

MONOTONICITY OF FIXED POINT AND NORMAL MAPPINGS ASSOCIATED WITH VARIATIONAL INEQUALITY AND ITS APPLICATION*

YUN-BIN ZHAO [†] AND DUAN LI[‡]

Abstract. We prove sufficient conditions for the monotonicity and the strong monotonicity of fixed point and normal maps associated with variational inequality problem over a general closed convex set. Sufficient conditions for the strong monotonicity of their perturbed versions are also shown. These results include some known ones in the literature as particular instances. Inspired by these results, we propose a modified Solodov and Svaiter iterative algorithm for the variational inequality problem whose fixed point map or normal map is monotone.

Key words. variational inequalities, co-coercive maps, (strongly) monotone maps, fixed point and normal maps, iterative algorithm

AMS subject classifications. 90C30, 90C33, 90C25

1. Introduction. Given a continuous function $f : R^n \rightarrow R^n$ and a closed convex set K in R^n , the well-known finite-dimensional variational inequality, denoted by $VI(K, f)$, is to find an element $x^* \in K$ such that

$$(x - x^*)^T f(x^*) \geq 0 \text{ for all } x \in K.$$

It is well-known that the above problem can be reformulated as nonsmooth equations such as the fixed point and normal equations (see e.g. [9, 18]). The fixed point equation is defined by

$$(1.1) \quad \pi_\alpha(x) = x - \Pi_K(x - \alpha f(x)) = 0,$$

and the normal equation is defined by

$$(1.2) \quad \Phi_\alpha(x) = f(\Pi_K(x)) + \alpha(x - \Pi_K(x)) = 0,$$

where $\alpha > 0$ is a positive scalar, and $\Pi_K(\cdot)$ denotes the projection operator on the convex set K , i.e.,

$$\Pi_K(x) = \arg \min\{\|z - x\| : z \in K\}.$$

Throughout the paper, $\|\cdot\|$ denotes the 2-norm (Euclidean norm) of the vector in R^n . It turns out that x^* solves $VI(K, f)$ if and only if $\pi_\alpha(x^*) = 0$, and that if x^* solves $VI(K, f)$, then $x^* - \frac{1}{\alpha}f(x^*)$ is a solution to $\Phi_\alpha(x) = 0$; conversely, if $\Phi_\alpha(u^*) = 0$, then $\Pi_K(u^*)$ is a solution to $VI(K, f)$.

Recently, several authors studied the P_0 property of fixed point and normal maps when K is a rectangular box in R^n , i.e., the Cartesian product of n one-dimensional

*This work was partially supported by Research Grants Council of Hong Kong under grant CUHK4392/99E.

[†]Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, Shatin, New Territory, Hong Kong, and Institute of Computational Mathematics and Scientific/Engineering Computing, Chinese Academy of Sciences, Beijing, China (Email: ybzhao@se.cuhk.edu.hk).

[‡]Corresponding author. Department of Systems Engineering and Engineering Management, The Chinese University of Hong Kong, Shatin, New Territory, Hong Kong (Fax: (852) 26035505; Tel: (852) 26098323; Email: dli@se.cuhk.edu.hk).

intervals. For such a K , Ravindran and Gowda [17] (respectively, Gowda and Tawhid [8]) showed that $\pi_\alpha(x)$ (respectively, $\Phi_\alpha(x)$) is a P_0 -function if f is. Notice that the monotone maps are very important special cases of the class of P_0 -functions. It is worth considering the problem:

(P) *When are the mappings $\pi_\alpha(x)$ and $\Phi_\alpha(x)$ monotone if K is a general closed convex set?*

Intuitively, we may conjecture that the fixed point map and the normal map are monotone if f is. However, this conjecture is not true. The following example shows that for a given $\alpha > 0$ the monotonicity of f , in general, does not imply the monotonicity of the fixed point map $\pi_\alpha(x)$ and the normal map $\Phi_\alpha(x)$.

EXAMPLE 1.1. Let K be a closed convex set given by

$$K = \{x \in R^2 : x_1 \geq 0, x_2 = 0\},$$

and

$$f(x) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}.$$

For any $x, y \in R^2$, we have that $(x - y)^T(f(x) - f(y)) = 0$. Hence the function f is monotone on R^2 . We now show that for an arbitrary scalar $\alpha > 0$ the fixed point mapping $\pi_\alpha(x) = x - \Pi_K(x - \alpha f(x))$ is not monotone in R^2 . Indeed, let $u = (0, 0)^T$ and $y = (1, \alpha/2)^T$. It is easy to verify that $\pi_\alpha(u) = (0, 0)^T$ and $\pi_\alpha(y) = (-\alpha^2/2, \alpha/2)^T$. Thus, we have

$$(u - y)^T(\pi_\alpha(u) - \pi_\alpha(y)) = -\alpha^2/2 < 0,$$

which implies that $\pi_\alpha(\cdot)$ is not monotone on R^n .

EXAMPLE 1.2. Let K be a closed convex set given by

$$K = \{x \in R^2 : x_1 \leq 0, x_2 = 0\},$$

and $f(x) : R^2 \rightarrow R^2$ be given as in Example 1.1. We now show that for an arbitrary $\alpha > 0$ the normal mapping $\Phi_\alpha(x) = f(\Pi_K(x)) + \alpha(x - \Pi_K(x))$ is not monotone in R^2 . Indeed, let $u = (0, 0)^T$ and $y = (-2\alpha^2, \alpha)^T$. We have that $\Phi_\alpha(u) = (0, 0)^T$ and $\Phi_\alpha(y) = (0, -\alpha^2)^T$. Thus, we have

$$(u - y)^T(\Phi_\alpha(u) - \Phi_\alpha(y)) = -\alpha^3 < 0,$$

which implies that $\Phi_\alpha(\cdot)$ is not monotone on R^n .

