Properties of a Homotopy Solution Path for Complementarity Problems with Quasi-monotone Mappings*

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Abstract. We study several properties of a homotopy solution path associated with continuous nonlinear quasi-monotone complementarity problems. We use Poincare-Bohn's homotopy invariance theorem of degree to derive an alternative theorem. Based on this result, a sufficient condition is established to assure the existence and boundedness of this homotopy solution path. Our results provide a theoretical basis to develop a new computational method for quasi-monotone complementarity problems.

Keywords: Complementarity problems, homotopy solution paths, μ-exceptional family, quasi-monotone mappings.

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1. Introduction

Throughout the paper, all vectors are column vectors and superscript T denotes the transpose of a vector. For any $x \in R^n$, we denote by $\|x\|$ the Euclidean norm of x, by x_i the ith component of x and by $[x]_+$ the vector whose ith component is $\max\{0, x_i\}$, where $i = 1, \cdots, n$. We denote by R^n_+ (respectively, R^n_{++}) the space of n-dimensional real vectors with nonnegative components (respectively, positive components). When $x \in R^n_+$ (respectively, R^n_{++}), we also write $x \geq 0$ (respectively, $x > 0$) for simplicity.

Consider the nonlinear complementarity problem (NCP(f) for short),

$$x \geq 0, \quad f(x) \geq 0, \quad x^T f(x) = 0,$$

which can be written as

$$\mathcal{F}(x,y) = \begin{pmatrix} y - f(x) \\ Xy \end{pmatrix} = 0, \quad (x,y) \geq 0,$$

where $(x, y) \in R^n \times R^n$, $X = \mathrm{diag}(x)$, and f is a continuous function from R^n into itself.

For the above problem, Newton equation-based methods have been proved successful when the Jacobian of $\mathcal{F}(x, y)$ is nonsingular. However, the nonsingularity of Jacobian of $\mathcal{F}(x, y)$ is not always assured. For some nonmonotone functions, it is well known that homotopy methods do not necessarily require the nonsingularity of Jacobian (see, [13]). A common feature of the homotopy methods is to construct a homotopy solution path leading to the solution set of the problem, and then to devise some computational methods to track this path. Different homotopy solution paths result in different homotopy continuation algorithms for NCP(f). In this paper, we consider the homotopy mapping $H : R^{4n+1} \to R^{2n}$ defined by

$$H(x, y, \mu, x^0, y^0) = \begin{pmatrix} \mu f(x) - y + (1-\mu) f(0) \\ Xy \end{pmatrix} - (1-\mu) \begin{pmatrix} f(0) - y^0 \\ X^0 y^0 \end{pmatrix}, \qquad (1)$$

where $\mu \in (0, 1)$, and (x^0, y^0) is a fixed vector in $R^n_{++} \times R^n_{++}$. In [16], Zhao and Li pointed out that this mapping is a special case of the mapping studied by Kojima et al. [8]. The homotopy solution path $\{(x(\mu), y(\mu)) : \mu \in (0, 1)\}$ is defined by

$$\{(x(\mu), y(\mu)) \in R^n_{++} \times R^n_{++} : \quad H(x(\mu), y(\mu), \mu, x^0, y^0) = 0, \quad \mu \in (0, 1)\}. \qquad (2)$$

If $(x(\mu), y(\mu))$ has an accumulation point (x^*, y^*) as $\mu \to 1$, it follows from (2) that

$$H(x^*, y^*, 1, x^0, y^0) = 0,$$

which implies that (x^*, y^*) is a solution to the NCP(f).

