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Existence of a solution to nonlinear variational inequality under generalized positive homogeneity

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Abstract

We establish several new existence theorems for nonlinear variational inequalities with generalized positively homogeneous functions. The results presented here are general enough to include two Moré existence theorems of complementarity problems as special cases. We also establish an existence result for nonlinear complementarity problems with an exceptional regularity map. The concept of an exceptional family for a variational inequality plays a key role in our analysis. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let f be a map from R^n into itself, and let K be an arbitrary closed convex set in R^n. The finite-dimensional variational inequality problem in R^n, denoted by VI(K, f), is to determine a vector $x^* \in R^n$ such that

$$x^* \in K, \ \ f(x^*)^{\mathrm{T}}(x - x^*) \geqslant 0 \quad \text{for all } x \in K.$$

This problem is important in various equilibrium settings and in applied mathematics, see for example [2,3]. When K is a closed convex cone, the variational inequality reduces to the well known complementarity problem (CP(f) for short), i.e.,

$$x^* \in K, \ \ f(x^*) \in K^*, \quad (x^*)^{\mathrm{T}} f(x^*) = 0,$$

where K^* is the dual cone of K, i.e.,

$$K^* = \{y \in R^n : y^{\mathrm{T}} x \geqslant 0 \text{ for all } x \in K\}.$$

Particularly, if $K = R^n_+ = \{x \in R^n : x \geqslant 0\}$, then $K = K^*$, the above CP(f) can be written as the following nonlinear complementarity problem (NCP(f) for short):

$$x^* \geqslant 0, \quad f(x^*) \geqslant 0, \quad (x^*)^{\mathrm{T}} f(x^*) = 0.$$

Since the existence of a solution to VI(K, f) is not always assured, the study of existence conditions is very important in the theory, algorithms and applications of VI(K, f). Our focus here is to develop new existence results for VI(K, f) under a generalized positively homogeneous assumption. We first review some previous results. Recall that a map $v : R^n \to R^n$ is said to be positively homogeneous of degree $\alpha > 0$ on a set $Q \subset R^n$ if $v(\lambda x) = \lambda^\alpha v(x)$ for all $\lambda > 0$ and $x \in Q$. The

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function f is said to be (strictly) copositive on the set $Q \subset R^n$ if $x^T(f(x) - f(0))(>) \geqslant 0$ for all $x \in Q$. Under strict copositivity of f and an assumption which is weaker than positive homogeneity of degree α of $G(x) = f(x) - f(0)$, Moré proved (Corollary 3.3 and its comments in [8], see also Theorem 3.7 in [3]) that CP(f) has a nonempty, compact solution set. Restricted to the rectangular cone, that is, the cartesian product of n intervals I_i, $i = 1, \dots, n$ where each I_i is either $[0, +\infty)$ or $(-\infty, 0]$, Moré further showed (Corollary 3.7 in [8]) an existence theorem under positive homogeneity of degree of α of $G(x)$, and a condition which is weaker than the strict copositivity of f. If the map $G(x)$ is positively homogeneous of degree α and $G(x)$ satisfies the so-called d-regularity for some $d \in R^n$ and $d > 0$, Karamardian proved (Theorem 3.1 in [6], see also Theorem 3.8 in [3]) that NCP(f) has a nonempty solution set.

The main purpose of the paper is to extend the first aforementioned Moré result to the general nonlinear variational inequalities and the second Moré result to the so-called extended rectangular sets which are broad enough to encompass the rectangular sets, and hence the rectangular cones. The other goal of this paper is to use the concept of an exceptional regularity of a map defined by Zhao and Isac [13], which is quite different from the notion of d-regularity due to Karamardian [6], to develop a new existence result for NCP(f). It is worth noting that the concept of an exceptional family of a variational inequality introduced by Zhao et al. [11,12,14] has played a key role in our proofs throughout the paper.

2. Existence for VI(K, f)

It is well known that VI(K, f) has a solution if K is bounded. Therefore, we assume that the set K is unbounded throughout the paper. The following definition plays a very important role in our analysis.

Definition 2.1. Let f be a mapping from R^n into itself, and let $\hat{x}$ be an arbitrary point in R^n. A sequence $\{x^r\}_{r>0}$ is said to be an *exceptional family with respect to* $\hat{x}$ for VI(K, f) if it satisfies the following two conditions:

(a) $\|x^r\| \to \infty$,

(b) for each x^r there exists a positive scalar α_r such that

$$\pi_r := (1 + \alpha_r)x^r - \alpha_r \hat{x} \in K \tag{1}$$

and

$$-f(x^r) - \alpha_r(x^r - \hat{x}) \in \mathcal{N}_K(\pi_r), \tag{2}$$

where $\mathcal{N}_K(\cdot)$ denotes the normal cone (see [1]) of the convex set K at the point π_r.

Remark 2.1 (see Zhao and Han [11, Definition 2.1]). If $K = \{x \in R^n : H(x) \leqslant 0\}$, where $H(x) = (H_1(x), \dots, H_m(x))^T$ and each $H_i(x)$ is a differentiable convex function from R^n into R, then relation (2) can be stated as follows: there exists some vector $\lambda_r \in R^m_+$ such that

$$-f(x^r) - \alpha_r(x^r - \hat{x}) = \tfrac{1}{2} \nabla H(\pi_r)^T \lambda^r, \tag{3}$$

$$\lambda_i^r H_i(\pi_r) = 0 \quad \text{for } i = 1, \dots, m, \tag{4}$$

where $\nabla H(\cdot)$ is the Jacobian matrix of H.

