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An alternative theorem for generalized variational inequalities and solvability of nonlinear quasi-P_*^M-complementarity problems

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Abstract

In this paper, an alternative theorem, and hence a sufficient solution condition, is established for generalized variational inequality problems. The concept of exceptional family for generalized variational inequality is introduced. This concept is general enough to include as special cases the notions of exceptional family of elements and the D-orientation sequence for continuous functions. Particularly, we apply the alternative theorem for investigating the solvability of the nonlinear complementarity problems with so-called quasi-P_*^M-maps, which are broad enough to encompass the quasi-monotone maps and P_*-maps as the special cases. An existence theorem for this class of complementarity problems is established, which significantly generalizes several previous existence results in the literature. © 2000 Elsevier Science Inc. All rights reserved.

Keywords: (Generalized) variational inequality; Complementarity problem; Exceptional family for variational inequality; Quasi-P_*^M-maps

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1. Introduction

Let f and g be two continuous functions from $\mathbb{R}^n$ into itself. The generalized variational inequality problem, denoted by $\mathrm{GVI}(K, f, g)$, is to find a vector $x \in \mathbb{R}^n$ such that

$$g(x) \in K, \quad (y - g(x))^{\mathrm{T}} f(x) \geqslant 0 \quad \text{for all } y \in K, \tag{1}$$

where K is a closed convex set in $\mathbb{R}^n$. This problem is important for it provides a unified formulation of several well-known problems such as the following variational inequality problem, denoted by $\mathrm{VI}(K, f)$

$$x^* \in K, \quad (y - x^*)^{\mathrm{T}} f(x^*) \geqslant 0 \quad \text{for all } y \in K; \tag{2}$$

and complementarity problem which is to determine a vector $x \in \mathbb{R}^n$ satisfying the following relation

$$x \geqslant 0, \quad f(x) \geqslant 0, \quad x^{\mathrm{T}} f(x) = 0. \tag{3}$$

See, Pang and Yao [16].

These problems arise in regional sciences, socio-economics analysis, energy modeling, transportation planning, game theory, control theory and mathematical programming (see for example, Refs. [1,5,6,8,13]). The variational inequality problem has become an important domain of applied mathematics. It has been extensively studied for several decades. A large number of existence conditions have been presented for (generalized) variational inequality problems including the complementarity problems [1,3–5,7,9,11,12,16,20–23]. Most of these existence results have been shown by using fixed-point theory, degree theory, min–max methods, etc. For nonlinear complementarity problems, quite different from these methods, Smith [18] and Isac et al. [7] proposed the argument methods of *exceptional sequence and exceptional family of elements* of a continuous function, respectively. Their main results claim that the condition "there exists no exceptional family of elements or exceptional sequence for the continuous functions" is sufficient for the existence of a solution to the complementarity problem. Isac and Obuchowska [9] showed that a variety of previous solution conditions, such as Karamardian conditions [11,12] and Schaible and Yao condition [17], imply this sufficient condition. Recently, Zhao [23] introduced the concept of *D-orientation sequences* for the continuous functions, and used this concept to investigate the solvability of nonlinear complementarity problems. Both concepts of exceptional family of elements and D-orientation sequence can be used to establish the alternative theorems for complementarity problems, and therefore a new solution condition on which many new existence results can be elicited. Because of the robustness of these concepts in studying the solvability of nonlinear complementarity problems, it has a sufficient reason to extend them to variational inequality problems. Zhao et al. [20–22] first introduced the concept of exceptional family for nonlinear variational inequality, and applied this concept to the study of the existence theorems for variational inequality problems. The exceptional family for variational inequality provides a powerful tool for investigating the solvability of this problem. However, it should be pointed out that, when specialized to nonlinear complementarity problems, this concept includes as a special case the Isac et al. concept, but it does not encompass the concept of D-orientation sequence for continuous functions.

