weaken such a condition such that a relaxed solution condition is not only sufficient but also necessary is answered in the next theorem. We make use of the following definition.

DEFINITION 3.2. A function $f : R^n \to R^n$ is said to be *weakly proper* on the set K if there is a point $\hat{x} \in K$ such that for each sequence $\{x^r\} \subset K$ with $\|x^r\| \to \infty$, there exists some r such that

$$(x^r - \hat{x})^T f(\hat{x}) \geq 0 \text{ and } \|x^r\| \neq \|\hat{x}\|.$$

THEOREM 3.2. *Let $f : R^n \to R^n$ be pseudo-monotone on K, then the variational inequality problem $VI(K, f)$ has a solution if and only if f is weakly proper on the set K.*

Proof. Assume that $VI(K, f)$ has a solution x^*, i.e.,

$$f(x^*)^T(x - x^*) \geq 0, \text{ for all } x \in K$$

which implies that f is weakly proper on the set K with $\hat{x} = x^*$. Conversely, assume that f is weakly proper on the set K. By Definition 3.2 there exists some $\hat{x} \in K$, such that for each sequence $\{x^r\} \subset K$ with $\|x^r\| \to \infty$, the following two relations hold for some r

$$(x^r - \hat{x})^T f(\hat{x}) \geq 0 \quad \text{and} \quad \|x^r\| \neq \|\hat{x}\|.$$

Since f is a pseudo-monotone map, we have

$$(x^r - \hat{x})^T f(x^r) \geq 0 \quad \text{and} \quad \|x^r\| \neq \|\hat{x}\|,$$

which implies that the problem $VI(K, f)$ has no exceptional family by Theorem 3.1, thus the problem $VI(K, f)$ has at least one solution.

From Theorem 2.1 (Corollary 2.1), Theorem 3.2 and the above proof of Theorem 3.2, we obtain the following immediate consequence.

THEOREM 3.3. *If $f : R^n \to R^n$ be pseudo-monotone on the set K, then the following two conditions are equivalent*

1. $VI(K, f)$ has no exceptional family

2. f is weakly proper on K

Therefore, the consequence of Theorem 3.2 can be restated as follows: $VI(K, f)$ has a solution if and only if it has no exceptional family.

However, it should be noted that in general the sufficient condition "without exceptional family" is not necessary for the existence of the solution to $VI(K, f)$. Such an example for $NCP(f)$ was given in [26] (see also [7]). On the other hand, one example is also given in [26] to show that the $NCP(f)$ may posses both a solution and an exceptional sequence.

The following result shows that strictly feasible condition is not a necessary for the solvability of $VI(K, f)$.

COROLLARY 3.2. *Under the pseudo-monotonicity assumption, the strict feasibility assumption implies that f is weakly proper on K, the converse is not true.*

Proof. By Theorem 3.3, the first consequence is obvious since the strictly feasibility implies that $VI(K, f)$ has a solution. Now we give an example to show that the converse is not true. Let $K = R^n_+$, $f \equiv 0$ on R^n, there exists no point $\hat{x} \in K$ such that $f(\hat{x}) \in \int(K^*)$, the strictly feasibility condition does not hold. However, it is easy to see that f is weakly proper on K.

We now develop an existence theorem for quasi-monotone $VI(K, f)$. A mapping is said to be quasi-monotone if for any distinct pair of point x, y in K , we have that

$$(y - x)^T f(x) > 0 \quad \text{implies that} \quad (y - x)^T f(y) \geq 0.$$

A pseudo-monotone function is quasi-monotone, but the converse is not true, some examples can be found in [12].

DEFINITION 3.3. The function $f : R^n \to R^n$ is said to be *strictly weakly proper* on the set K, if there exists some $\hat{x} \in K$ such that for every sequence $\{x^r\} \subset K$ with $\|x^r\| \to \infty$, there exists some r such that

$$(x^r - \hat{x})^T f(\hat{x}) > 0 \quad \text{and} \quad \|x^r\| \neq \|\hat{x}\|.$$

Similar to the second part of the proof of Theorem 3.2, we can show the following result that further extends the aforementioned existence results on pseudo-monotonicity.