If f is a continuously differentiable function satisfying certain semi-monotone property, Kojima et al. [8] showed that for almost all $(x^0, y^0) > 0$ (in Lebesgue measure) there exists a smooth homotopy solution path that is also bounded for some complementarity problems. In this paper, we further study the properties of the homotopy solution path. Given a continuous function f and a scalar $\mu \in (0, 1)$, the existence of a solution $(x(\mu), y(\mu))$ of the homotopy equation $H(x, y, \mu, x^0, y^0) = 0$ is not always assured, and even if a solution exists, the solution might be multiple. Thus, we define a (multivalued) map $\mathcal{T} : (0, 1) \to R^{2n}_{++}$ as follows:

$$\mathcal{T}(\mu) = \{(x, y) > 0 : H(x, y, \mu, x^0, y^0) = 0\}. \tag{3}$$

Since the mapping $\mathcal{T}(\mu)$ is not necessarily to be single-valued for some nonmontone complementarity problems, an interesting problem is how to construct a homotopy continuation method to solve such a nonmonotone complementarity problem. The following two properties are necessary for developing such a method:

(P_1) For each $\mu \in (0, 1), \mathcal{T}(\mu) \neq \emptyset$.
(P_2) For any $\hat{\mu} \in (0, 1)$, the set $\cup_{\mu \in [\hat{\mu}, 1)} \mathcal{T}(\mu)$ is bounded.

This two properties also imply the solvability of NCP(f). Indeed, if the above two conditions are satisfied, then for any $\mu_k \to 1$, by continuity all the accumulation points of $(x(\mu_k), y(\mu_k)) \in \mathcal{T}(\mu)$ are solutions to NCP(f). When $\mathcal{T}(\mu)$ is multivalued, we can refer to the set $\cup_{\mu \in (0,1)} \mathcal{T}(\mu)$ as a path or "band" connecting the starting point $(x^0, y^0) > 0$ to the solution set of the problem.

In this paper, we develop a sufficient condition to guarantee the properties (P_1) and (P_2) of the homotopy solution mapping $\mathcal{T}(\mu)$. We use a concept of μ-exceptional family and the homotopy invariance theorem of degree of a continuous function to establish a general alternative theorem. We use this result to develop a sufficient condition for ensuring (P_1) and (P_2) of $\mathcal{T}(\mu)$ for quasi-monotone complementarity problems. Our analysis is very different from previous methods such as the maximal monotone operators methods [9, 3], minimization methods [10], and homeomorphism technique [7, 11].

This paper is organized as follows: In section 2, we show an equivalent formulation of the homotopy formulation (1), and establish an alternative theorem concerning the existence of an element of $\mathcal{T}(\mu)$. In sections 3 , we establish a sufficient condition to guarantee the properties (P_1) and (P_2) of $\mathcal{T}(\mu)$. Conclusions are given in section 4.

2. Alternative theorem for nonemptyness of $\mathcal{T}(\mu)$

We first introduce a definition and a classical result.

Definition 2.1. [5, 6] A function $f : R^n \to R^n$ is said to be quasi-monotone if for any distinct points x, y in R^n, $f(y)^T(x-y) > 0$ implies that $f(x)^T(x-y) \geq 0$.

Let O be an open bounded set of R^n, we denote by $\overline{O}$ and $\partial(O)$ the closure and boundary of O, respectively. Let f be a continuous function from $\overline{O}$ into R^n. For $y \in R^n$ such that $y \notin f(\partial(O))$, the notation $\deg(f, O, y)$ is the topological degree associated with f, O and y (see, [12]).

Lemma 2.1. [12] *Let $O \subset R^n$ be an open bounded set and F, G be two continuous functions from $\overline{O}$ into R^n. The homotopy $H(x,t)$ is defined as follows: $H(x,t) = tG(x) + (1-t)F(x)$, $0 \leq t \leq 1$. Let y be an arbitrary point in R^n. If y satisfies the condition $y \notin \{H(x,t) : x \in \partial O \text{ and } t \in [0,1]\}$, then deg$(G, O, y) =$ deg(F, O, y).*

Lemma 2.2. [12] *Let O and F be given as in Lemma 2.1. If $y \notin F(\partial(O))$ and deg$(F, O, y) \neq 0$, then the equation $F(x) = y$ has a solution in O.*

In this section, we use Lemma 2.1 to establish an alternative result for the existence of an element of $\mathcal{T}(\mu)$ defined by (3). To accomplish this, we will make use of the following fact that gives an equivalent formulation of the homotopy formulation (1).