One purpose of this paper is to establish a unified definition of exceptional family for $\mathrm{GVI}(K, f, g)$ such that this concept is general enough to include the exceptional sequence, exceptional family of elements and the D-orientation sequence when applied to nonlinear complementarity problems, and we use this concept to establish an alternative theorem for $\mathrm{GVI}(K, f, g)$. Another purpose of this paper is to use this alternative theorem to establish a new existence result for a class of nonlinear complementarity problems with the so-called quasi-P_*^M-maps. This result remarkably relaxes the solution condition of Karamardian [11], Hadjisavvas and Schaible [4] (but restricted to nonlinear complementarity problem), and Zhao and Han [22], etc. It is of interest that the class of nonlinear quasi-P_*^M-maps is significantly larger than the class of quasimonotone maps [10] (in particular, the monotone and pseudo-monotone maps) and the nonlinear P_*-maps [22].

In the remainder of this paper, Section 2 introduces a unified concept of exceptional family for generalized variational inequality and shows a general alternative theorem. Several special cases of the unified concept are also discussed. In Section 3, we define the class of nonlinear quasi-P_*^M-maps and establish a new existence result for nonlinear complementarity problems with such a class of maps. Conclusions are given in Section 4.

2. Alternative theorem

In the remainder of this section, we assume that the feasible set K is given as follows

$$K = \{x \in \mathbb{R}^n\colon\ E_i(x) \leqslant 0,\ i = 1, \ldots, m;\ H_j(x) = 0,\ j = 1, \ldots, l\},$$

where $E_i (i = 1, \ldots, m)$ and $H_j (j = 1, \ldots, l)$ are assumed to be convex and affine real-valued differentiable functions, respectively. In addition, *we assume that K is an unbounded set and satisfies some standard constrained qualifications. For instance, K satisfies the Slater's condition, i.e., there exists at least one $x^* \in K$ such that $E_i(x^*) < 0$ for all $i = 1, \ldots, m$. (when $K = \mathbb{R}^n_+$, the Slater's condition holds trivially).*

Let $E(x) = (E_1(x), \ldots, E_m(x))^{\mathrm{T}}$ and $H(x) = (H_1(x), \ldots, H_m(x))^{\mathrm{T}}$ be two mappings from $\mathbb{R}^n$ into $\mathbb{R}^m$ and $\mathbb{R}^l$, respectively. Then

$$K = \{x \in \mathbb{R}^n: \ E(x) \leqslant 0, \ H(x) = 0\}.$$

Throughout the paper, $\|\cdot\|$ denotes the Euclidean norm. Let $\nabla E(x)$ and $\nabla H(x)$ denote the Jacobian matrices of the mappings E and H, respectively.

Given $\beta \in [0, 1]$ and $d \in \mathbb{R}^n$ with the following restriction: *β can be prescribed to be zero only when $K = \mathbb{R}^n_+$ and d belongs to the positive orthant* (*i.e.*, $d > 0$). For fixed β and d, the function

$$c(x) = \beta x^{\mathrm{T}} x + (1 - \beta) x^{\mathrm{T}} d \tag{4}$$

is convex, and for each real $r > 0$ the level set $B_r = \{x \in \mathbb{R}^n: c(x) \leqslant r\}$ is nonempty. When $d = 0$ and $\beta = 1$, B_r is reduced to the Euclidean ball. Denote by K_r the intersection between the sets K and B_r, i.e.,

$$K_r = K \cap B_r = \{x \in \mathbb{R}^n: \ E(x) \leqslant 0; \ H(x) = 0; c(x) \leqslant r\}. \tag{5}$$

It is not difficult to see that there exists a real $r_0 > 0$ such that for each $r \geqslant r_0$, K_r is a nonempty compact convex set. By the assumption on K, K_r also satisfies some constrained qualifications.

The following two lemmas characterize the properties of the solution to a generalized variational inequality.