THEOREM 3.4. *Let $f : R^n \to R^n$ be quasi-monotone on K, and f be strictly weakly proper on K, then $VI(K, f)$ has no exceptional family, and hence $VI(K, f)$ is solvable.*

We now consider a special case. If $g_i(x)$ $(i = 1, ..., m)$ is positively homogeneous of degree $p_i > 0$ $(i = 1, ..., m)$ (i.e., $g_i(rx) = r^{p_i} g(x)$ for all $r > 0$), then the feasible set $K = \{x \in R^n : g(x) \leq 0\}$ is a closed convex cone in R^n.

COROLLARY 3.3. *Let f be quasi-monotone on K and K be pointed solid closed convex cone in R^n (see Definition 2.1 in [11]), if there exist some $\hat{x} \in K$ such that $f(\hat{x}) \in \int(K^*)$, then $VI(K, f)$ has no exceptional family.*

Proof. Under the above assumption, by Lemma 2.1 in [11] (let $d = f(\hat{x}) \in \mathrm{int}(K^*)$, $\alpha = f(\hat{x})^T \hat{x} > 0$), it is easy to see that the following set is bounded

$$S(\hat{x}) = \{x \in K : (x - \hat{x})^T f(\hat{x}) \leq 0\}.$$

For each unbounded sequence $\{x^r\} \subset K$, since $S(\hat{x})$ is bounded, there must exist some r with $\|x^r\| \neq \|\hat{x}\|$ such that $x^r \notin S(\hat{x})$, i.e.,

$$(x - \hat{x})^T f(\hat{x}) > 0.$$

Hence, f is strictly weakly proper on the set K. The desired result follows by Theorem 3.3.

So far, we have developed several new existence theorems for $VI(K, f)$, and some of these results relax several classical existence conditions. The proposed sufficient condition, i.e., "without exceptional family" is so weak that it becomes a necessary condition for the solvability pseudo-monotone $VI(K, f)$. In the rest of the paper we establish a solution condition for a class of nonlinear complementarity problems, where the mappings are so-called nonlinear P_*-mappings that include monotone functions and a subclass of uniform P-functions as the particular cases, but in general a P_*-mapping is neither a monotone mapping nor a uniform P-function.

DEFINITION 3.4. A mapping $f : R^n \to R^n$ is said to be a P_*-mapping, if there exists a nonnegative constant γ such that the following inequality holds for any distinct points $x, y \in R^n$

$$(1+\gamma) \sum_{j \in I_+(x,y,f)} (x_j - y_j)(f_j(x) - f_j(y)) + \sum_{j \in I_-(x,y,f)} (x_j - y_j)(f_j(x) - f_j(y)) \geq 0, \tag{16}$$

where

$$I_+(x, y, f) = \{j : (x_j - y_j)(f_j(x) - f_j(y)) > 0\}. \tag{17}$$

$$I_-(x, y, f) = \{j : (x_j - y_j)(f_j(x) - f_j(y)) < 0\}. \tag{18}$$

It is easy to see that (16) can be written as follows

$$(x - y)^T (f(x) - f(y)) \geq -\gamma \sum_{j \in I_+(x,y,f)} (x_j - y_j)(f_j(x) - f_j(y)). \tag{19}$$

PROPOSITION 3.1. *Let* $f = (f_1, ..., f_n)^T \in R^n$ *be a monotone map, where* $f_i (i = 1, ..., n)$ *is a real-valued function from* R^n *into* R^1. *Then* f *is a* P_**-mapping. Moreover, if the scalar* $d_i \neq 0$ *for all* $i = 1, ..., n$, *then each scaled mapping*

$$F^{(i)} = (f_1, ..., f_{i-1}, d_i f_i, f_{i+1}, ..., f_n)^T, (i = 1, ..., n)$$

is also a P_* *-mapping.*

Proof. Since (19) holds for monotone mapping with constant $\gamma = 0$, any monotone mappings must be P_*-mappings. It is evident that each scaled mapping is not necessary monotone, however, we now show that $F^{(i)}$ is also a P_*-mapping. Let x, y be any distinct points in R^n. We have

$$\begin{aligned}
&(x-y)^T(F^{(i)}(x)-F^{(i)}(y)) \\
&\quad = (x-y)^T(f(x)-f(y)) + (d_i-1)(f_i(x)-f_i(y))(x_i-y_i) \\
&\quad \geq \frac{d_i-1}{d_i}(x_i-y_i)(F_i^{(i)}(x)-F_i^{(i)}(y)).
\end{aligned} \tag{20}$$

Three cases are possible.