The following alternative result is very important for us to establish the main results of this paper. The proof method of it has been used in several papers. See [4,11,14].

Theorem 2.1. *Let K be a closed convex set in R^n. Let f be a continuous function from R^n into itself. Given any $\hat{x} \in R^n$, there exists either a solution to* VI(K, f) *or an exceptional family with respect to $\hat{x}$ for* VI(K, f).

Proof. We assume that VI(K, f) has no solution. We show that there exists an exceptional family with respect to $\hat{x}$ for VI(K, f). Denote $\phi : R^n \to R^n$ by

$$\phi(x) = x - \Pi_K(x - f(x)),$$

where $\Pi_K(\cdot)$ is the projection operator onto the set K, that is,

$$\Pi_K(x) = \arg\min_{z \in K} \|z - x\|.$$

It is known that x^* is a solution to VI(K, f) if and only if $\phi(x^*) = 0$. By the continuity of f and the property of projection operator, the map $\phi(x)$ must be a continuous function from R^n into itself. We now consider the homotopy between the mapping $g(x) := x - \hat{x}$ and $\phi(x)$, that is,

$$H(x, t) = t(x - \hat{x}) + (1 - t)\phi(x), \quad t \in [0, 1].$$

We conclude that the set

$$\mathscr{F} = \{x \in R^n : H(x,t) = 0 \text{ for some } t \in [0,1]\}$$

is unbounded. Assume by the way of contradiction that $\mathscr{F}$ is bounded. Then there exists a bounded open set $D \supset \mathscr{F}$ such that $x \notin \mathscr{F}$ for all $x \in \partial D$, the boundary of D. Thus, by the injectivity of the map $g(x) = x - \hat{x}$ and the well known homotopy invariance theorem of degree [7, Theorems 3.3.3 and 2.1.2], we have

$$1 = \deg(g(x), D, 0) = \deg(\phi(x), D, 0).$$

Therefore, $\phi(x) = 0$ has a solution in D [7, Theorem 2.1.1]. This is in contradiction with the assumption at the beginning of the proof. Thus the set $\mathscr{F}$ is an unbounded set. That is, there exists a sequence contained in $\mathscr{F}$, denoted by $\{x^r\}$ with $\|x^r\| \to \infty$. Without loss of generality, we may assume that $\|x^r\| > \|\hat{x}\|$ for all r. By the definition of $\mathscr{F}$, for each x^r there is a scalar $t^r \in [0,1]$ such that

$$\begin{aligned} 0 &= H(x^r, t^r) \\ &= t^r(x^r - \hat{x}) + (1 - t^r)(x^r - \Pi_K(x^r - f(x^r))). \end{aligned} \quad (5)$$

Noting that $\mathrm{VI}(K,f)$ has no solution, and hence $\phi(x^r) \neq 0$, and noting that $\|x^r\| > \|\hat{x}\|$, we deduce from (5) that $t^r \neq 0$ and 1. Thus in the rest of the proof, it is sufficient to consider the case: $t^r \in (0,1)$. We show in this case that $\mathrm{VI}(K,f)$ has an exceptional family with respect to $\hat{x}$. Indeed, from (5) we have

$$\frac{t^r}{1-t^r}(x^r - \hat{x}) + x^r = \Pi_K(x^r - f(x^r)).$$

Let $\alpha_r = t^r/(1-t^r)$ and

$$\pi_r := \alpha_r(x^r - \hat{x}) + x^r.$$

Then

$$\pi_r = \Pi_K(x^r - f(x^r)),$$

which implies that $\pi_r \in K$, and that π_r is the unique solution to the following convex program

$$\min_{y \in K} Q(y) := \tfrac{1}{2}\|y - (x^r - f(x^r))\|^2.$$

Clearly, $Q(y)$ is differentiable and therefore must be locally Lipschitz continuous, and thus by the corollary of Proposition 2.4.3 of [1], we have

$$\nabla Q(\pi_r) = \pi_r - (x^r - f(x^r)) \in -\mathscr{N}_K(\pi_r).$$

Therefore,

$$-f(x^r) - \alpha_r(x^r - \hat{x}) \in \mathscr{N}_K(\pi_r)),$$

and hence $\{x^r\}$ is an exceptional family with respect to $\hat{x}$ for $\mathrm{VI}(K,f)$. □

The above alternative theorem provides a sufficient condition for the existence of a solution to $\mathrm{VI}(K,f)$, that is, "there exists no exceptional family with respect to $\hat{x}$ for $\mathrm{VI}(K,f)$". Definition 2.1 is general enough to include as special cases the concepts of exceptional family of elements for variational inequality introduced in [11,12,14], and for continuous functions introduced by Isac et al. [4,5], and the notion of exceptional sequence for continuous functions defined by Smith [10]. Indeed, setting $\hat{x} = 0$, it is easy to see that Definition 2.1 reduces to the following concept for $\mathrm{NCP}(f)$.

Definition 2.2 (Isac et al. [4]). A sequence $\{x^r\}$ is said to be an exceptional family of elements for a continuous function f if $\|x^r\| \to \infty$ and for each x^r there exists $\alpha_r > 0$ such that

$$f_i(x^r) = -\alpha_r x_i^r \quad \text{if } x_i^r > 0, \quad (6)$$

$$f_i(x^r) \geqslant 0 \quad \text{if } x_i^r = 0. \quad (7)$$

Moreover, if $\|x^r\| = r$ and $r = 1, 2, \ldots$, then the above concept is just the one defined by Smith [10].