Lemma 2.3. *For each $\mu \in (0,1)$, the point $(x(\mu), y(\mu))$ satisfies the relation (2), i.e., $(x(\mu), y(\mu)) \in \mathcal{T}$, if and only if $x(\mu)$ is a solution to the following equation*

$$F(x, \mu) = 0,$$

where $F(x,\mu) = (F_1(x,\mu), \cdots, F_n(x,\mu))^T$, and for each i

$$\begin{aligned} F_i(x,\mu) &= x_i + \mu f_i(x) + (1-\mu)y_i^0 \\ &\quad - \sqrt{x_i^2 + (\mu f_i(x) + (1-\mu)y_i^0)^2 + 2(1-\mu)x_i^0 y_i^0}. \end{aligned} \tag{4}$$

Proof. $x(\mu)$ is a solution to $F(x,\mu) = 0$, that is,

$$\begin{aligned} &x_i(\mu) + \mu f_i(x(\mu)) + (1-\mu)y_i^0 \\ &= \sqrt{x_i^2(\mu) + (\mu f_i(x(\mu)) + (1-\mu)y_i^0)^2 + 2(1-\mu)x_i^0 y_i^0} \end{aligned}$$

for all $i = 1, \cdots, n$, if and only if the following two relations hold

$$2x_i(\mu)(\mu f_i(x(\mu)) + (1-\mu)y_i^0) = 2(1-\mu)x_i^0 y_i^0 \quad \text{for all } i = 1, \cdots, n$$

and

$$x_i(\mu) + \mu f_i(x(\mu)) + (1-\mu)y_i^0 > 0 \quad \text{for all } i = 1, \cdots, n.$$

That is,

$$x_i(\mu)y_i(\mu) = (1-\mu)x_i^0 y_i^0, \tag{5}$$

$$y_i(\mu) = \mu f_i(x(\mu)) + (1-\mu)y_i^0, \tag{6}$$

$$(x_i(\mu), y_i(\mu)) > 0, \tag{7}$$

for all $i = 1, \cdots, n$. Noting that $H(x(\mu), y(\mu), \mu, x^0, y^0) = 0$ coincides with (5) and (6). The desired result follows. $\square$

From the above proof, it is worth noting that $(x(\mu), y(\mu)) \in \mathcal{T}(\mu)$ if and only if it satisfies (5)–(7). This fact will be used in the remainder of this paper. We are now ready to establish an alternative theorem concerning the nonemptyness of $\mathcal{T}(\mu)$. The following concept plays a very important role in the proof of our results.

Definition 2.2. Let $(x^0, y^0) > 0$ be a given vector. For each scalar $\mu \in (0,1)$, the sequence $\{x^r\} \subseteq R^n_{++}$ with $\|x^r\| \to \infty$ as $r \to \infty$ is said to be a μ-exceptional family (μ-EF for short) for f if for each x^r there exists a scalar $\beta_r \in (0,1)$ such that

$$\mu f_i(x^r) = \frac{1}{2}\left(\beta_r - \frac{1}{\beta_r}\right)x_i^r + \frac{(1-\mu)\beta_r x_i^0 y_i^0}{x_i^r} - (1-\mu)y_i^0 \tag{8}$$

for all $i = 1, \cdots, n$.

This concept can be regarded as a variant of the concept of interior-point-ε-exceptional family, which was originally introduced by Zhao and Isac [15]. By using the concept of μ-EF and Lemmas 2.1, 2.2 and 2.3, we can prove the following result which states that either $\mathcal{T}(\mu) \neq \emptyset$ or there exists a μ-EF for f.

Theorem 2.1. *Let $(x^0, y^0) > 0$ be a given vector, and let $f : R^n \to R^n$ be a continuous function. Then for each $\mu \in (0,1)$ there exists either a pair $(x(\mu), y(\mu)) > 0$ such that*

$$H(x(\mu), y(\mu), \mu, x^0, y^0) = 0 \tag{9}$$

or a μ-EF for f.