Case 1. $d_i \geq 1$. Let $\gamma = d_i - 1 \geq 0$. If $(x_i - y_i)(F_i^{(i)}(x) - F_i^{(i)}(y)) \geq 0$, from (20) we have that

$$(x-y)^T(F^{(i)}(x)-F^{(i)}(y)) \geq 0.$$

Hence,

$$\begin{aligned}
&(x-y)^T(F^{(i)}(x)-F^{(i)}(y)) \\
&\quad \geq -\gamma \sum_{j\in I_+(x,y,F^{(i)})} (x_j-y_j)(F_j^{(i)}(x)-F_j^{(i)}(y))
\end{aligned} \tag{21}$$

holds trivially. If $(x_i - y_i)(F_i^{(i)}(x) - F_i^{(i)}(y)) < 0$, note that in the case

$$\begin{aligned}
&\sum_{j\in I_+(x,y,F^{(i)})} (x_j-y_j)(F_j^{(i)}(x)-F_j^{(i)}(y)) \\
&\quad = \sum_{j\in I_+(x,y,f)} (x_j-y_j)(f_j(x)-f_j(y)).
\end{aligned} \tag{22}$$

By the monotonicity of f, we have

$$\begin{aligned}
&\sum_{j\in I_+(x,y,f)} (x_j-y_j)(f_j(x)-f_j(y)) \\
&\qquad + \sum_{j\in I_-(x,y,f)} (x_j-y_j)(f_j(x)-f_j(y)) \geq 0
\end{aligned} \tag{23}$$

Therefore, by (22) and (23)

$$\begin{aligned}
0 &> (x_i-y_i)(F_i^{(i)}(x)-F_i^{(i)}(y)) \\
&= d_i(x_i-y_i)(f_i(x)-f_i(y)) \\
&\geq -d_i \sum_{j\in I_+(x,y,f)} (x_j-y_j)(f_j(x)-f_j(y)) \\
&= -d_i \sum_{j\in I_+(x,y,F^{(i)})} (x_j-y_j)(F_j^{(i)}(x)-F_j^{(i)}(y)).
\end{aligned}$$

Combining (20) and the above inequality, we again obtain (21).

Case 2. $0 < d_i < 1$. Let $\gamma = (1-d_i)/d_i > 0$. If $(x_i-y_i)(F_i^{(i)}(x)-F_i^{(i)}(y)) > 0$, then from (20)

$$\begin{aligned}(x-y)^T(F^{(i)}(x) - F^{(i)}(y)) \\ \geq -\gamma(x_i - y_i)(F_i^{(i)}(x) - F_i^{(i)}(y)) \\ \geq -\gamma \sum_{j\in I_+(x,y,F^{(i)})} (x_j - y_j)(F_j^{(i)}(x) - F_j^{(i)}(y)).\end{aligned}$$

If $(x_i - y_i)(F_i^{(i)}(x) - F_i^{(i)}(y)) \leq 0$, then from (20), the above inequality holds trivially.

Case 3. $d_i < 0$. Let $\gamma = 1 - d_i > 0$. The desired results can be shown by the similar argument to the above proof.

PROPOSITION 3.2. *Let f be a uniform P-function with modulus $c > 0$, that is,*

$$\max_{1\leq j\leq n} (x_j - y_j)(f_j(x) - f_j(y)) \geq c\|x-y\|^2 \text{ for all } x, y \in R^n. \tag{24}$$

If there exists a number λ (possibly negative)

$$(x-y)^T(f(x) - f(y)) \geq \lambda\|x-y\|^2 \text{ for all } x, y \in R^n, \tag{25}$$

then f is a P_*-mapping.

Proof. By (24) and (25), we have

$$\begin{aligned}(x-y)^T(f(x) - f(y)) &\geq \lambda\|x-y\|^2 \\ &\geq -|\lambda|\|x-y\|^2 \\ &\geq -\frac{|\lambda|}{c} \max_{1\leq j\leq n} (x_j - y_j)(f_j(x) - f_j(y)) \\ &\geq -\frac{|\lambda|}{c} \sum_{j\in I_+(x,y,f)} (x_j - y_j)(f_j(x) - f_j(y)),\end{aligned}$$

thus f is a P_*-mapping.