To show the main results, we first make some preparations. We say that the map $v(x)$ is a *generalized positively homogeneous mapping* on a set $Q \subset R^n$ if there exists a map $c : R_+ \to R_1$ such that $c(\lambda) \to \infty$ as $\lambda \to \infty$ and $v(\lambda x) = c(\lambda)v(x)$ for all $x \in Q$ and sufficiently large $\lambda > 0$. Clearly, if $v(x)$ is positively homogeneous of degree of $\alpha > 0$ on the set Q, then $v(x)$ must be a generalized positively homogeneous map with $c(\lambda) = \lambda^\alpha$. In particular, positively homogeneous maps on Q, i.e., $v(\lambda x) = \lambda v(x)$ for all $\lambda > 0$ and $x \in Q$, are generalized positively homogeneous.

For the continuous function f, if $G(x) = f(x) - f(0)$ is a generalized positively homogeneous map on $\mathrm{Cone}(K)$, the cone generated by K, then there exists two constants $\mu^*_{\inf}$ and $\mu^*_{\sup}$ such that

$$0 \leqslant \mu^*_{\inf} = \liminf_{x \in \mathrm{Cone}(K), \|x\| \to \infty} \frac{|x^{\mathrm{T}} f(x)|}{\|x\| c(\|x\|)}, \quad (8)$$

$$\limsup_{x \in \mathrm{Cone}(K), \|x\| \to \infty} \frac{|x^{\mathrm{T}} f(x)|}{\|x\| c(\|x\|)} = \mu^*_{\sup} < +\infty. \quad (9)$$

Indeed,

$$\frac{x^{\mathrm{T}}f(x)}{\|x\|c(\|x\|)}=\frac{x^{\mathrm{T}}(f(x)-f(0))+x^{\mathrm{T}}f(0)}{\|x\|c(\|x\|)}$$

$$=(x/\|x\|)^{\mathrm{T}}(f(x/\|x\|)-f(0))$$

$$+\frac{(x/\|x\|)^{\mathrm{T}}f(0)}{c(\|x\|)}.$$

Therefore

$$\limsup_{x\in \mathrm{Cone}(K),\|x\|\to\infty} |x^{\mathrm{T}}f(x)|/(\|x\|c(\|x\|))$$

$$\leqslant \max_{\|v\|=1}|v^{\mathrm{T}}(f(v)-f(0))|<\infty.$$

In what follows, we denote by 0^+K the recession cone of the convex set K [9], and we denote $B=\{x\in R^n : \|x\|=1\}$. We are now ready to show the first existence result.

Theorem 2.2. *Let $\hat{x}$ be an arbitrary point in the closed convex set K. Let f be a continuous function and $G(x)=f(x)-f(0)$ be a generalized positively homogeneous map on* Cone(K). *Let $\mu^*_{\inf}$ and $\mu^*_{\sup}$ be given by* (8) *and* (9), *respectively. If the equality $G(x)^{\mathrm{T}}x=-\hat{\mu}$ has no solution $(x,\hat{\mu})\in\bar{S}\times[\mu^*_{\inf},\mu^*_{\sup}]$, where $\bar{S}=B\cap 0^+K$, then there exists no exceptional family with respect to $\hat{x}$ for* VI(K,f), *and hence there exists a solution to* VI(K,f). *Moreover, if f is strictly copositive on $\bar{S}$, then the solution set is compact.*

Proof. Assume by way of contradiction that there exists an exceptional family with respect to $\hat{x}$ for VI(K,f), denoted by $\{x^r\}$. Then by definition there exists a positive scalar $\{\alpha_r>0\}$ such that (1) and (2) hold. Since $G(x)$ is a generalized positively homogeneous map on Cone(K), for sufficiently large r we deduce that

$$f(x^r)=f(0)+c(\|x^r\|)(f(x^r/\|x^r\|)-f(0)).$$

Therefore

$$\frac{(x^r-\hat{x})^{\mathrm{T}}f(x^r)}{\|x^r\|c(\|x^r\|)}$$

$$=\frac{(x^r-\hat{x})^{\mathrm{T}}(f(x^r/\|x^r\|)-f(0))}{\|x^r\|}$$

$$+\frac{(x^r-\hat{x})^{\mathrm{T}}f(0)}{\|x^r\|c(\|x^r\|)}$$

$$=\left(\frac{(x^r-\hat{x})^{\mathrm{T}}(f(x^r/\|x^r\|)-f(0))}{\|x^r-\hat{x}\|}\right.$$

$$\left.+\frac{(x^r-\hat{x})^{\mathrm{T}}f(0)}{\|x^r-\hat{x}\|c(\|x^r\|)}\right)\frac{\|x^r-\hat{x}\|}{\|x^r\|}. \tag{10}$$

Without loss of generality, we assume that $(x^r-\hat{x})/\|x^r-\hat{x}\|\to x^*$. Then

$$\frac{x^r}{\|x^r\|}=\frac{x^r-\hat{x}}{\|x^r-\hat{x}\|}\frac{\|x^r-\hat{x}\|}{\|x^r\|}+\frac{\hat{x}}{\|x^r\|}\to x^*.$$