Lemma 2.1. *x^r solves* $\mathrm{GVI}(K_r, f, g)$, *which is equivalent to*

$$g(x^r) = P_{K_r}(g(x^r) - f(x^r)),$$

where P_{K_r} is the projection operator on K_r with respect to the Euclidean norm, if and only if there exist two vectors $\lambda^r \in \mathbb{R}^m_+$ (*m-dimensional nonnegative orthant*) *and $u^r \in \mathbb{R}^l$ and some nonnegative scalar μ_r such that*

$$f(x^r) = -\mu_r[\beta g(x^r) + \tfrac{1}{2}(1 - \beta) d] - \tfrac{1}{2}(\nabla E(g(x^r))^{\mathrm{T}} \lambda^r + \nabla H(g(x^r))^{\mathrm{T}} u^r), \tag{6}$$

$$\lambda_i^r E_i(g(x^r)) = 0 \quad \textit{for all } i = 1, \ldots, m, \tag{7}$$

$$\mu_r(c(g(x^r)) - r) = 0. \tag{8}$$

Proof. Let y be an arbitrary vector in $\mathbb{R}^n$. By the property of projection operator, the projection $P_{K_r}(y)$ is the unique solution to the following problem

$$\min\{\|x - y\|^2: \ x \in K_r\}.$$

Therefore, $g(x^r) = P_{K_r}(g(x^r) - f(x^r))$ if and only if $g(x^r)$ is the unique solution to the following problem

$$\min_{x \in \mathbb{R}^n}\{\|x - (g(x^r) - f(x^r))\|^2: \ E(x) \leqslant 0; \ H(x) = 0; c(x) \leqslant r\}.$$

Since the above problem is a convex program, the Karush–Kukn–Tucker optimality conditions completely characterize the solution of the above problem. Therefore, there exist some vectors $\lambda^r \in \mathbb{R}^m_+$ and $u^r \in \mathbb{R}^l$ and some scalar $\mu_r \geqslant 0$ such that the following system holds

$$2f(x^r) + \nabla E(g(x^r))^{\mathrm{T}}\lambda^r + \nabla H(g(x^r))^{\mathrm{T}}u^r + \mu_r[2\beta g(x^r) + (1-\beta)d] = 0, \tag{9}$$

$$E(g(x^r)) \leqslant 0; \quad H(g(x^r)) = 0, \quad c(g(x^r)) \leqslant r, \tag{10}$$

$$\lambda_i^r E_i(g(x^r)) = 0 \quad \text{for all } i = 1, \ldots, m, \tag{11}$$

$$\mu_r(c(g(x^r)) - r) = 0. \tag{12}$$

Since $g(x^r) \in K_r$, the condition (10) holds. Hence the system (9)–(12) is equivalent to the system (6)–(8). □

Similarly, by the same proof as the above, we have the following lemma.

Lemma 2.2. *x^* is a solution to* GVI(K, f, g) *if and only if there exist two vectors $\lambda^* \in \mathbb{R}^m_+$ and $u^* \in \mathbb{R}^l$ such that the following system holds*

$$f(x^*) = -\tfrac{1}{2}(\nabla E(g(x^*))^{\mathrm{T}}\lambda^* + \nabla H(g(x^*))^{\mathrm{T}}u^*),$$

$$\lambda_i^* E_i(g(x^*)) = 0 \quad \textit{for all } i = 1, \ldots, m.$$

We now introduce the notion of exceptional family for generalized variational inequality.

Definition 2.1. Let g and f be two mappings from $\mathbb{R}^n$ into itself. We say that a set of points $\{x^r\}_{r>0} \subset \mathbb{R}^n$ is an exceptional family for GVI(K, f, g), if the sequence satisfies the following two conditions:

(p_1) the sequence $\{g(x^r)\}_{r>0} \subset K$ and $\|g(x^r)\| \to \infty$ as $r \to \infty$,

(p_2) for every $r > 0$ there exist some positive scalar $\mu_r > 0$ and two vectors $\lambda^r \in \mathbb{R}^m_+$ and $u^r \in \mathbb{R}^l$ such that

$$f(x^r) = -\mu_r[\beta g(x^r) + \tfrac{1}{2}(1-\beta)d] - \tfrac{1}{2}(\nabla E(g(x^r))^{\mathrm{T}}\lambda^r + \nabla H(g(x^r))^{\mathrm{T}}u^r), \tag{13}$$

$$\lambda_i^r E_i(g(x^r)) = 0 \quad \text{for all } i = 1, \ldots, m. \tag{14}$$

The above general concept for generalized variational inequality includes several important special cases. Let g be the identity mapping, then we have the following concept for the nonlinear variational inequality problem (2).