From the above examples, we conclude that certain condition stronger than the monotonicity of f is required to guarantee the monotonicity of $\pi_\alpha(x)$ and $\Phi_\alpha(x)$. One such condition is so-called co-coercivity condition. We recall that f is said to be co-coercive with modulus $\beta > 0$ on a set $S \subset R^n$ if there exists a constant $\beta > 0$ such that

$$(x - y)^T(f(x) - f(y)) \geq \beta \|f(x) - f(y)\|^2 \quad \text{for all } x, y \in S.$$

The co-coercivity condition was used in several works, such as Bruck [1], Gabay [7] (in which this condition is used implicitly), Tseng [25], Marcotte and Wu [15], Magnanti and Perakis [13, 14], and Zhu and Marcotte [29, 30]. It is also used to study the strict feasibility of complementarity problems [27]. It is interesting to note that in an affine

case the co-coercivity has a close relation to the property of psd-plus matrices [12, 30]. A special case of the co-coercive map is the strongly monotone and Lipschitzian map. We recall that a mapping f is said to be strongly monotone with modulus $c > 0$ on the set S if there is a scalar $c > 0$ such that

$$(x - y)^T(f(x) - f(y)) \geq c\|x - y\|^2 \text{ for all } x, y \in S.$$

It is evident that any co-coercive map on the set S must be monotone and Lipschitz continuous (with constant $L = 1/\beta$), but not necessarily strongly monotone (for instance, the constant mapping) on the same set.

In fact, the aforementioned problem (P) is not completely unknown. By using the co-coercivity condition implicitly and using properties of nonexpansive maps, Gabay [7] actually showed (but did not explicitly state) that $\pi_\alpha(x)$ and $\Phi_{1/\alpha}(x)$ are monotone if the scalar α is chosen such that the map $I - \alpha f$ is nonexpansive. Furthermore, for strongly monotone and Lipschitzian map f , Gabay [7] and Sibony [20] actually showed that $\pi_\alpha(x)$ and $\Phi_{1/\alpha}(x)$ are strongly monotone if the scalar α is chosen such that the map $I - \alpha f$ is contractive. Throughout this paper, we use the standard concept “nonexpansive” map and “contractive” map in the literature to mean a Lipschitzian map with constant $L = 1$ and $L < 1$, respectively.

However, it is easy to give an example to show that $\pi_\alpha(x)$ and $\Phi_\alpha(x)$ are still monotone (strongly monotone) even when α is chosen such that $I - \alpha f$ is not nonexpansive (contractive). For instance, let $K = \mathbb{R}_+^n$ and $f(x) = x$. We see that the function f is co-coercive with modulus $\beta = 1$. While $I - \alpha f$ is not nonexpansive for $\alpha > 2$, the map $\pi_\alpha(x)$ remains monotone. As a result, the main purpose of this paper is to expand the results of Sibony [20] and Gabay [7]. We show that if f is co-coercive (strongly monotone and Lipschitz continuous, respectively), the monotonicity (strong monotonicity, respectively) of the maps $\pi_\alpha(x)$ and $\Phi_\alpha(x)$ can be ensured when α lies in a larger interval in which the map $I - \alpha f$ may not be nonexpansive (contractive, respectively). The results derived in this paper are not obtainable by the proof based on the nonexpansiveness and contractiveness of maps.

The other purpose of the paper is to introduce an application of the monotonicity of $\pi_\alpha(x)$ and $\Phi_\alpha(x)$. This application (see Section 3) is motivated by the globally convergent inexact Newton method for the system of monotone equations proposed by Solodov and Svaiter [21]. See also [22, 23, 24]). We propose a modified Solodov and Svaiter method to solve the monotone equations $\pi_\alpha(x) = 0$ or $\Phi_\alpha(x) = 0$. This modified algorithm requires no projection operations in the line-search step.

2. Monotonicity of $\pi_\alpha(x)$ and $\Phi_\alpha(x)$. It is known (see Sibony [20] and Gabay [7]) that if f is strongly monotone with modulus $c > 0$ and Lipschitz continuous with constant $L > 0$, then $I - \alpha f$ is contractive when $0 < \alpha < 2c/L^2$. Since Π_K is nonexpansive, this in turn implies that $\pi_\alpha(x)$ and $\Phi_{1/\alpha}(x)$ are both strongly monotone for $0 < \alpha < 2c/L^2$. Similarly, it follows from Gabay [7] (see Theorem 6.1 therein) that if f is co-coercive with modulus $\beta > 0$, then $I - \alpha f$ is nonexpansive for $0 < \alpha \leq 2\beta$, and thus we can easily verify that $\pi_\alpha(x)$ and $\Phi_{1/\alpha}(x)$ are monotone for $0 < \alpha \leq 2\beta$.

In this section, we prove an improved version of the above-mentioned results. We prove that i) when α lies outside of the interval $(0, 2c/L^2)$, for instance, $2c/L^2 \leq \alpha \leq 4c/L^2$, $\pi_\alpha(x)$ and $\Phi_{1/\alpha}(x)$ are still strongly monotone although $I - \alpha f$, in this case, is not contractive, and ii) when α lies outside of the interval $(0, 2\beta]$, for instance, $2\beta < \alpha \leq 4\beta$, $\pi_\alpha(x)$ and $\Phi_{1/\alpha}(x)$ remain monotone although $I - \alpha f$ is not nonexpansive. This new result on monotonicity (strongly monotonicity) of $\pi_\alpha(x)$ and $\Phi_{1/\alpha}(x)$ for $\alpha > 2\beta$ ($\alpha \geq 2c/L^2$) is not obtainable by using the nonexpansive (contractive) property

of $I - \alpha f$. The reason goes as follows: Let f be co-coercive with modulus $\beta > 0$ on the set $S \subseteq R^n$, where

$$\beta = \sup\{\gamma > 0 : (x - y)^T(f(x) - f(y)) \geq \gamma\|f(x) - f(y)\|^2 \text{ for all } x, y \in S\}.$$