Proof. The proof method has been used in [15, 16]. Given a $\mu \in (0,1)$, we assume that there exists no solution to the equation (9) in $R^n_{++} \times R^n_{++}$. We show that there must exist a μ-EF for f.

Consider the homotopy between the identity mapping and $F(x,\mu)$ defined by (4), that is,

$$G(x,t) = tx + (1-t)F(x,\mu), \quad 0 \le t \le 1.$$

Let $r > 0$ be an arbitrary positive scalar. As r varies, the family of bounded open sets is defined by $O_r = \{x \in R^n : \|x\| < r\}$, and hence the boundary $\partial O_r = \{x \in R^n : \|x\| = r\}$. Under the assumption at the beginning of the proof, we show first that for each $r > 0$ there exists some point $x^r \in \partial O_r$ and $t_r \in [0,1]$ such that

$$0 = G(x^r, t_r) = t_r x^r + (1-t_r)F(x^r, \mu). \tag{10}$$

Indeed, if there exists some scalar $r^* > 0$ such that

$$0 \notin \{G(x,t) : x \in \partial O_{r^*} \text{ and } t \in [0,1]\},$$

then by Lemma 2.1 we deduce that

$$\deg(F(x,\mu), O_{r^*}, 0) = \deg(I, O_{r^*}, 0),$$

where I denotes the identity mapping. Since $|\deg(I, O_{r^*}, 0)| = 1$, it follows from Lemma 2.2 that $F(x,\mu) = 0$ has a solution in the set O_{r^*}, and hence from Lemma 2.3 that there exists a point $(x(\mu), y(\mu)) > 0$ being a solution to the equation (9). This contradicts our assumption. Therefore, for each $r > 0$, (10) holds for some $x^r \in \partial O_r$ and $t_r \in [0,1]$. We point out that $t_r \neq 0$ and 1. Clearly, since $x^r \in \partial O_r$, it follows from (10) that $t_r \neq 1$. Under our assumption and by using Lemma 2.3 again, we know that $t_r \neq 0$ in (10).

Therefore, it is sufficient to consider the case: $0 < t_r < 1$. We show that there exists a μ-EF for f. Indeed, the equation (10) can be written as follows

$$\begin{aligned} & x_i^r + (1-t_r)(\mu f_i(x^r) + (1-\mu)y_i^0) \\ & = (1-t_r)\sqrt{(x_i^r)^2 + (\mu f_i(x^r) + (1-\mu)y_i^0)^2 + 2(1-\mu)x_i^0 y_i^0} \end{aligned} \tag{11}$$

for all $i = 1, \cdots, n$. Squaring both sides of (11) and denoting $\beta_r = 1 - t_r$, we have

$$\beta_r x_i^r(\mu f_i(x^r) + (1-\mu)y_i^0) = \frac{1}{2}\left(\beta_r - \frac{1}{\beta_r}\right)(x_i^r)^2 + \beta_r(1-\mu)x_i^0 y_i^0$$

for all $i = 1, \cdots, n$. Since $(x^0, y^0) > 0$, $\mu \in (0,1)$ and $\beta_r \in (0,1)$, we deduce from the above equation that $x_i^r \neq 0$ for all $i = 1, \cdots, n$. Thus

$$\mu f_i(x^r) = \frac{1}{2}\left(\beta_r - \frac{1}{\beta_r}\right) x_i^r + \frac{\beta_r(1-\mu)x_i^0 y_i^0}{x_i^r} - (1-\mu)y_i^0.$$

Since $x^r \in \partial O_r$, it follows that $\|x^r\| \to \infty$ as $r \to \infty$. We now show that $\{x^r\} \subseteq R^n_{++}$. Actually, if $x_i^r < 0$ for some i , we have from the above equation that

$$x_i^r + \beta_r(\mu f_i(x^r) + (1-\mu)y_i^0) = \frac{1}{2}(\beta_r^2 + 1)x_i^r + \frac{\beta_r^2(1-\mu)x_i^0 y_i^0}{x_i^r} < 0.$$

This contradicts (11) which shows that the left is nonnegative. Therefore, the sequence $\{x^r\}$ is a μ-EF for f. □

An immediate consequence of the above result is the following.