Definition 2.2. Let f be a mapping from $\mathbb{R}^n$ into itself, the sequence $\{x^r\}_{r>0} \subset K$ is said to be an exceptional family for VI(K, f) if $\|x^r\| \to \infty$ as $r \to \infty$ and for each $r > 0$ there exist some positive scalar $\mu_r > 0$ and two vectors $\lambda^r \in \mathbb{R}^m_+$ and $u^r \in \mathbb{R}^l$ such that

$$f(x^r) = -\mu_r[\beta x^r + \tfrac{1}{2}(1-\beta)d] - \tfrac{1}{2}(\nabla E(x^r)^{\mathrm{T}}\lambda^r + \nabla H(x^r)^{\mathrm{T}}u^r),$$

$$\lambda_i^r E_i(x^r) = 0 \quad \text{for all } i = 1, \ldots, m.$$

If $E(x) = -x$ and there exists no equation system $H(x) = 0$, then $K = \mathbb{R}^n_+$. The $\mathrm{GVI}(K, f, g)$ reduces to the generalized complementarity problem

$$g(x) \geqslant 0, \quad f(x) \geqslant 0, \quad g(x)^{\mathrm{T}}f(x) = 0.$$

In this case, Eqs. (13) and (14) can be written as

$$f(x^r) = -\mu_r[\beta g(x^r) + \tfrac{1}{2}(1-\beta)d] + \tfrac{1}{2}\lambda^r,$$

$$\lambda_i^r g_i(x^r) = 0 \quad \text{for all } i = 1, \ldots, m.$$

Therefore, we have the following notion.

Definition 2.3. Let f and g be two mappings from $\mathbb{R}^n$ into $\mathbb{R}^n$. We say that a set of points $\{x^r\}_{r>0} \subset \mathbb{R}^n$ is an exceptional family for the generalized complementarity problem

$$g(x) \geqslant 0, \quad f(x) \geqslant 0, \quad g(x)^{\mathrm{T}}f(x) = 0$$

if the sequence satisfies

(c_1) $\|g(x^r)\| \to \infty$ as $r \to \infty$ and $g(x^r) \geqslant 0$ for all r,

(c_2) for each $r > 0$ there exists a positive scalar $\mu_r > 0$ such that

$$f_i(x^r) = -\mu_r[\beta g_i(x^r) + \tfrac{1}{2}(1-\beta)d_i] \quad \text{if } g_i(x^r) > 0, \tag{15}$$

$$f_i(x^r) \geqslant -\mu_r[\beta g_i(x^r) + \tfrac{1}{2}(1-\beta)d_i] \quad \text{if } g_i(x^r) = 0. \tag{16}$$

Furthermore, let $\beta = 1$ and $g(x) = x$, we can reduce the above notion to the following concept introduced in Ref. [7].

Definition 2.4 (Ref. [7]). A sequence $\{x^r\}_{r>0} \subset \mathbb{R}^n_+$ is an exceptional family of elements for the problem (3) if $\|x^r\| \to \infty$ as $r \to \infty$ and for every $r > 0$ there exists $\mu_r > 0$ such that

$$f_i(x^r) = -\mu_r x_i^r \quad \text{if } x_i^r > 0,$$

$$f_i(x^r) \geqslant 0 \qquad \text{if } x_i^r = 0.$$

Definition 2.3 includes two special cases, i.e., $\beta = 0$ and $\beta = 1$. When $\beta = 1$, then Eqs. (15) and (16) are reduced to

$$f_i(x^r) = -\mu_r g_i(x^r) \quad \text{if } g_i(x^r) > 0,$$

$$f_i(x^r) \geqslant -\mu_r g_i(x^r) \quad \text{if } g_i(x^r) = 0.$$

This notion of exceptional family is different from the one presented by Isac et al. [7]. One difference is that their concept requires the sequence $\|x^r\| \to \infty$ as $r \to \infty$ instead of $\|g(x^r)\| \to \infty$ as $r \to \infty$ (see, Definition 5 in Ref. [7]). The other difference is that their concept requires "$f_i(x^r) \geqslant 0$ when $g_i(x^r) = 0$". Their concept is derived from the topological degree theory.