Since $x^r-\hat{x}=(\pi_r-\hat{x})/(1+\alpha_r)$ and $\|x^r\|\to\infty$, we have that $\|\pi_r\|\to+\infty$, and hence $(\pi_r-\hat{x})/\|\pi_r-\hat{x}\|\to x^*$. Notice that $\hat{x},\pi_r\in K$, it follows that $x^*\in 0^+K$. Taking the limit, it follows from (10) that

$$\lim_{r\to\infty}\frac{(x^r-\hat{x})^{\mathrm{T}}f(x^r)}{\|x^r\|c(\|x^r\|)}=(x^*)^{\mathrm{T}}G(x^*). \tag{11}$$

On the other hand, from (2), there must be a point $u_r\in\mathcal{N}_K(\pi_r)$ such that

$$f(x^r)=-\alpha_r(x^r-\hat{x})-u^r.$$

Thus

$$(x^r-\hat{x})^{\mathrm{T}}f(x^r)=-\alpha_r\|x^r-\hat{x}\|^2-(u^r)^{\mathrm{T}}(x^r-\hat{x}).$$

Since $\hat{x},\ \pi_r\in K$ and by the property of $\mathcal{N}_K(\cdot)$, we have

$$(1+\alpha_r)(\hat{x}-x^r)^{\mathrm{T}}z=z^{\mathrm{T}}(\hat{x}-\pi_r)\leqslant 0$$

for all $z\in\mathcal{N}_K(\pi_r)$. Thus for a sufficiently large r

$$(x^r-\hat{x})^{\mathrm{T}}f(x^r)\leqslant -\alpha\|x^r-\hat{x}\|^2<0.$$

From (11) and the above relation, we deduce that there exists a nonnegative scalar β such that

$$(x^*)^{\mathrm{T}}G(x^*)=-\beta.$$

We now show that $\beta\in[\mu^*_{\inf},\mu^*_{\sup}]$. Indeed,

$$\frac{(x^r-\hat{x})^{\mathrm{T}}f(x^r)}{\|x^r\|c(\|x^r\|)}$$

$$=\frac{(x^r)^{\mathrm{T}}f(x^r)}{\|x^r\|c(\|x^r\|)}-\frac{(\hat{x})^{\mathrm{T}}(f(x^r)-f(0))+f(0)^{\mathrm{T}}\hat{x}}{\|x^r\|c(\|x^r\|)}$$

$$=\frac{(x^r)^{\mathrm{T}}f(x^r)}{\|x^r\|c(\|x^r\|)}-\frac{(\hat{x})^{\mathrm{T}}(f(x^r/\|x^r\|)-f(0))}{\|x^r\|}$$

$$-\frac{f(0)^{\mathrm{T}}\hat{x}}{\|x^r\|c(\|x^r\|)}$$

which implies that $\mu^*_{\inf}\leqslant\beta\leqslant\mu^*_{\sup}$. Therefore, the equality $G(x)^{\mathrm{T}}x=-\hat{\mu}$ has a solution $(x,\hat{\mu})=(x^*,\beta)\in$

$\bar{S} \times [\mu^*_{\inf}, \mu^*_{\sup}]$. This is a contradiction. Thus $\mathrm{VI}(K, f)$ must have no exceptional family, and hence there exists a solution according to Theorem 2.1.

Moreover, if f is strictly copositive on $\bar{S}$, we now show that the solution set is compact. By the continuity, the solution set is closed, it suffices to show the boundedness. Assume the contrary, that is, there exists an unbounded solution sequence for $\mathrm{VI}(K, f)$, say $\{x^p\}_{p=1}^{\infty}$ satisfying $\|x^p\| \to \infty$. Then for each p, we have $f(x^p)^{\mathrm{T}}(x' - x^p) \geqslant 0$, where $x' \in K$ is a given vector. That is

$$-(f(x^p) - f(0))^{\mathrm{T}}(x^p - x') - f(0)^{\mathrm{T}}(x^p - x') \geqslant 0.$$

By the generalized positive homogeneity of $G(x)$, we have

$$-\frac{(f(x^p/\|x^p\|) - f(0))^{\mathrm{T}}(x^p - x')}{\|x^p - x'\|} - \frac{f(0)^{\mathrm{T}}(x^p - x')}{\|x^p - x'\| c(\|x^p\|)} \geqslant 0.$$

Assume that $(x^p - x')/\|x^p - x'\| \to \bar{x}$, then $\bar{x} \in \bar{S}$, and $x^p/\|x^p\| \to \bar{x}$, thus the above inequality implies that $-G(\bar{x})^{\mathrm{T}}\bar{x} \geqslant 0$, which is a contradiction. □

The following two corollaries are the immediate consequences of Theorem 2.2.

Corollary 2.1. *Let K be a closed convex set in R^n. Let f be a continuous function and $\hat{x}$ be a point in K. Let the set $\bar{S}$ be given as Theorem* 2.2. *If f is strictly copositive on $\bar{S}$, and $G(x)$ is a generalized positively homogeneous map on* Cone(K), *then the* $\mathrm{VI}(K, f)$ *has a nonempty, compact solution set.*

Corollary 2.2. *Let K, $\hat{x}$, $\bar{S}$ and f be given as in Corollary* 2.1. *If f is copositive on the set $\bar{S}$, and $G(x)$ is a generalized positively homogeneous map on* Cone(K), *and the constant $\mu^*_{\inf} > 0$, then there exists a solution to* $\mathrm{VI}(K, f)$.