If f is Lipschitz continuous, i.e., there exists a scalar L such that $\|f(x) - f(y)\| \leq L\|x-y\|$, then we can verify (25) holding with $\lambda = -L$. Indeed, it follows from

$$|(x-y)^T(f(x) - f(y))| \leq \|x-y\|\|f(x) - f(y)\| \leq L\|x-y\|^2$$

that

$$(x-y)^T(f(x) - f(y)) \geq -L\|x-y\|^2.$$

When f is a linear function, that is, $f = Mx+q$, where $M \in R^{n\times n}$ and $q \in R^n$, then (25) holds trivially. Indeed,

$$\begin{aligned}(x-y)^T(f(x)-f(y)) &= (x-y)^T M(x-y)\\ &= \frac{1}{2}(x-y)^T[M+M^T](x-y)\\ &\geq \lambda_{\min}\|x-y\|^2,\end{aligned}$$

where $\lambda_{\min}$ is the least eigenvalue of $(M+M^T)/2$. For linear mapping, (24) is equivalent to saying that M is P-matrix (the class of matrices with all principal minors positive). Therefore we have the following corollary.

COROLLARY 3.4 ([14]). *Each linear mapping $f = Mx+q$, where M is a P-matrix, is a P^*-mapping.*

Since the class of P_*-mappings includes monotone mappings, the feasibility condition can not guarantee the solvability of $VI(K, f)$ (see [16]). The next consequence declares that the strictly feasibility can guarantee the solvability of nonlinear P_*-complementarity problem.

THEOREM 3.5. *For nonlinear P_*-mapping $f : R^n \to R^n$, if there exists a point $u \in R^n_+$ such that $f(u) > 0$, i.e, $f(u) \in \operatorname{int}(R^n_+)$, then the complementarity problem $NCP(f)$ has no exceptional family, and hence the problem $NCP(f)$ has a solution.*

Proof. Assume the contrary, let $\{x^r\} \subset K = R^n_+$ be an exceptional family for $NCP(f)$, then there are vector sequence $\{\lambda_r\} \subset R^n_+$ and scalar sequence $\{\alpha_r > 1\}$ such that (3) and (4) hold. Thus, for each $i \in \{1, ..., n\}$, by (3) and (4), we have

$$(f_i(x^r)-f_i(u))(x_i^r-u_i) = \begin{cases} -[(\alpha_r-1)x_i^r + f_i(u)](x_i^r-u_i) & \text{if } x_i^r > 0,\\ -[(\lambda_r)_i/2 - f_i(u))u_i] & \text{if } x_i^r = 0.\end{cases} \tag{26}$$

Since $\|x^r\| \to \infty$, if necessary we choose a subsequence, there exists one component index i_0 such that $x^r_{i_0} \to +\infty$ as $r \to +\infty$. Note that $\alpha_r > 1$ for all r and $f(u) > 0$, from (26), we have

$$(f_{i_0}(x^r)-f_{i_0}(u))(x^r_{i_0}-u_{i_0}) = -[(\alpha_r-1)x^r_{i_0} + f_{i_0}(u)](x^r_{i_0}-u_{i_0}) \to -\infty. \tag{27}$$

Therefore, the cardinality $|I_+(x^r, u, f)| \leq n-1$ and $|I_-(x^r, u, f)| \geq 1$ for sufficiently large r. Hence, we have

$$\begin{aligned}&\max_{1\leq i\leq n}(f_i(x^r)-f_i(u))(x_i^r-u_i)\\ &\quad\geq \frac{1}{n-1}\sum_{i\in I_+(x^r,u,f)}(f_i(x^r)-f_i(u))(x_i^r-u_i).\end{aligned} \tag{28}$$

There is a subsequence, denoted by $\{x^{r_j}\}(r_j \to +\infty$ as $j \to +\infty. j = 1, 2, ...)$, such that for some fixed index k, the following equation

$$(f_k(x^{r_j}) - f_k(u))(x_k^{r_j} - u_k) = \max_{1\le i\le n}(f_i(x^{r_j}) - f_i(u))(x_i^{r_j} - u_i) \tag{29}$$

holds for all $r_j(j = 1, 2, ...)$. On the other hand, Since f is a P_*-mapping, there exists some constant $\gamma \ge 0$ such that

$$(1+\gamma)\sum_{i\in I_+(x^{r_j},u,f)}(f_i(x^{r_j}) - f_i(u))(x_i^{r_j} - u_i) + \sum_{i\in I_-(x^{r_j},u,f)}(f_i(x^{r_j}) - f_i(u))(x_i^{r_j} - u_i) \ge 0. \tag{30}$$