When $\beta = 0$ (in the case, $d > 0$ and $K = \mathbb{R}^n_+$), conditions (15) and (16) reduce to the following form.

$$f_i(x^r) = -\mu_r d_i \quad \text{if } g_i(x^r) > 0,$$

$$f_i(x^r) \geqslant -\mu_r d_i \quad \text{if } g_i(x^r) = 0.$$

The above version of exceptional family for generalized complementarity problem can be viewed as the generalization of *D-orientation sequence for a continuous function* introduced in Ref. [23]. Actually, when $g(x) = x$, the above concept reduces to the D-orientation sequence for f.

We are now ready to establish an alternative theorem for $\mathrm{GVI}(K, f, g)$. This theorem claims that the condition "there exists no exceptional family for $\mathrm{GVI}(K, f, g)$" is sufficient for the existence of a solution of $\mathrm{GVI}(K, f, g)$. To accomplish this, we will make use of the following lemma which establishes the relations between $\mathrm{GVI}(K, f, g)$ and $\mathrm{GVI}(K_r, f, g)$, where K_r is given by Eq. (5).

Lemma 2.3. *Let f and g be two functions from $\mathbb{R}^n$ into itself. $c(x)$ is given by Eq.* (4), *then the generalized variational inequality* $\mathrm{GVI}(K, f, g)$ *has at least one solution if and only if there exists some $r > 0$ such that* $\mathrm{GVI}(K_r, f, g)$ *has a solution x^r with property $c(g(x^r)) < r$.*

Proof. If x^* solves $\mathrm{GVI}(K, f, g)$, we have $g(x^*) \in K$ and

$$(y - g(x^*))^{\mathrm{T}} f(x^*) \geqslant 0 \quad \text{for all } y \in K.$$

Let $r > 0$ be a scalar such that $c(g(x^*)) < r$. From Eq. (5) and the above we have

$$g(x^*) \in K_r, \quad (y - g(x^*))^{\mathrm{T}} f(x^*) \geqslant 0 \quad \text{for all } y \in K_r,$$

which implies that x^* solves $\mathrm{GVI}(K_r, f, g)$.

Conversely, suppose that there exists some scalar $r > 0$ such that $\mathrm{GVI}(K_r, f, g)$ has a solution x^r with property $c(g(x^r)) < r$. Then

$$g(x^r) \in K_r, \quad (y - g(x^r))^{\mathrm{T}} f(x^r) \geqslant 0 \quad \text{for all } y \in K_r. \tag{17}$$

It suffices to show that

$$(y - g(x^r))^{\mathrm{T}} f(x^r) \geqslant 0 \quad \text{for all } y \in K \setminus K_r. \tag{18}$$

In fact, let y be an arbitrary vector in $K \setminus K_r$. Since K is convex and $g(x^r) \in K_r \subset K$, we have

$$p(\lambda) = \lambda y + (1-\lambda) g(x^r) \in K \quad \text{for all } \lambda \in [0,1].$$

Since $c(g(x^r)) < r$ and $c(x)$ is continuous, there exists a sufficiently small scalar $\lambda^* > 0$ such that $c(p(\lambda^*)) < r$, hence $p(\lambda^*) \in K_r$. By Eq. (17) we have

$$\begin{aligned} 0 \leqslant (p(\lambda^*) - g(x^r))^{\mathrm{T}} f(x^r) &= [\lambda^* y + (1-\lambda^*) g(x^r) - g(x^r)]^{\mathrm{T}} f(x^r) \\ &= \lambda^* (y - g(x^r))^{\mathrm{T}} f(x^r), \end{aligned}$$

which implies that Eq. (18) holds. $\square$

The following mild assumption is also needed.

Assumption 2.1. There exists at least one positive sequence $\{r_j\}$ with $r_j \to \infty$ as $j \to \infty$ such that the generalized variational inequality $\mathrm{GVI}(K_{r_j}, f, g)$ is solvable (has a solution) for each j.