If we impose some conditions on K, for instance, $0 \in K$, the above generalized positive homogeneity of $G(x)$ can be relaxed. Actually, we have the following result.

Theorem 2.3. *Let K be a closed convex set in R^n. Let f be a continuous function from R^n into itself and $\bar{S}$ be given as Theorem* 2.2. *If f is strictly copositive on $\bar{S}$ and there exists a mapping $c : R_+ \to R$ such that $c(\lambda) \to \infty$ as $\lambda \to \infty$ such that*

$$(f(\lambda x) - f(0))^{\mathrm{T}} x \geqslant c(\lambda)(f(x) - f(0))^{\mathrm{T}} x \tag{12}$$

for all $x \in$ Cone(K) and sufficiently large $\lambda > 0$ and if $0 \in K$, then there exists no exceptional family with respect to $\hat{x} = 0$, and hence there exists a solution to $\mathrm{VI}(K, f)$, *and the solution set is compact.*

Proof. Assume the contrary, that is, there exists an exceptional family with respect to $\hat{x} = 0$ for $\mathrm{VI}(K, f)$, denoted it by $\{x^r\}$, then there exists a corresponding positive sequence $\{\alpha_r\}$ such that

$$(1 + \alpha_r)x^r \in K$$

and

$$-f(x^r) - \alpha_r x^r \in \mathcal{N}_K((1 + \alpha_r)x^r). \tag{13}$$

It follows from (12) that

$$(x^r)^{\mathrm{T}}(f(x^r) - f(0)) \geqslant c(\|x^r\|)(f(x^r/\|x^r\|) - f(0))^{\mathrm{T}} x.$$

Assume without loss of generality that $x^r/\|x^r\| \to x^*$, from the above inequality we have

$$\begin{aligned}\liminf_{r\to\infty} \frac{(x^r)^{\mathrm{T}} f(x^r)}{\|x^r\| c(\|x^r\|)} &\geqslant \lim_{r\to\infty} \frac{(f(x^r/\|x^r\|) - f(0))^{\mathrm{T}} x^r}{\|x^r\|} + \lim_{r\to\infty} \frac{(x^r)^{\mathrm{T}} f(0)}{\|x^r\| c(\|x\|)} \\ &= (x^*)^{\mathrm{T}} G(x^*).\end{aligned} \tag{14}$$

Clearly, $x^* \in$ Cone(K) and $\|x^*\| = 1$. Since $0 \in K$ and $(1 + \alpha_r)x^r \in K$, (13) implies that

$$(0 - (1 + \alpha_r)x^r)^{\mathrm{T}}(-f(x^r) - \alpha_r x^r) \leqslant 0.$$

That is,

$$(x^r)^{\mathrm{T}} f(x^r) \leqslant -\alpha_r \|x^r\|^2 < 0. \tag{15}$$

From (14) and (15), there exists a number $\hat{\mu} \geqslant 0$ such that

$$-\hat{\mu} = \liminf_{r\to\infty} \frac{(x^r)^{\mathrm{T}} f(x^r)}{\|x^r\| c(\|x^r\|)} \geqslant (x^*)^{\mathrm{T}} G(x^*).$$

Since $0 \in K$ and $(1+\alpha)x^r \in K$, it is easy to see that $x^* \in 0^+K$. The above inequality implies that f is impossible to be strictly copositive on $\bar{S}$. This is a contradiction. The compactness of the solution set follows from the strict copositivity as we have showed in Theorem 2.2. □

Particularly, if K be a closed convex cone, we have the following result

Corollary 2.3 (Moré [8, Corollary 3.3], see also Harker and Pang [3, Theorem 3.7]). *Let f be a continuous function and K be a closed convex cone. If $f(x)$ is strictly copositive with respect to K, and there exists a map $c: R_+ \to R$ that satisfies $c(\lambda) \to \infty$ as $\lambda \to \infty$, and*

$$x^{\mathrm{T}}(f(\lambda x) - f(0)) \geqslant c(\lambda) x^{\mathrm{T}}(f(x) - f(0))$$

for all $\lambda > 0$ and $x \in K$, then the VI(K, f) *has a nonempty, compact solution set.*

An important special case of the convex set is that K is a rectangular set, that is, $K = \prod_{i=1}^{n} I_i$, where I_i is either $[a,b]$, $[a,\infty)$, $(-\infty,b]$, or $(-\infty,\infty)$. In particular, if each I_i is either $[0,\infty)$ or $(-\infty,0]$, then K is a rectangular cone. In what follows we will generalize the second Moré result to the rectangular sets. Clearly, each rectangular set can be formulated as $K = \{x \in R^n : g(x) \leqslant 0\}$, where $g(x) = (g_1(x_1), \ldots, g_n(x_n))^{\mathrm{T}}$, each $g_i(x_i)$ is either a quadratic convex function or a linear function. It is easy to check that the Jacobian $\nabla g(x)$ satisfies the property

$$\lambda_i [\nabla g(x)(x-y)]_i = (x-y)_i [\nabla g(x)^{\mathrm{T}} \lambda]_i,$$

for each $i = 1, \ldots, n$ and for all $x, y \in K$ and $\lambda \in R^n_+$. Motivated by this observation, we now define a class of sets which includes the rectangular sets as special cases.