We now consider the component sequence $\{x_{i_0}^{r_j}\}$. From (27), we see that $i_0 \in I_-(x^{r_j}, u, f)$ for sufficiently large j, Hence from (30), (28) and (29), we deduce that

$$\begin{aligned}
&(f_{i_0}(x^{r_j}) - f_{i_0}(u))(x_{i_0}^{r_j} - u_{i_0})\\
&\quad\ge -(1+\gamma)\sum_{i\in I_+(x^{r_j},u,f)}(f_i(x^{r_j}) - f_i(u))(x_i^{r_j} - u_i)\\
&\quad\ge -(1+\gamma)(n-1)\max_{1\le i\le n}(f_i(x^{r_j}) - f_i(u))(x_i^{r_j} - u_i)\\
&\quad= -(1+\gamma)(n-1)(f_k(x^{r_j}) - f_k(u))(x_k^{r_j} - u_k).
\end{aligned}$$

Combining (27) and the above inequality yields

$$\begin{aligned}
&-[(\alpha_{r_j}-1)x_{i_0}^{r_j} + f_{i_0}(u)](x_{i_0}^{r_j} - u_{i_0})\\
&\quad\ge -(1+\gamma)(n-1)(f_k(x^{r_j}) - f_k(u))(x_k^{r_j} - u_k)
\end{aligned} \tag{31}$$

Without loss of generality, assume that j is sufficiently large. For each j, three cases are possible.

Case 1. $x_k^{r_j} = 0$. It follows from (26) and $\lambda_{r_j} \in R_+^m$ that

$$(f_k(x^{r_j}) - f_k(u))(x_k^{r_j} - u_k) = -[(\lambda_{r_j})_k/2 - f_k(u)]u_k \le f_k(u)u_k. \tag{32}$$

By using (31) and (32), we deduce that

$$-(1+\gamma)(n-1)f_k(u)u_k \le -[(\alpha_r - 1)x_{i_0}^{r_j} + f_{i_0}(u)](x_{i_0}^{r_j} - u_{i_0}) \to -\infty$$

which is a contradiction.

Case 2. $0 < x_k^{r_j} \le u_k$. By using (31) and (26), we have

$$\begin{aligned}
&-[(\alpha_{r_j}-1)x_{i_0}^{r_j} + f_{i_0}(u)](x_{i_0}^{r_j} - u_{i_0})\\
&\quad\ge -(1+\gamma)(n-1)[(\alpha_{r_j}-1)x_k^{r_j} + f_k(u)](u_k - x_k^{r_j})\\
&\quad\ge -(1+\gamma)(n-1)[(\alpha_{r_j}-1)u_k + f_k(u)]u_k
\end{aligned}$$

which is a contradiction since $x_{i_0}^{r_j} \to +\infty$ and $f_{i_0}(u) > 0$.

Case 3. $x_k^{r_j} > u_k$. By using (26) and (31)

$$\begin{aligned} &- [(\alpha_{r_j} - 1)x_{i_0}^{r_j} + f_{i_0}(u)](x_{i_0}^{r_j} - u_{i_0}) \\ &\quad \geq (1+\gamma)(n-1)[(\alpha_{r_j} - 1)x_k^{r_j} + f_k(u)](x_k^{r_j} - u_k) \end{aligned}$$

which is also a contradiction since the left-hand side tends to $-\infty$ as $j \to \infty$, but the right-hand side is positive.

Therefore, the problem $NCP(f)$ has no exceptional family, and hence there exists a solution to $NCP(f)$ according to Theorem 2.1.

4. Final remarks

In the last there decades, the solvability of variational inequality problems including complementarity problems has been studied extensively by many authors. A large number of solution conditions have been developed in the literature. A comprehensive survey in this field can be found in reference [3]. In this paper, we have extended the concepts of exceptional sequence and exceptional family of elements for a continuous function introduced in [26] and [7] to nonlinear variational inequality problems. We think that the proposed concept provides some unified argument for the solvability of variational inequality problems from the point of view that many known existence theorems can be generalized or reobtained by the proposed concept of exceptional family. The condition "without exceptional family" is a sufficient condition for the solvability of variational inequality problem provided that the mapping f is continuous in R^n. This sufficient condition is also necessary for pseudo-monotone problems. It is worth noting that for quasi-monotone variational inequalities we have developed a sufficient condition for its solvability. Is there a solution condition which is not only sufficient but also necessary for the solvability of quasi-monotone variational inequality problems? Such a topic is also worth investigating in the future.

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