Actually, if Assumption 2.1 does not hold, then there exists some $r^* > 0$ such that for all $r \geqslant r^* > 0$, $\mathrm{GVI}(K_r, f, g)$ is not solvable, therefore $\mathrm{GVI}(K, f, g)$ has no solution by Lemma 2.3. Since K_r is bounded for all $r > 0$, each $\mathrm{VI}(K_r, f)$ is solvable (Theorem 3.4, [5]), *Assumption* 2.1 *holds trivially for variational inequality* $\mathrm{VI}(K, f)$ *and complementarity problem* (3).

Theorem 2.1. *Suppose that f and g are two continuous functions from $\mathbb{R}^n$ into itself, then the generalized variational inequality problem* $\mathrm{GVI}(K, f, g)$ *either has a solution or has an exceptional family.*

Proof. We assume that $\mathrm{GVI}(K, f, g)$ has no solution. In what follows, we prove that $\mathrm{GVI}(K, f, g)$ has an exceptional family. By Assumption 2.1, there exists an infinite positive sequence $r_j \to \infty$ as $j \to \infty$ such that $\mathrm{GVI}(K_{r_j}, f, g)$ is solvable for each j. Without loss of generality, we assume that x^{r_j} is a solution to $\mathrm{GVI}(K_{r_j}, f, g)$. Then by Lemma 2.3, the sequence $\{x^{r_j}\}$ satisfies $c(g(x^{r_j})) = r_j$ which implies $\|g(x^{r_j})\| \to \infty$ as $j \to \infty$ by the definition of the function $c(x)$. Since x^{r_j} solves $\mathrm{GVI}(K_{r_j}, f, g)$, by Lemma 2.1, there exist two vectors $\lambda^{r_j} \in \mathbb{R}^m_+$ and $u^{r_j} \in \mathbb{R}^l$ and some nonnegative scalar μ_{r_j} such that

$$f(x^{r_j}) = -\mu_{r_j}[\beta g(x^{r_j}) + \tfrac{1}{2}(1-\beta) d] - \tfrac{1}{2}(\nabla E(g(x^{r_j})))^{\mathrm{T}} \lambda^{r_j} + \nabla H(g(x^{r_j})))^{\mathrm{T}} u^{r_j}), \tag{19}$$

$$\lambda_i^{r_j} E_i(g(x^{r_j})) = 0 \quad \text{for all } 1 = 1, \ldots, m. \tag{20}$$

If there exists some j such that $\mu_{r_j} = 0$, then Eq. (19) reduces to

$$f(x^{r_j}) = -\tfrac{1}{2}(\nabla E(g(x^{r_j}))^{\mathrm{T}}\lambda^{r_j} + \nabla H(g(x^{r_j}))^{\mathrm{T}}u^{r_j}). \tag{21}$$

By Lemma 2.2, Eqs. (20) and (21) implies that x^{r_j} is a solution to $\mathrm{GVI}(K, f, g)$. A contradiction. Therefore $\mu_{r_j} > 0$ holds for all j and from Eqs. (19) and (20) we deduce that the sequence $\{x^{r_j}\}_{j\to\infty}$ is an exceptional family for $\mathrm{GVI}(K, f, g)$. $\square$

Corollary 2.1. *Let f and g be two continuous mappings from $\mathbb{R}^n$ into itself, if $\mathrm{GVI}(K, f, g)$ has no exceptional family, then it has a solution.*

The above results establish a new sufficient condition for the solvability of generalized variational inequality. Corollary 2.1 makes it possible for us to investigate the solvability of generalized variational inequality via studying the nonexistence conditions for the exceptional family. It should be pointed out that the Theorem 2.1 needs the Assumption 2.1, however, for variational inequality problem (2) and complementarity problem (3), this assumption is not needed.

3. Solvability of the complementarity problem

In this section, we define a new class of nonlinear mappings called quasi-P_*^M-maps, which is significantly larger than the class of nonlinear quasi-monotone maps and nonlinear P_*-maps. Under the strictly feasible condition, that is, there exists a point $u \in \mathbb{R}^n_+$ such that $f(u) > 0$, we will show that the nonlinear quasi-P_*^M-complementarity problem has a solution. The argument method is by using the unified concept of exceptional family for the nonlinear complementarity problem introduced in Section 2.