Definition 2.3. Let

$$K = \{x \in R^n : H(x) \leqslant 0\}, \tag{16}$$

where $H(x) = (h_1(x), \ldots, h_m(x))^{\mathrm{T}}$, each $h_i(x)$ is a convex and differentiable function from R^n into R. We say that K is an *extended rectangular set*, if

$$\lambda_i [\nabla H(x)(x-y)]_i = (x-y)_i [\nabla H(x)^{\mathrm{T}} \lambda]_i,$$

holds for all $i = 1, \ldots, n$ and for all $x, y \in K$ and $\lambda \in R^m_+$.

Similar to (8) and (9), if $G(x)$ is generalized positively homogeneous map on Cone(K), then for each i, there exist two constants $\mu^{(i)}_{\inf}$ and $\mu^{(i)}_{\sup}$ such that

$$0 \leqslant \mu^{(i)}_{\inf} = \liminf_{x \in \mathrm{Cone}(K), \|x\| \to \infty} \frac{|x_i f_i(x)|}{\|x\| c(\|x\|)}, \tag{17}$$

$$\limsup_{x \in \mathrm{Cone}(K), \|x\| \to \infty} \frac{|x_i f_i(x)|}{\|x\| c(\|x\|)} = \mu^{(i)}_{\sup} < \infty. \tag{18}$$

Theorem 2.4. *Let f be a continuous function, and K be an extended rectangular set, and let $\hat{x} \in K$. Assume that $G(x) = f(x) - f(0)$ is a generalized positively homogeneous map on* Cone(K). *If there exists some i such that*

$$G_i(x) x_i = -\mu_i, \quad x_i \neq 0$$

has no solution in the set

$$x \in \bar{S} = B \cap 0^+K, \quad \mu_i \in [\mu^{(i)}_{\inf}, \mu^{(i)}_{\sup}],$$

then there exists no exceptional family with respect to $\hat{x}$ for VI(K, f), *and hence there exists a solution to* VI(K, f). *Moreover, if $\max_{1 \leqslant i \leqslant n} x_i G_i(x) > 0$ for all $x \in \bar{S}$, the solution set is compact.*

Proof. Let K be given as (16) and $\hat{x} \in K$. We show the result by contradiction, that is, assume that there exists an exceptional family with respect to $\hat{x}$ for VI(K, f), denoted by $\{x^r\}$. Then $\|x^r\| \to \infty$ and for each r there exists a positive scalar $\alpha_r > 0$ and a vector $\lambda \in R^m_+$, such that (1), (3) and (4) hold. By the generalized positive homogeneity of $G(x)$, we deduce that the equality

$$f_i(x^r) = f_i(0) + c(\|x^r\|)(f_i(x^r/\|x^r\|) - f_i(0))$$

holds for all sufficiently large r.

Assume that $(x^r - \hat{x})/\|x^r - \hat{x}\| \to x^*$, then by the same argument of (10) and (11), we deduce that $x^* \in 0^+K$ and

$$\lim_{r \to \infty} \frac{(x^r_i - \hat{x}_i) f_i(x^r)}{\|x^r\| c(\|x\|)} = x^*_i G_i(x^*). \tag{19}$$

Noting that K is given by (16), and that each $H_j(x)$ is a convex function, we have

$$H(\hat{x}) \geqslant H(\pi_r) + \nabla H(\pi_r)(\hat{x} - \pi_r).$$

Since $\hat{x}$ and $\pi_r \in K$ and $\lambda_i H_i(\pi_r) = 0$, we have that

$$\lambda_i [\nabla H(\pi_r)(\hat{x} - \pi_r)]_i \leqslant 0.$$

Since K is an extended rectangular set, from the above we have

$$(\hat{x}-\pi_r)_i[\nabla H(\pi_r)^{\mathrm{T}}\lambda]_i=\lambda_i[\nabla H(\pi_r)(\hat{x}-\pi_r)]_i\leqslant 0.$$

Thus from (3) and noting that $x^r-\hat{x}=(\pi_r-\hat{x})/(1+\alpha)$, we have

$$\begin{aligned}(x^r-\hat{x})_i f_i(x^r) &=(x^r-\hat{x})_i(-\alpha_r(x^r-\hat{x})_i-\tfrac{1}{2}[\nabla H(\pi_r)^{\mathrm{T}}\lambda]_i)\\ &=-\alpha_r(x^r-\hat{x})_i^2\\ &\quad+\frac{1}{2(1+\alpha_r)}(\hat{x}-\pi_r)_i[\nabla H(\pi_r)^{\mathrm{T}}\lambda]_i\\ &\leqslant -\alpha_r(x^r-\hat{x})_i^2. \end{aligned} \tag{20}$$

Therefore, for each $x_i^*\neq 0$, from $(x^r-\hat{x})_i/\|x^r-\hat{x}\|\to x_i^*$ we have that $(x^r-\hat{x})_i f_i(x^r)<0$ for all sufficiently large r. From (19) and (20), we deduce that there exists a nonnegative scalar β_i such that

$$-\beta_i=\lim_{r\to\infty}\frac{(x^r-\hat{x})_i f_i(x^r)}{\|x^r\|c(\|x^r\|)}=x_i^*G_i(x^*).$$

We now verify that $\beta_i\in[\mu_{\inf}^{(i)},\mu_{\sup}^{(i)}]$. Actually, by the analogous proof of Theorem 2.2, we can show that

$$\beta_i=\lim_{r\to\infty}\frac{|(x^r-\hat{x})_i f_i(x^r)|}{\|x^r\|c(\|x^r\|)}=\lim_{r\to\infty}\frac{|x_i^r f_i(x^r)|}{\|x^r\|c(\|x^r\|)}.$$

Thus $\beta\in[\mu_{\inf}^{(i)},\mu_{\sup}^{(i)}]$. Therefore for each $x_i^*\neq 0$, the equation $G_i(x^*)x_i^*=-\beta_i$ has a solution, where $x^*\in\bar{S}$ and $\beta_i\in[\mu_{\inf}^{(i)},\mu_{\sup}^{(i)}]$. This is impossible by the assumption of the theorem. Therefore VI(K,f) has a solution.