Clearly, such a scalar β is unique and $0 < \beta < \infty$ provided that f is not a constant mapping. We now verify that $I - \alpha f$ is nonexpansive on S if and only if $0 < \alpha \leq 2\beta$. It is sufficient to show that if $\alpha > 0$ is chosen such that $I - \alpha f$ is nonexpansive on S , then we must have $\alpha \leq 2\beta$. In fact, if $I - \alpha f$ is nonexpansive, then for any x, y in S we have

$$\begin{aligned} \|x - y\|^2 &\geq \|(I - \alpha f)(x) - (I - \alpha f)(y)\|^2 \\ &= \|x - y\|^2 - 2\alpha(x - y)^T(f(x) - f(y)) + \alpha^2\|f(x) - f(y)\|^2, \end{aligned}$$

which implies that

$$(x - y)^T(f(x) - f(y)) \geq (\alpha/2)\|f(x) - f(y)\|^2.$$

By the definition of β , we deduce that $\alpha/2 \leq \beta$, the desired consequence. Similarly, let f be strongly monotone with modulus $c > 0$ and Lipschitz continuous with constant $L > 0$ on the set S , where

$$c = \sup\{\gamma > 0 : (x - y)^T(f(x) - f(y)) \geq \gamma\|x - y\|^2 \text{ for all } x, y \in S\}$$

and

$$L = \inf\{\gamma > 0 : \|f(x) - f(y)\| \leq \gamma\|x - y\| \text{ for all } x, y \in S\}.$$

We can easily see that $0 < c < \infty$ and $L > 0$ provided that S is not a single point set. It is also easy to show that $I - \alpha f$ is contractive if and only if $0 < \alpha < 2c/L^2$.

Since the map $I - \alpha f$ is not contractive (nonexpansive, respectively) for $\alpha \geq 2c/L^2$ ($\alpha > 2\beta$, respectively), our result established in this section cannot follow directly from the proof of Sibony [20] and Gabay [7].

We also study the strong monotonicity of the perturbed fixed point and normal maps defined by

$$\pi_{\alpha, \varepsilon}(x) := x - \Pi_K(x - \alpha(f(x) + \varepsilon x)),$$

and

$$\Phi_{\alpha, \varepsilon}(x) := f(\Pi_K(x)) + \varepsilon\Pi_K(x) + \alpha(x - \Pi_K(x)),$$

respectively. This is motivated by the well-known Tikhonov regularization method for complementarity problems and variational inequalities. See for example, Isac [10, 11], Venkateswaran [26], Facchinei [3], Facchinei and Kanzow [4], Facchinei and Pang [5], Gowda and Tawhid [8], Qi [16], Ravindran and Gowda [17], Zhao and Li [28], etc. It is worth mentioning that Gowda and Tawhid [8] showed that when $\alpha = 1$ the perturbed mapping $\Phi_{1, \varepsilon}(x)$ is a P-function if f is a P_0 -function and K is a rectangular set. We show in this paper a sufficient condition for the strong monotonicity of $\pi_{\alpha, \varepsilon}(x)$ and $\Phi_{\alpha, \varepsilon}(x)$. The following lemma is helpful.

LEMMA 2.1. (i) Denote

$$(2.1) \quad u_z = z - \Pi_K(z) \text{ for all } z \in R^n.$$

Then

$$(z - w)^T(u_z - u_w) \geq \|u_z - u_w\|^2.$$

(ii) For any $\alpha > 0$ and vector $b \in R^n$, the following inequality holds for all $v \in R^n$,

$$\alpha\|v\|^2 + v^T b \geq -\frac{\|b\|^2}{4\alpha}.$$

Proof. By the property of projection operator, we have

$$(\Pi_K(z) - \Pi_K(w))^T(\Pi_K(w) - w) \geq 0 \text{ for all } z, w \in R^n,$$

$$(\Pi_K(w) - \Pi_K(z))^T(\Pi_K(z) - z) \geq 0 \text{ for all } z, w \in R^n.$$

Adding the above two inequalities leads to

$$(\Pi_K(z) - \Pi_K(w))^T(z - \Pi_K(z) - (w - \Pi_K(w))) \geq 0 \text{ for all } z, w \in R^n,$$

i.e.,

$$[z - u_z - (w - u_w)]^T(u_z - u_w) \geq 0 \text{ for all } z, w \in R^n.$$

This proves the result (i).

Given $\alpha > 0$ and $b \in R^n$, it is easy to check that the minimum value of $\alpha\|v\|^2 + v^T b$ is $-\|b\|^2/(4\alpha)$. This proves the result (ii). \square

We are ready to prove the main result in this section.

THEOREM 2.1. *Let K be an arbitrary closed convex set in R^n and $K \subseteq S \subseteq R^n$. Let $f : R^n \rightarrow R^n$ be a function.*

(i) *If f is co-coercive with modulus $\beta > 0$ on the set S , then for any fixed scalar α satisfying $0 < \alpha \leq 4\beta$, the fixed point map $\pi_\alpha(x)$ defined by (1.1) is monotone on the set S .*

(ii) *If f is strongly monotone with modulus $c > 0$ on the set S , and f is Lipschitz continuous with constant $L > 0$ on S , then for any fixed scalar α satisfying $0 < \alpha < 4c/L^2$, the fixed point map $\pi_\alpha(x)$ is strongly monotone on the set S .*

(iii) *If f is co-coercive with modulus $\beta > 0$ on the set S , then for any $0 < \alpha < 4\beta$ and $0 < \varepsilon \leq 2\left(\frac{1}{\alpha} - \frac{1}{4\beta}\right)$ the perturbed map $\pi_{\alpha,\varepsilon}(x)$ is strongly monotone in x on the set S .*

Proof. Let $\alpha > 0$ and $0 \leq \varepsilon \leq 2/\alpha$ be two scalars. For any vector x, y in S , denote

$$z = x - \alpha(f(x) + \varepsilon x), \quad w = y - \alpha(f(y) + \varepsilon y).$$