Corollary 2.1. *Let $(x^0, y^0) > 0$ and f be a continuous function from R^n into R^n. If for each $\mu \in (0,1)$ there exists no μ-EF for f, then $\mathcal{T}(\mu) \neq \emptyset$ for each $\mu \in (0,1)$.*

The above result provides a generic sufficient condition to guarantee $\mathcal{T}(\mu) \neq \emptyset$ for each $\mu \in (0,1)$. It is worth noting that the only assumption in the above result is the continuity of f. Because of this result, it is of interest to study the conditions under which a given continuous function has no μ-EF for each $\mu \in (0,1)$. To do this, we can establish some other practical sufficient conditions to assure $\mathcal{T}(\mu) \neq \emptyset$. The next section is devoted to developing such conditions for quasi-monotone complementarity problems.

3. Quasi-monotone problem

In this section, we show a sufficient condition for the properties (P_1) and (P_2) of mapping $\mathcal{T}(\mu)$ for quasi-monotone complementarity problems. We first introduce some properness conditions for the function f.

Condition 3.1. For any sequence $\{x^k\} \subseteq R^n_{++}$ such that $\|x^k\| \to \infty$ and $\{f(x^k)\}$ is bounded, it holds

$$\max_{1 \le i \le n} x_i^{k_l} f_i(x^{k_l}) \to \infty$$

for some subsequence $\{x^{k_l}\}$.

Condition 3.2. For any $\{x^k\} \subseteq R^n_{++}$ such that $\|x^k\| \to \infty$ and $[-f(x^k)]_+ < \infty$, and for each i such that $x_i^k \to \infty$, the sequence $\{f_i(x^k)\}$ is bounded, it holds

$$\max_{1\leq i\leq n} x_i^{k_l} f_i(x^{k_l}) \to \infty$$

for some subsequence $\{x^{k_l}\}$.

Clearly, Condition 3.2 is slightly stronger than Condition 3.1. But both two conditions are weaker than several existing conditions such as Condition 3.8 in [1], and conditions used in Theorem 1 in [2].

Lemma 3.1. *Assume that Condition 3.2 is satisfied. Then for each scalar* $\hat{\mu} \in (0,1)$, *the set* $\mathcal{T}_{\hat{\mu}} = \cup_{\mu\in[\hat{\mu},1)}\mathcal{T}(\mu)$ *is bounded.*

Proof. Let

$$X_{\hat{\mu}} = \{x(\mu) : (x(\mu), y(\mu)) \in \mathcal{T}(\mu) \text{ for some } y(\mu) \text{ and } \mu \in [\hat{\mu}, 1)\},$$

$$Y_{\hat{\mu}} = \{y(\mu) : (x(\mu), y(\mu)) \in \mathcal{T}(\mu) \text{ for some } x(\mu) \text{ and } \mu \in [\hat{\mu}, 1)\}.$$

Clearly, $\mathcal{T}_{\hat{\mu}} \subseteq X_{\hat{\mu}} \times Y_{\hat{\mu}}$. It suffices to show the set $X_{\hat{\mu}} \times Y_{\hat{\mu}}$ is bounded. Noting that $(x(\mu), y(\mu)) \in \mathcal{T}(\mu)$ if and only if it satisfies (5)–(7). By continuity of f it suffices to show the set $X_{\hat{\mu}}$ is bounded. Suppose that there exists some sequence $\{x(\mu_k)\}$, where $\mu_k \in [\hat{\mu}, 1)$ such that $\|x(\mu_k)\| \to \infty$, we derive a contradiction. Indeed, it follows from $y(\mu_k) = \mu_k f(x(\mu_k)) + (1-\mu_k)y^0 > 0$ that

$$f_i(x(\mu_k)) \geq -\frac{(1-\mu_k)y_i^0}{\mu_k} \geq -\frac{(1-\hat{\mu})y_i^0}{\hat{\mu}} \quad \text{for } i = 1, \cdots, n.$$