We now prove the solution set is bounded if $\max_{1\leqslant i\leqslant n}x_iG_i(x)>0$ for all $x\in\bar{S}$. Assume the contrary, that is, there exists a solution sequence denoted by $\{x^l\}_{l=1}^{\infty}$ satisfying $r_l:=\|x^l\|\to\infty$. Each x^l can be viewed as a solution to the following variational inequality:

$$(x-x^l)^{\mathrm{T}}f(x^l)\geqslant 0$$

for all $x\in K_l=K\cap\{x\in R^n:\|x\|\leqslant r_l\}$. Thus x^l is the solution to the projection equation $x^l=\prod_{K_l}(x^l-f(x^l))$ (see [3]), which implies that x^l is the unique solution to the following convex program:

$$\min_{y\in K_l}\|y-(x^l-f(x^l))\|^2.$$

Thus there exists a vector $\lambda^l\in R_+^m$ and a scalar $\mu^l\geqslant 0$ such that the following KKT condition is satisfied:

$$f(x^l)=-\mu^l x^l-\tfrac{1}{2}\nabla H(x^l)^{\mathrm{T}}\lambda^l,$$

$$\lambda_i^l H_i(x^l)=0\quad\text{for } i=1,\ldots,m.$$

Let $\hat{x}$ be an arbitrary point in K. Assume $(x^l-\hat{x})/\|x^l-\hat{x}\|\to x^*$, then $x^*\in\bar{S}$. By the same proof of (19) and (20), we can show that, for each $x_i^*\neq 0$,

$$\lim_{r\to\infty}\frac{(x^l-\hat{x})_i f_i(x^l)}{\|x^l\|c(\|x^l\|)}=x_i^*G_i(x^*) \tag{21}$$

and

$$(x^l-\hat{x})_i f_i(x^l)\leqslant -\mu^l x_i^l(x^l-\hat{x})_i.$$

Notice that for each $x_i^*\neq 0$, we have $[x_i^l(x^l-\hat{x})_i]/\|x^l-\hat{x}\|^2\to(x_i^*)^2>0$. Thus, for sufficiently large l, we have $(x^l-\hat{x})_i f_i(x^l)\leqslant 0$. It follows from (21) that $x_i^*G_i(x^*)\leqslant 0$ holds for all $x_i^*\neq 0$, this is in contradiction with the condition "$\max_{1\leqslant i\leqslant n}x_iG_i(x)>0$ for all $x\in\bar{S}$". □

Corollary 2.4. *Let f be a continuous function and K be a rectangular set, and let $\hat{x}\in K$. Assume $G(x)$ is a generalized positively homogeneous map on* Cone(K). *If*

$$\max_{1\leqslant i\leqslant n}x_iG_i(x)>0\quad \textit{for all } x\in\bar{S},$$

then VI(K,f) *has a nonempty, compact set.*

Since a rectangular cone is a special case of the rectangular sets. An immediate consequence of Corollary 2.4 is the following result.

Corollary 2.5 (Moré [18, Corollary 3.7])**.** *Let $f: R^n\to R^n$ be a continuous mapping, and let K be a rectangular cone, and assume that for each $x\neq 0$ in K*

$$\max_{1\leqslant i\leqslant n}x_iG_i(x)>0.$$

If the mapping $G: R^n\to R^n$ is positively homogeneous of degree $\alpha>0$ on K, then VI(K,f) *has a nonempty, compact set.*

3. Exceptional regularity and NCP(f)

In this section, we use the concept of exceptional regularity of a function to establish an existence result for NCP(f). The following definition makes the concept precise.

Definition 3.1 (Zhao and Isac [13]). The function $G(x)=f(x)-f(0)$ is exceptional regular if there exists no $(x,\pi)\in R_+^{n+1}$ where $x\in R^n$ and $\|x\|=1$ such that

$$G_i(x)/x_i=-\pi \quad \text{for } x_i>0,$$

$$G_i(x)\geqslant 0 \quad \text{for } x_i=0.$$

The following is the main result of this section, which is a generalized version of Theorem 4.1 of [13].

Theorem 3.1. *Let f be a continuous function. If $G(x)=f(x)-f(0)$ is a generalized positively homogeneous map on R_+^n, and $G(x)$ is exceptional regular, then there exists no exceptional family of elements for f, and therefore the* NCP(f) *has a solution.*

Proof. Assume the contrary, suppose that there exists an exceptional family of elements for the function f, denoted by $\{x^r\}$, then $\{x^r\}\subset R_+^n$, $\|x^r\|\to\infty$ and there exists a positive sequence $\{\mu_r>0\}$ such that (6) and (7) hold.