By using the notation of (2.1) and Lemma 2.1, we have

$$\begin{aligned} & (x - y)^T(\pi_{\alpha,\varepsilon}(x) - \pi_{\alpha,\varepsilon}(y)) \\ &= (x - y)^T[(z - \Pi_K(z) - (w - \Pi_K(w)) + \alpha(f(x) + \varepsilon x) - \\ & \quad \alpha(f(y) + \varepsilon y))] \\ &= (x - y)^T(u_z - u_w) + \alpha\varepsilon\|x - y\|^2 + \alpha(x - y)^T(f(x) - f(y)) \\ &= [z + \alpha(f(x) + \varepsilon x) - (w + \alpha(f(y) + \varepsilon y))]^T(u_z - u_w) + \\ & \quad \alpha\varepsilon\|x - y\|^2 + \alpha(x - y)^T(f(x) - f(y)) \end{aligned}$$

$$\begin{aligned}
&= (z - w)^T(u_z - u_w) + \alpha[f(x) + \varepsilon x - (f(y) + \varepsilon y)]^T(u_z - u_w) + \\
&\quad \alpha\varepsilon\|x - y\|^2 + \alpha(x - y)^T(f(x) - f(y)) \\
&\geq \|u_z - u_w\|^2 + \alpha[f(x) + \varepsilon x - (f(y) + \varepsilon y)]^T(u_z - u_w) + \\
&\quad \alpha\varepsilon\|x - y\|^2 + \alpha(x - y)^T(f(x) - f(y)) \\
&\geq -(\alpha^2/4)\|f(x) + \varepsilon x - (f(y) + \varepsilon y)\|^2 + \alpha\varepsilon\|x - y\|^2 + \\
&\quad \alpha(x - y)^T(f(x) - f(y)) \\
&= (\alpha\varepsilon - \alpha^2\varepsilon^2/4)\|x - y\|^2 - (\alpha^2/4)\|f(x) - f(y)\|^2 + \\
(2.2) \quad &(\alpha - \alpha^2\varepsilon/2)(x - y)^T(f(x) - f(y)).
\end{aligned}$$

If f is co-coercive with modulus $\beta > 0$, using $\varepsilon \leq 2/\alpha$ we see from the above that

$$\begin{aligned}
&(x - y)^T(\pi_{\alpha,\varepsilon}(x) - \pi_{\alpha,\varepsilon}(y)) \\
&\geq (\alpha\varepsilon - \alpha^2\varepsilon^2/4)\|x - y\|^2 - (\alpha^2/4)\|f(x) - f(y)\|^2 + \\
&\quad (\alpha - \alpha^2\varepsilon/2)\beta\|f(x) - f(y)\|^2 \\
&= \alpha\varepsilon(1 - \alpha\varepsilon/4)\|x - y\|^2 + \alpha^2\beta\left(\frac{1}{\alpha} - \frac{1}{4\beta} - \frac{\varepsilon}{2}\right)\|f(x) - f(y)\|^2.
\end{aligned}$$

Setting $\varepsilon = 0$ in the above inequality, we see that for $0 < \alpha \leq 4\beta$ the right-hand side is nonnegative, showing that π_α is monotone on the set S . This proves the result (i). Also, if $\alpha < 4\beta$ and $0 < \varepsilon \leq 2(\frac{1}{\alpha} - \frac{1}{4\beta})$, the right-hand side of the above inequality is greater than or equal to $r\|x - y\|^2$, where $r = \alpha\varepsilon(1 - \alpha\varepsilon/4) > 0$, showing that $\pi_{\alpha,\varepsilon}$ is strongly monotone on the set S . The proof of the result (iii) is complete.

Assume that f is strongly monotone with modulus $c > 0$ and Lipschitz continuous with constant $L > 0$. We now prove the result (ii). For this case, setting $\varepsilon = 0$ in (2.2), we have that

$$\begin{aligned}
&(x - y)^T(\pi_\alpha(x) - \pi_\alpha(y)) \\
&\geq -(\alpha^2/4)\|f(x) - f(y)\|^2 + \alpha(x - y)^T(f(x) - f(y)) \\
&\geq -(\alpha^2L^2/4)\|x - y\|^2 + \alpha c\|x - y\|^2 \\
&= (\alpha c - \alpha^2L^2/4)\|x - y\|^2.
\end{aligned}$$

For $\alpha < 4c/L^2$, it is evident that the scalar

$$r = \alpha c - \frac{\alpha^2L^2}{4} = \frac{\alpha L^2}{4} \left(\frac{4c}{L^2} - \alpha \right) > 0.$$

Result (ii) is proved. \square

Similarly, we have the following result for $\Phi_\alpha(x)$.

THEOREM 2.2. *Let f be a function from R^n into itself and K be a closed convex set and $K \subseteq S \subseteq R^n$.*

(i) *If f is co-coercive with modulus $\beta > 0$ on the set S , then for any constant α such that $\alpha > 1/(4\beta)$, the normal map $\Phi_\alpha(x)$ given by (1.2) is monotone on the set S .*

(ii) *If f is strongly monotone with modulus $c > 0$ and Lipschitz continuous with constant $L > 0$ on the set S , then for any α satisfying $\alpha > L^2/(4c)$, the normal map $\Phi_\alpha(x)$ given by (1.2) is strongly monotone on the set S .*

(iii) *If f is co-coercive with modulus $\beta > 0$ on the set S , then for any constant $\alpha > 1/(4\beta)$, the perturbed normal map $\Phi_{\alpha,\varepsilon}(x)$, where $0 < \varepsilon < \alpha$, is strongly monotone in x on the set S .*

Proof. Let α, ε, r be given such that $\alpha > \varepsilon \geq r \geq 0$. For any vector x, y in S , let u_x and u_y be defined by (2.1) with $z = x$ and $z = y$, respectively. Then, by Lemma 2.1 we have

$$(2.3) \quad \begin{aligned} & (\alpha - r)\|u_x - u_y\|^2 + (u_x - u_y)^T(f(\Pi_K(x)) - f(\Pi_K(y))) \\ & \geq -\frac{1}{4(\alpha - r)}\|f(\Pi_K(x)) - f(\Pi_K(y))\|^2, \end{aligned}$$