Thus

$$[-f_i(x(\mu_k))]_+ \leq \frac{(1-\hat{\mu})y_i^0}{\hat{\mu}} \quad \text{for all } i = 1, \cdots, n.$$

On the other hand, for each i such that $x_i(\mu_k) \to \infty$, from (5)–(7), we have

$$0 < \mu_k f_i(x(\mu_k)) + (1-\mu_k)y_i^0 = y_i(\mu_k) = \frac{(1-\mu_k)x_i^0 y_i^0}{x_i(\mu_k)} \to 0 \text{ as } k \to \infty.$$

Hence $\{f_i(x(\mu_k))\}$ is bounded for each i such that $x_i(\mu_k) \to \infty$. We deduce from Condition 3.2 that there exists a subsequence of $\{x(\mu_k)\}$, denoted also by $\{x(\mu_k)\}$, such that

$$x_i(\mu_k) f_i(x(\mu_k)) \to \infty.$$

From (5)–(7), however, we have

$$x_i(\mu_k)f_i(x(\mu_k)) = -\frac{(1-\mu_k)y_i^0}{\mu_k}x_i(\mu_k) + \frac{(1-\mu_k)x_i^0y_i^0}{\mu_k} \le \frac{(1-\hat{\mu})x_i^0y_i^0}{\hat{\mu}}$$

for all $i = 1, \cdots, n$. This is a contradiction. $\square$

The following two technical results are used to prove our main result.

Lemma 3.2 *Let $\{x^r\}$ be a μ-EF for the continuous function f. If condition 3.1 is satisfied and there exists a subsequence $\{x^{r_k}\}, k = 1, 2, \cdots,$ such that*

$$\|x^{r_k}\|\left(\beta_{r_k} - \frac{1}{\beta_{r_k}}\right) \to 0 \quad as \quad k \to \infty, \tag{12}$$

then we have

$$\min_{1\le i\le n} x_i^{r_k} \to 0 \quad as \quad k \to \infty. \tag{13}$$

Proof. For any constant $\beta \in (0,1)$ and $\eta > 0$, the function $\phi(t) : R_1 \to R_1$ defined by

$$\phi(t) = \frac{1}{2}\left(\beta - \frac{1}{\beta}\right)t + \frac{\eta}{t}$$

is monotonically decreasing on $(0,\infty)$ since the derivative $\phi'(t) < 0$ for all $t \in (0,\infty)$. Let $\{x^r\} \subseteq R^n_{++}$ be a μ-EF for f, Then it follows from (8) and the property of $\phi(t)$ that

$$\mu f_i(x^r) \le \frac{1}{2}\left(\beta_r - \frac{1}{\beta_r}\right)\left(\min_{1\le j\le n} x_j^r\right) + \frac{\beta_r(1-\mu)x_i^0y_i^0}{\min_{1\le j\le n} x_j^r} - (1-\mu)y_i^0, \tag{14}$$

and

$$\mu f_i(x^r) \ge \frac{1}{2}\left(\beta_r - \frac{1}{\beta_r}\right)\left(\max_{1\le j\le n} x_j^r\right) + \frac{\beta_r(1-\mu)x_i^0y_i^0}{\max_{1\le j\le n} x_j^r} - (1-\mu)y_i^0 \tag{15}$$

for all $i = 1, \cdots, n$.