It follows from the generalized positive homogeneity of $G(x)$ that

$$f(x^r)=f(0)+c(\|x^r\|)(f(x^r/\|x^r\|)-f(0))$$

for all sufficiently large r. Assume that $x^r/\|x^r\|\to\hat{x}$, thus $\hat{x}\in R_+^n$ and $\|\hat{x}\|=1$. Therefore,

$$\begin{aligned}\lim_{r\to+\infty} & f(x^r)/c(\|x^r\|)\\ &=\lim_{r\to+\infty}(f(x^r/\|x^r\|)-f(0))+\frac{f(0)}{c(\|x^r\|)}\\ &=G(\hat{x}).\end{aligned} \tag{22}$$

If $i\in I_+(\hat{x})=\{i:\hat{x}_i>0\}$, then $x_i^r>0$ for all sufficiently large r, thus by (6) and (22), we have

$$\begin{aligned}-G_i(\hat{x})/\hat{x}_i&=\lim_{r\to+\infty}(-f_i(x^r)/c(\|x^r\|))(\|x^r\|/x_i^r)\\ &=\lim_{r\to+\infty}(\mu_r x_i^r/c(\|x^r\|))(\|x^r\|/x_i^r)\\ &=\lim_{r\to+\infty}\mu_r\|x^r\|/c(\|x^r\|)\\ &=\pi\geqslant 0.\end{aligned} \tag{23}$$

Therefore, for all $i\in I_+(\hat{x})$,

$$G_i(\hat{x})/\hat{x}_i=-\pi. \tag{24}$$

We now consider the case $i\in I_0(\hat{x})=\{i:\hat{x}_i=0\}$. In this case, for any $i\in I_0(\hat{x})$, we have that $x_i^r/\|x^r\|\to\hat{x}_i=0$. By (23) we have

$$-\frac{\mu_r x_i^r}{c(\|x^r\|)}=\frac{\mu_r\|x^r\|}{c(\|x^r\|)}\cdot\frac{x_i^r}{\|x^r\|}\to 0.$$

Thus for each $i\in I_0(\hat{x})$, we have

$$\begin{aligned}G_i(\hat{x})&=\lim_{r\to\infty} f_i(x^r)/c(\|x^r\|)\\ &=\begin{cases}\lim_{r\to\infty}\frac{f_i(x^r)}{c(\|x^r\|)} & \text{if } x_i^r=0\\ \lim_{r\to\infty}-\frac{\mu_r x_i^r}{c(\|x^r\|)} & \text{if } x_i^r>0\end{cases}\\ &\geqslant 0.\end{aligned}$$

Combining (24) and the above relation implies that G is not exceptional regular. This is a contradiction. □

Clearly, each of the following two conditions imply the exceptional regularity of $G(x)$

(a) $\max_{1\leqslant i\leqslant n} x_i(f_i(x)-f_i(0))>0$ for all $x^r\in R_+^n$ and $x\neq 0$.

(b) $\max_{1\leqslant i\leqslant n} x_i(f_i(x)-f_i(0))\geqslant 0$ for all $x^r\in R_+^n$ and $x\neq 0$, and there exists at least one component i such that $x_i>0$ and $f_i(x)\neq f_i(0)$.

We end this paper by presenting two open problems: Can the condition "$0\in K$" of Theorem 2.3 be removed? and What is the relationship between d-regularity [6] and exceptional regularity?

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References

[1] F.H. Clarke, Optimization and Nonsmooth Analysis, Wiley Interscience, New York, 1983.
[2] M.C. Ferris, J.S. Pang (Eds.), Complementarity and Variational Problems, State of the Art, SIAM Publication, Philadelphia, PA, 1997.
[3] P.T. Harker, J.S. Pang, Finite-dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms and applications, Math. Programming 48 (1990) 161–220.

[4] G. Iasc, V. Bulavski, V. Kalashnikov, Exceptional families, topological degree and complementarity problems, J. Global Optim. 10 (1997) 207–225.

[5] G. Isac, W.T. Obuchowska, Functions without exceptional family of elements and complementarity problems, J. Optim. Theory Appl. 99 (1998) 147–163.

[6] S. Karamardian, The complementarity problem, Math. Programming 2 (1972) 107–129.

[7] N.G. Lloyd, Degree Theory, Cambridge University Press, Cambridge, 1978.

[8] J.J. Moré, Coercivity conditions in nonlinear complementarity problems, SIAM Rev. 16 (1974) 1–16.

[9] R.T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, NJ, 1970.

[10] T.E. Smith, A solution condition for complementarity problem with an application to spatial price equilibrium, Appl. Math. Comput. 15 (1984) 61–69.

[11] Y.B. Zhao, Exceptional family and finite-dimensional variational inequality over polyhedral convex sets, Appl. Math. Comput. 87 (1997) 111–126.

[12] Y.B. Zhao, J. Han, Exceptional family of elements for a variational inequality and its applications, J. Global Optim., in press.

[13] Y.B. Zhao, G. Isac, Quasi-P_* and $P(\tau, \alpha, \beta)$-maps, exceptional family of elements and complementarity problems, Research Report 98/2, Institute of Applied Mathematics, Chinese Academy of Sciences, Beijing, January 1998.

[14] Y.B. Zhao, D. Sun, Alternative theorems for generalized projection equations and their applications to generalized complementarity problems, Institute of Computational Mathematics and Scientific/Engineering Computing, Chinese Academy of Sciences, Beijing, March 1998.