and

$$(2.4) \quad (x - y)^T(u_x - u_y) \geq \|u_x - u_y\|^2,$$

that further implies

$$\|x - y\| \geq \|u_x - u_y\|.$$

By using the above three inequalities, we have

$$(2.5) \quad \begin{aligned} & (x - y)^T(\Phi_{\alpha, \varepsilon}(x) - \Phi_{\alpha, \varepsilon}(y)) - r\|x - y\|^2 \\ & = (x - y)^T[f(\Pi_K(x)) + \varepsilon\Pi_K(x) + \alpha u_x - f(\Pi_K(y)) - \varepsilon\Pi_K(y) - \alpha u_y] \\ & \quad - r\|x - y\|^2 \\ & = \alpha(x - y)^T(u_x - u_y) + \varepsilon(x - y)^T(\Pi_K(x) - \Pi_K(y)) - r\|x - y\|^2 + \\ & \quad (x - y)^T(f(\Pi_K(x)) - f(\Pi_K(y))) \\ & = (\alpha - \varepsilon)(x - y)^T(u_x - u_y) + (\varepsilon - r)\|x - y\|^2 + \\ & (2.5) \quad (x - y)^T(f(\Pi_K(x)) - f(\Pi_K(y))) \\ & = (\alpha - \varepsilon)(x - y)^T(u_x - u_y) + (\varepsilon - r)\|x - y\|^2 + (u_x - u_y)^T(f(\Pi_K(x)) \\ & \quad - f(\Pi_K(y))) + (\Pi_K(x) - \Pi_K(y))^T(f(\Pi_K(x)) - f(\Pi_K(y))) \\ & \geq (\alpha - \varepsilon)\|u_x - u_y\|^2 + (\varepsilon - r)\|u_x - u_y\|^2 + (u_x - u_y)^T(f(\Pi_K(x)) \\ & \quad - f(\Pi_K(y))) + (\Pi_K(x) - \Pi_K(y))^T(f(\Pi_K(x)) - f(\Pi_K(y))) \\ & = (\alpha - r)\|u_x - u_y\|^2 + (u_x - u_y)^T(f(\Pi_K(x)) - f(\Pi_K(y))) + \\ & \quad (\Pi_K(x) - \Pi_K(y))^T(f(\Pi_K(x)) - f(\Pi_K(y))) \\ & \geq -\frac{1}{4(\alpha - r)}\|f(\Pi_K(x)) - f(\Pi_K(y))\|^2 + \\ (2.6) \quad & (\Pi_K(x) - \Pi_K(y))^T(f(\Pi_K(x)) - f(\Pi_K(y))). \end{aligned}$$

Let f be co-coercive with modulus $\beta > 0$ on the set S . Setting $\varepsilon = r = 0$ in the above inequality, and using the co-coercivity of f , we have

$$\begin{aligned} (x - y)^T(\Phi_\alpha(x) - \Phi_\alpha(y)) & \geq -\frac{1}{4\alpha}\|f(\Pi_K(x)) - f(\Pi_K(y))\|^2 + \\ & \quad (\Pi_K(x) - \Pi_K(y))^T(f(\Pi_K(x)) - f(\Pi_K(y))) \\ & \geq \left(\beta - \frac{1}{4\alpha}\right)\|f(\Pi_K(x)) - f(\Pi_K(y))\|^2. \end{aligned}$$

For $\alpha > 1/(4\beta)$, the right-hand side is nonnegative, and hence the map Φ_α is monotone on the set S . This proves the result (i).

Let $\alpha > 1/(4\beta)$, $0 < \varepsilon < \alpha$, and $0 < r < \min\{\varepsilon, \alpha - 1/(4\beta)\}$. By the co-coercivity of f , the inequality (2.6) can be further written as

$$\begin{aligned} & (x - y)^T(\Phi_{\alpha, \varepsilon}(x) - \Phi_{\alpha, \varepsilon}(y)) - r\|x - y\|^2 \\ & \geq \left(\beta - \frac{1}{4(\alpha - r)}\right)\|f(\Pi_K(x)) - f(\Pi_K(y))\|^2. \end{aligned}$$

Since $0 < r < \alpha - 1/(4\beta)$, the right-hand side of the above is nonnegative, and thus the map $\Phi_{\alpha,\varepsilon}$ is strongly monotone on the set S . Result (iii) is proved.

Finally, we prove the result (ii). Assume that f is strongly monotone with modulus $c > 0$ and Lipschitz continuous with constant $L > 0$. For any vector x, y in S , we note that the equation (2.5) holds for any $\alpha > 0, \varepsilon \geq 0$ and $r \geq 0$. Setting $\varepsilon = 0$, the equation (2.5) reduces to

$$(2.7) \quad \begin{aligned} & (x - y)^T(\Phi_\alpha(x) - \Phi_\alpha(y)) - r\|x - y\|^2 \\ &= \alpha(x - y)^T(u_x - u_y) - r\|x - y\|^2 + \\ & (x - y)^T(f(\Pi_K(x)) - f(\Pi_K(y))). \end{aligned}$$

Given $\alpha > L^2/(4c)$. Let r be a scalar such that $0 < r < \alpha/2$ and $2r + \frac{L^2}{4(\alpha - 2r)} < c$. Notice that

$$\begin{aligned} \|x - y\|^2 &= \|\Pi_K(x) - \Pi_K(y) + u_x - u_y\|^2 \\ &\leq 2(\|\Pi_K(x) - \Pi_K(y)\|^2 + \|u_x - u_y\|^2). \end{aligned}$$

Substituting the above into (2.7) and using inequalities (2.3) and (2.4), we have

$$\begin{aligned} & (x - y)^T(\Phi_\alpha(x) - \Phi_\alpha(y)) - r\|x - y\|^2 \\ &\geq \alpha\|u_x - u_y\|^2 - 2r(\|\Pi_K(x) - \Pi_K(y)\|^2 + \|u_x - u_y\|^2) + \\ & (x - y)^T(f(\Pi_K(x)) - f(\Pi_K(y))) \\ &= (\alpha - 2r)\|u_x - u_y\|^2 + (u_x - u_y)^T(f(\Pi_K(x)) - f(\Pi_K(y))) + \\ & (\Pi_K(x) - \Pi_K(y))^T(f(\Pi_K(x)) - f(\Pi_K(y))) - 2r\|\Pi_K(x) - \Pi_K(y)\|^2 \\ &\geq -\frac{1}{4(\alpha - 2r)}\|f(\Pi_K(x)) - f(\Pi_K(y))\|^2 - 2r\|\Pi_K(x) - \Pi_K(y)\|^2 + \\ & (\Pi_K(x) - \Pi_K(y))^T(f(\Pi_K(x)) - f(\Pi_K(y))) \\ &\geq \left(-\frac{L^2}{4(\alpha - 2r)} - 2r + c\right)\|\Pi_K(x) - \Pi_K(y)\|^2, \end{aligned}$$

where the last inequality follows from the Lipschitz continuity and strong monotonicity of f . The right-hand side of the above is nonnegative. Thus, the map Φ_α is strongly monotone on the set S . This proves the result (ii). \square

The following result is an immediate consequence of Theorems 2.1 and 2.2.