Suppose that there exists a positive number $\alpha > 0$ such that there exists a subsequence of $\{x^{r_k}\}$ denoted also by $\{x^{r_k}\}$ satisfying $\min_{1\le i\le n} x_i^{r_k} \ge \alpha > 0$ for all k. We now derive a contradiction. Indeed, since $\mu \in (0,1), y^0 > 0$, and the first and third terms of the right-hand side of inequality (15) are negative, we have

$$\mu f_i(x^{r_k}) \le \frac{\beta_{r_k}(1-\mu)x_i^0y_i^0}{\min_{1\le j\le n} x_j^{r_k}} \le \frac{(1-\mu)x_i^0y_i^0}{\alpha} \quad \text{for all } i = 1, \cdots, n.$$

Combining the above inequality with (15) yields

$$\frac{(1-\mu)x_i^0y_i^0}{\alpha} \geq \mu f_i(x^{r_k}) \geq \frac{1}{2}\left(\beta_{r_k}-\frac{1}{\beta_{r_k}}\right)\left(\max_{1\leq j\leq n} x_j^{r_k}\right)-(1-\mu)y_i^0 \tag{16}$$

for all $i = 1, \cdots, n$. By (12) and (16), we deduce that the sequence $\{f(x^{r_k})\}$ is bounded. From Condition 3.1, choosing a subsequence if necessary we have

$$\max_{1\leq i\leq n} x_i^{r_k} f_i(x^{r_k}) \to \infty.$$

However, from (8), we have

$$x_i^{r_k} f_i(x^{r_k}) \leq \beta_k(1-\mu)x_i^0y_i^0/\mu < (1-\mu)x_i^0y_i^0/\mu$$

for all $i = 1, \cdots, n$. This contradiction shows that there exists no $\alpha > 0$ satisfying the property of the assumption. Hence, (13) must hold. □

We now show the next result.

Lemma 3.3. *Let $u > 0$ be an arbitrary vector in R^n_{++}. Let $\mu \in (0,1)$ and $\{x^r\}$ be a μ-EF for the continuous function f satisfying Condition 3.1. Then for any subsequence $\{x^{r_k}\}$ with $\|x^{r_k}\| \to \infty$, there exists a subsequence of $\{x^{r_k}\}$ denoted also by $\{x^{r_k}\}$ such that $f(x^{r_k})^T(x^{r_k}-u) < 0$ for all sufficiently large k.*

Proof. Let $\{x^{r_k}\}$ with $\|x^{r_k}\| \to \infty$ be an arbitrary subsequence of $\{x^r\}$, the μ-EF for f. Noting that $y^0 > 0$, $x^{r_k} > 0$, $\mu \in (0,1)$ and using (8), we have

$$\begin{aligned}
&\mu f(x^{r_k})^T(x^{r_k}-u)\\
&= \frac{1}{2}\left(\beta_{r_k}-\frac{1}{\beta_{r_k}}\right)(x^{r_k})^T(x^{r_k}-u) + \sum_{i=1}^{n}\left(\frac{\beta_{r_k}(1-\mu)x_i^0y_i^0}{x_i^{r_k}}(x_i^{r_k}-u_i)\right)\\
&-(1-\mu)(y^0)^T(x^{r_k}-u) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (17)\\
&\leq \frac{1}{2}\left(\beta_{r_k}-\frac{1}{\beta_{r_k}}\right)(\|x^{r_k}\|^2-(x^{r_k})^Tu) + (1-\mu)(x^0)^Ty^0 + (1-\mu)(y^0)^Tu.
\end{aligned}$$

We show the desired result by contradiction. Suppose that $f(x^{r_k})^T(x^{r_k}-u) \geq 0$ for sufficiently large k. From the inequality above we have

$$0 \leq \frac{1}{2}\left(\beta_{r_k}-\frac{1}{\beta_{r_k}}\right)(\|x^{r_k}\|^2-(x^{r_k})^Tu) + (1-\mu)(y^0)^T(u+x^0).$$