COROLLARY 2.1. *Assume that f is monotone and Lipschitz continuous with constant $L > 0$ on a set $S \supseteq K$.*

(i) *If $0 < \varepsilon < \infty$ and $0 < \alpha < \frac{4\varepsilon}{(L+\varepsilon)^2}$, then the perturbed map $\pi_{\alpha,\varepsilon}(x)$ is strongly monotone in x on the set S .*

(ii) *If $0 < \varepsilon < \infty$ and $\alpha > \frac{(L+\varepsilon)^2}{4\varepsilon}$, then the perturbed normal map $\Phi_{\alpha,\varepsilon}(x)$ is strongly monotone in x on the set S .*

Proof. Let $\varepsilon \in (0, \infty)$ be a fixed scalar. It is evident that under the condition of the corollary, the function $F(x) = f(x) + \varepsilon x$ is strongly monotone with modulus $\varepsilon > 0$ and Lipschitz continuous with constant $L + \varepsilon$. Therefore, from Theorem 2.1(ii) we deduce that if $0 < \alpha < 4\varepsilon/(L + \varepsilon)^2$, the map $\pi_{\alpha,\varepsilon}(x)$ is strongly monotone on S . Similarly, the strong monotonicity of $\Phi_{\alpha,\varepsilon}(x)$ follows from Theorem 2.2(ii). \square

The Item (iii) of both Theorem 2.1 and Theorem 2.2 shows that for any sufficiently small parameter ε , the perturbed fixed point and normal maps are strongly monotone. This result is quite different from Corollary 2.1. When α is a fixed constant, Corollary

2.1 does not cover the case where ε can be sufficiently small. Indeed, for a fixed $\alpha > 0$, the inequalities $0 < \alpha < \frac{4\varepsilon}{(L+\varepsilon)^2}$ and $\alpha > \frac{(L+\varepsilon)^2}{4\varepsilon}$ fail to hold when $\varepsilon \rightarrow 0$.

Up to now, we have shown that the fixed point map $\pi_\alpha(x)$ (respectively, the normal map $\Phi_\alpha(x)$) is monotone if f is co-coercive with modulus $\beta > 0$ and $\alpha \in (0, 4\beta]$ (respectively, $\alpha \in (1/(4\beta), \infty)$). This result includes the known ones from Sibony [20] and Gabay [7] as special cases. Under the same assumption on f and α , we deduce from the Item (iii) of Theorems 2.1 and 2.2 that the perturbed forms $\pi_{\alpha,\varepsilon}$ and $\Phi_{\alpha,\varepsilon}$ are strongly monotone provided that the scalar ε is sufficiently small. In the succeeding sections, we will introduce an application of the above results on globally convergent iterative algorithms for $\text{VI}(K, f)$ whose fixed point map or normal map is monotone.

3. Application: Iterative algorithm for $\text{VI}(K, f)$. Since $\pi_\alpha(x)$ and $\Phi_\alpha(x)$ are monotone if the function f is co-coercive and α lies in certain interval, we can solve the co-coercive variational inequity problems via solving the system of monotone equation $\pi_\alpha(x) = 0$ or $\Phi_\alpha(x) = 0$. Recently, Solodov and Svaiter [21] (see also, [22, 23, 24]) proposed a class of inexact Newton methods for monotone equations. Let $\mathcal{F}(x)$ be a monotone mapping from R^n into R^n . The Solodov and Svaiter's algorithm for the equation $\mathcal{F}(x) = 0$ proceeds as follows:

ALGORITHM SS. [21] Choose any $x^0 \in R^n, t \in (0, 1)$ and $\lambda \in (0, 1)$. Set $k := 0$.

Inexact Newton step. Choose a positive semidefinite matrix G_k . Choose $\mu_k > 0$ and $\gamma_k \in [0, 1)$. Compute $d^k \in R^n$ such that

$$0 = \mathcal{F}(x^k) + (G_k + \mu_k I)d^k + e^k,$$

where $\|e^k\| \leq \gamma_k \mu_k \|d^k\|$. Stop if $d^k = 0$; Otherwise,

Line-search step. Find $y^k = x^k + \alpha_k d^k$, where $\alpha_k = t^{m_k}$ with m_k being the smallest nonnegative integer m such that

$$-\mathcal{F}(x^k + t^m d^k)^T d^k \geq \lambda(1 - \gamma_k) \mu_k \|d^k\|^2.$$

Projection step. Compute

$$x^{k+1} = x^k - \frac{\mathcal{F}(y^k)^T (x^k - y^k)}{\|\mathcal{F}(y^k)\|^2} \mathcal{F}(y^k).$$

Set $k := k + 1$, and repeat.

As pointed out in [21], the above inexact Newton step is motivated by the idea of proximal point algorithm [2, 6, 19]. Algorithm SS has an advantage over other Newton methods that the whole iteration sequence is globally convergent to a solution of the system of equations, provided a solution exists, under no assumption on \mathcal{F} other than continuity and monotonicity. Setting $\mathcal{F}(x) = \pi_\alpha(x)$ or $\Phi_\alpha(x)$, from Theorems 2.1 and 2.2 in this paper and Theorem 2.1 of [21], we have the following result.

THEOREM 3.1. *Let f be a co-coercive map with constant $\beta > 0$. Substitute $\mathcal{F}(x)$ in Algorithm SS by $\pi_\alpha(x)$ (respectively, $\Phi_\alpha(x)$) where $0 < \alpha \leq 4\beta$ (respectively, $\alpha > 1/4\beta$). If μ_k is chosen such that $C_2 \geq \mu_k \geq C_1 \|\mathcal{F}(x^k)\|$, where C_1 and C_2 are two constants, then Algorithm SS converges to a solution of variational inequality provided that a solution exists.*

While Algorithm SS can be used to solve the monotone equations $\pi_\alpha(x) = 0$ and $\Phi_\alpha(x) = 0$, each line-search step needs to compute the values of $\pi_\alpha(x^k + \beta^m d^k)$ and $\Phi_\alpha(x^k + \beta^m d^k)$ that represents a major cost of the algorithm in calculating projection operations. Hence, in general cases, Algorithm SS has high computational

cost per iteration when applied to solve $\Phi_\alpha(x) = 0$ or $\pi_\alpha(x) = 0$. To reduce this major computational burden, we propose the following algorithm which needs no projection operations other than the evaluation of the function f in line-search steps.

ALGORITHM 3.1. Choose $x^0 \in R^n, t \in (0, 1)$ and $\gamma \in [0, 1)$. Set $k := 0$.

Inexact Newton Step: Choose a positive semidefinite matrix G_k . Choose $\mu_k > 0$. Compute $d^k \in R^n$ such that

$$(3.1) \quad 0 = \pi_\alpha(x^k) + (G_k + \mu_k I)d^k + e^k,$$

where $\|e^k\| \leq \gamma\mu_k\|d^k\|$. Stop if $d^k = 0$; Otherwise,

Line-search step. Find $y^k = x^k + s_k d^k$, where $s_k = t^{m_k}$ with m_k being the smallest nonnegative integer m such that

$$(3.2) \quad \|f(x^k + t^m d^k) - f(x^k)\| < \frac{(1-\gamma)\mu_k - 4t^m}{2\alpha} \|d^k\|.$$

Projection step. Compute

$$x^{k+1} = x^k - \frac{\pi_\alpha(y^k)^T(x^k - y^k)}{\|\pi_\alpha(y^k)\|^2} \pi_\alpha(y^k).$$

Set $k := k + 1$. Return.

The above algorithm has the following property.

LEMMA 3.1. *Let $\pi_\alpha(x)$ be given as (1.1). At k th iteration, if m_k is the smallest nonnegative integer such that (3.2) holds, then $y^k = x^k + t^{m_k} d^k$ satisfies the following estimation:*

$$-\pi_\alpha(y^k)^T d^k \geq \frac{1}{2}(1-\gamma)\mu_k \|d^k\|^2.$$

Proof. By the definition of $\pi_\alpha(x)$, the nonexpansiveness of projection operator and (3.2), we have

$$\begin{aligned} & \|\pi_\alpha(x^k + t^{m_k} d^k) - \pi_\alpha(x^k)\| \\ &= \|x^k + t^{m_k} d^k - \Pi_K(x^k + t^{m_k} d^k - \alpha f(x^k + t^{m_k} d^k)) \\ & \quad - (x^k - \Pi_K(x^k - \alpha f(x^k)))\| \\ &\leq t^{m_k} \|d^k\| + \|\Pi_K(x^k + t^{m_k} d^k - \alpha f(x^k + t^{m_k} d^k)) \\ & \quad - \Pi_K(x^k - \alpha f(x^k))\| \\ &\leq t^{m_k} \|d^k\| + \|x^k + t^{m_k} d^k - \alpha f(x^k + t^{m_k} d^k) \\ & \quad - (x^k - \alpha f(x^k))\| \\ &\leq 2t^{m_k} \|d^k\| + \alpha \|f(x^k + t^{m_k} d^k) - f(x^k)\| \\ (3.3) \quad &\leq \frac{1}{2}(1-\gamma)\mu_k \|d^k\|. \end{aligned}$$

Also,

$$\begin{aligned} & -\pi_\alpha(x^k + t^{m_k} d^k)^T d^k \\ &= -[\pi_\alpha(x^k + t^{m_k} d^k) - \pi_\alpha(x^k)]^T d^k - \pi_\alpha(x^k)^T d^k \\ (3.4) \quad &\geq -\|\pi_\alpha(x^k + t^{m_k} d^k) - \pi_\alpha(x^k)\| \|d^k\| - \pi_\alpha(x^k)^T d^k. \end{aligned}$$

By (3.1) and positive semi-definiteness of G_k , we have

$$\begin{aligned}
 -\pi_\alpha(x^k)^T d^k &= (d^k)^T (G_k + \mu_k I) d^k + (e^k)^T d^k \\
 &\geq \mu_k \|d^k\|^2 - \gamma \mu_k \|d^k\|^2 \\
 (3.5) \qquad &= (1 - \gamma) \mu_k \|d^k\|^2.
 \end{aligned}$$

Combining (3.3), (3.4) and (3.5) yields

$$-\pi_\alpha(x^k + t^{m_k} d^k)^T d^k \geq \frac{1}{2} (1 - \gamma) \mu_k \|d^k\|^2.$$

The proof is complete. \square

Using Lemma 3.1 and following the line of the proof of Theorem 2.1 in [21], it is not difficult to prove the following convergence result.

THEOREM 3.2. *Let $f : R^n \rightarrow R^n$ be a continuous function such that there exists a constant $\alpha > 0$ such that $\pi_\alpha(x)$ defined by (1.1) is monotone. Choose G_k and μ_k such that $\|G_k\| \leq C'$ and $\mu_k = C \|\pi_\alpha(x^k)\|^p$, where C', C and p are three fixed positive numbers and $p \in (0, 1]$. Then the sequence $\{x^k\}$ generated by Algorithm 3.1 converges to a solution of the variational inequality provided that a solution exists.*

Algorithm 3.1 can solve the variational inequality whose fixed point mapping $\pi_\alpha(x)$ is monotone for some $\alpha > 0$. Since the co-coercivity of f implies the monotonicity of the functions $\pi_\alpha(x)$ and $\Phi_\alpha(x)$ for suitable choices of the value of α , Algorithm 3.1 can locate a solution of any solvable co-coercive variational inequality problem. This algorithm has an advantage over Algorithm SS in that it does not carry out any projection operation in the line-search step, and hence the computational cost is significantly reduced.

4. Conclusions . In this paper, we show some sufficient conditions for the monotonicity (strong monotonicity) of the fixed point and normal maps associated with the variational inequality problem. The results proved in the paper encompass some known results as particular cases. Based on these results, an iterative algorithm for a class of variational inequalities is proposed. This algorithm can be viewed as a modified Solodov and Svaiter's method but has lower computational cost than the latter.