Thus, for sufficiently large k, we have

$$\begin{aligned}
0 &\geq \frac{1}{2}\left(\beta_{r_k} - \frac{1}{\beta_{r_k}}\right)\|x^{r_k}\| \\
&= \frac{1}{2}\left(\beta_{r_k} - \frac{1}{\beta_{r_k}}\right)(\|x^{r_k}\|^2 - (x^{r_k})^T u).\frac{\|x^{r_k}\|}{\|x^{r_k}\|^2 - (x^{r_k})^T u} \\
&\geq -\frac{(1-\mu)(y^0)^T(u+x^0)\|x^{r_k}\|}{\|x^{r_k}\|^2 - (x^{r_k})^T u} \to 0,
\end{aligned}$$

which shows that relation (12) holds. By Lemma 3.2, it follows that (13) holds. Since $\|x^{r_k}\| \to \infty$ and the first and third terms of (17) are negative numbers when k is sufficiently large, from (17) and (13), we have for sufficiently large k that

$$\begin{aligned}
f(x^{r_k})^T(x^{r_k} - u) &\leq \sum_{i=1}^n \beta_{r_k}(1-\mu)x_i^0 y_i^0 \left(1 - \frac{u_i}{x_i^{r_k}}\right) \\
&\leq (1-\mu)\beta_{r_k}\left((x^0)^T y^0 - \sum_{i=1}^n \frac{x_i^0 y_i^0 u_i}{x_i^{r_k}}\right) \\
&\leq (1-\mu)\beta_{r_k}\left((x^0)^T y^0 - \frac{\min_{1\leq i\leq n}(x_i^0 y_i^0 u_i)}{\min_{1\leq i\leq n} x_i^{r_k}}\right) < 0.
\end{aligned}$$

This is a contradiction. □

We have all ingredients to prove the main result of this section.

Theorem 3.1. *Let (x^0, y^0) be an arbitrary point in $R^n_{++} \times R^n_{++}$. Let $f : R^n \to R^n$ be a continuous quasi-monotone function. If Condition 3.1 is satisfied and there exists a vector $u > 0$ such that $f(u) > 0$, then $\mathcal{T}(\mu) \neq \emptyset$ for each $\mu \in (0,1)$. Moreover, if Condition 3.2 is satisfied, then for any $\hat{\mu} \in (0,1)$, the set $\cup_{\mu\in[\hat{\mu},1)}\mathcal{T}(\mu)$ is bounded.*

Proof. We show this result by contrary. Suppose that there exists a μ-EF for f, denoted by $\{x^r\}$. Then, by definition, we have $\{x^r\} \subseteq R^n_{++}$ and $\|x^r\| \to \infty$ as $r \to \infty$. Since $f(u) > 0$, we have $f(u)^T(x^r - u) > 0$ for sufficiently large r. By the quasi-monotonicity of f, this inequality implies that $f(x^r)^T(x^r - u) \geq 0$ for all sufficiently large r. This is impossible according to Lemma 3.3. Thus f has no μ-EF for any $\mu \in (0,1)$, and hence we have from Theorem 2.1 that $\mathcal{T}(\mu) \neq \emptyset$ for each $\mu \in (0,1)$. Moreover, if Condition 3.2 holds, the boundedness of the set $\cup_{\mu\in[\hat{\mu},1)}\mathcal{T}(\mu)$ follows directly from Lemma 3.3. □

While for quasi-monotone functions, we have developed a sufficient condition for properties (P_1) and (P_2), we are not clear, however, whether or not $\mathcal{T}(\mu)$ is single-valued and continuous on $(0,1)$. This is an interesting research topic in the future. It should be pointed out that (P_1) and (P_2) also imply the existence of a solution to NCP(f), but if we only consider the existence of a solution, the assumption like Condition 3.1 is superfluous. It is well known (see, Hadjisavaas and Schaible [4]) that quasi-monotonicity, combined with strict feasibility, implies the existence of a solution to the NCP(f). This result holds, in fact, for more general functions beyond quasi-monotonicity (see, Zhao and Isac [14]).

4. Conclusions

We establish a sufficient condition for the nonemptyness and boundedness of the mapping $\mathcal{T}(\mu)$ associated with quasi-monotone complementarity problems. The concept of μ-exceptional family and the alternative result "Theorem 2.1" have played a key role in our analysis throughout the paper. Choosing suitable version of μ-exceptional family, the argument method can be extended to the study of other homotopy solution paths. The result established in the paper might provide a theoretical basis to develope a new homotopy method for quasi-monotone problems.

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