Acknowledgements. The authors would like to thank two anonymous referees for their incisive comments and helpful suggestions which help us to improve many aspects of the paper. They also thank Professor O. L. Mangasarian for encouragement and one referee for pointing out refs. [7, 20].

REFERENCES

- [1] R. E. BRUCK, JR., *An iterative solution of a variational inequality for certain monotone operators in Hilbert space*, Bull. Amer. Math. Soc., 81 (1975), pp. 890-892.
- [2] J. ECKSTEIN AND D. P. BERTSEKAS, *On the Douglas-Rachford splitting method and the proximal point algorithm for maximal monotone operators*, Math. Programming, 55 (1992), pp. 293-318.
- [3] F. FACCHINEI, *Structural and stability properties of P_0 nonlinear complementarity problems*, Math. Oper. Res., 23 (1998), pp. 735-749.
- [4] F. FACCHINEI AND C. KANZOW, *Beyond monotonicity in regularization methods for nonlinear complementarity problems*, SIAM J. Control Optim., 37 (1999), pp. 1150-1161.
- [5] F. FACCHINEI AND J. S. PANG, *Total Stability of Variational Inequalities*, Dipartimento di Informatica e Sistemistica, Università di Roma, Via Buonarroti, Roma, Italy, 1998.

- [6] M. C. FERRIS, *Finite termination of the proximal point algorithm*, Math. Programming, 50 (1991), pp. 359-366.
- [7] D. GABAY, *Applications of the method of multipliers to variational inequalities*, M. Fortin, R. Glowinski, eds., Augmented Lagrangian Methods: Application to the Numerical Solution of Boundary-Value Problems, North-Holland, Amsterdam, 1983, pp. 229-332.
- [8] M. S. GOWDA AND M. A. TAWHID, *Existence and limiting behavior of trajectories associated with P_0 equations*, Comput. Optim. Appl., 12 (1999), pp. 229-251.
- [9] P. T. HARKER AND J. S. PANG, *Finite-dimensional variational inequality and nonlinear complementarity problems: A survey of theory, algorithms and applications*, Math. Programming, 48 B (1990), pp. 161-220.
- [10] G. ISAC, *Tikhonov's regularization and the complementarity problem in Hilbert spaces*, J. Math. Anal. Appl., 174 (1993), pp. 53-66.
- [11] G. ISAC, *A generalization of Karamardian's condition in complementarity theory*, Nonlinear Anal. Forum, 4 (1999), pp. 49-63.
- [12] Z. Q. LUO AND P. TSENG, *A decomposition property for a class of square matrices*, Appl. Math. Letters, 4 (1991), pp. 67-69.
- [13] T. L. MAGNANTI AND G. PERAKIS, *A unifying geometric solution framework and complexity analysis for variational inequalities*, Math. Programming, 71 (1996), pp. 327-352.
- [14] T. L. MAGNANTI AND G. PERAKIS, *The orthogonality theorem and the strong- f -monotonicity condition for variational inequality algorithms*, SIAM J. Optim., 7 (1997), pp. 248-273.
- [15] P. MARCOTTE AND J. H. WU, *On the convergence of projection methods: application to the decomposition of affine variational inequalities*, J. Optim. Theory Appl., 85 (1995), pp. 347-362.
- [16] H. D. QI, *Tikhonov regularization methods for variational inequality problems*, J. Optim. Theory Appl., 102 (1999), pp. 193-201.
- [17] G. RAVINDRAN AND M. S. GOWDA, *Regularization of P_0 -functions in box variational inequality problems*, SIAM J. Optim. (to appear).
- [18] S. M. ROBINSON, *Normal maps induced by linear transformations*, Math. Oper. Res., 16 (1992), pp. 292-309.
- [19] R. T. ROCKAFELLAR, *Monotone operators and proximal point algorithm*, SIAM J. Control Optim., 14 (1976), pp. 877-898.
- [20] M. SIBONY, *Méthodes itératives pour les équations et inéquations aus dérivées partielles non-linéaires de type monotone*, Calcolo, 7 (1970), pp. 65-183.
- [21] M. V. SOLODOV AND B. F. SVAITER, *A globally convergence inexact Newton method for systems of monotone equations*, in *Reformulation: Nonsmooth, Piecewise Smooth, Semismooth and Smoothing Methods*, (eds. by M. Fukushima and L. Qi), Kluwer Academic Publishers, pp. 355-369, 1998.
- [22] M. V. SOLODOV AND B. F. SVAITER, *A truly globally convergent Newton-type method for the monotone nonlinear complementarity problem*, SIAM J. Optim., 10 (2000), pp. 605-625.
- [23] M. V. SOLODOV AND B. F. SVAITER, *A hybrid projection-proximal point algorithm*, J. Convex Anal., 6 (1999), pp. 59-70.
- [24] M. V. SOLODOV AND B. F. SVAITER, *A new projection method for variational inequality problems*, SIAM J. Control Optim., 37 (1999), pp. 765-776.
- [25] P. TSENG, *Further applications of a matrix splitting algorithm to decomposition in variational inequalities and convex programming*, Math. Programming, 48 (1990), pp. 249-264.
- [26] V. VENKATESWARAN, *An algorithm for the linear complementarity problem with a P_0 -matrix*, SIAM J. Matrix Anal. Appl., 14 (1993), pp. 967-977.
- [27] Y. B. ZHAO AND D. LI, *Strict feasibility conditions in nonlinear complementarity problems*, J. Optim. Theory Appl., 107 (2000), pp. 641-664.
- [28] Y. B. ZHAO AND D. LI, *On a new homotopy continuation trajectory for nonlinear complementarity problems*, Math. Oper. Res., 26 (2001), pp. 119-146.
- [29] D. ZHU AND P. MARCOTTE, *New classes of generalized monotonicity*, J. Optim. Theory Appl., 87 (1995), pp. 457-471.
- [30] D. ZHU AND P. MARCOTTE, *Co-coercivity and its role in the convergence of iterative schemes for solving variational inequalities*, SIAM J. Optim., 6 (1996), pp. 714-726.