Recall that a map $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be a quasi-monotone map if for any distinct pair $(x, y) \in \mathbb{R}^{2n}$, $f(y)^{\mathrm{T}}(x - y) > 0$ implies $f(x)^{\mathrm{T}}(x - y) \geqslant 0$. (see Ref. [10]). A map $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be a nonlinear P_*-mapping [22]. If there exists a constant $\kappa \geqslant 0$ such that

$$(1+\kappa)\sum_{i\in I_+(x,y)}(x_i - y_i)(f_i(x) - f_i(y)) + \sum_{i\in I_-(x,y)}(x_i - y_i)(f_i(x) - f_i(y)) \geqslant 0, \tag{22}$$

where

$$I_+(x,y) = \{i:\ (x_i - y_i)(f_i(x) - f_i(y)) > 0\},$$

$$I_-(x,y) = \{i:\ (x_i - y_i)(f_i(x) - f_i(y)) \leqslant 0\}.$$

The following concept is broader than the above two classes of maps.

Definition 3.1. We say the function $f: \mathbb{R}^n \to \mathbb{R}^n$ is a quasi-P_*^M-map if there exists some constant $\tau \geqslant 0$ such that

$$f(y)^{\mathrm{T}}(x-y) - \tau \max_{1 \leqslant i \leqslant n}(x_i - y_i)(f_i(x) - f_i(y)) > 0 \text{ implies } f(x)^{\mathrm{T}}(x-y) \geqslant 0 \tag{23}$$

for any distinct pair $x, y \in \mathbb{R}^n$.

Remark 3.1. Clearly, quasi-monotone functions are included in the class of quasi-P_*^M-maps. Indeed, the quasi-P_*^M-map with the constant $\tau = 0$ is just the quasi-monotone map. The P_*-maps are also encompassed in quasi-P_*^M-maps. Actually, it is evident that Eq. (22) can be written as

$$(f(x) - f(y))^{\mathrm{T}}(x-y) + \kappa \sum_{i \in I_+(x,y)} (x_i - y_i)(f_i(x) - f_i(y)) \geqslant 0. \tag{24}$$

We consider two possible cases. The first case is that $I_+(x,y) \neq \emptyset$, then

$$\sum_{i \in I_+(x,y)} (x_i - y_i)(f_i(x) - f_i(y)) \leqslant n \max_{1 \leqslant i \leqslant n}(x_i - y_i)(f_i(x) - f_i(y)).$$

Thus Eq. (24) implies that

$$(f(x) - f(y))^{\mathrm{T}}(x-y) + (\kappa n) \max_{1 \leqslant i \leqslant n}(x_i - y_i)(f_i(x) - f_i(y)) \geqslant 0.$$

Let $\tau = \kappa n$, the above inequality implies that the implication (23) holds.

The second case is that $I_+(x,y) = \emptyset$. In this case, it is easy to see from Eq. (24) that

$$\max_{1 \leqslant i \leqslant n}(x_i - y_i)(f_i(x) - f_i(y)) = 0 \quad \text{for all } i = 1, \ldots, n.$$

Thus the implication (23) holds trivially. Therefore, a P_*-map must be a quasi-P_*^M-map.

Remark 3.2. In the linear case, $f = Mx + q$, where $M \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$, is a P_*-map if and only if M is a P_*-matrix. The concept of P_*-matrix was first defined by Kojima et al. [14], later, Väliaho [19] showed that it is equivalent to the concept of sufficient matrix introduced by Cottle et al. [2].

Since for the nonlinear complementarity problem the Assumption 2.1 in Theorem 2.1 and Corollary 2.1 is not necessary, the condition that f is without exceptional family is sufficient for the existence of a solution to the problem. The main result of the section is proved by using such a fact.

Theorem 3.1. *Let f be a continuous function from $\mathbb{R}^n$ into $\mathbb{R}^n$. Assume f being a quasi-P_*^M-map, i.e., there exists $\tau \geqslant 0$ such that the implication* (23) *holds. If there exists a point $u \geqslant 0$ such that $f(u) > 0$, then there exists no exceptional family* *for the nonlinear complementarity problem* (3), *thus there exists a solution for the problem.*

Proof. Assume the contrary, that is, there exists an exceptional family for the nonlinear complementarity problem, denoted by $\{x^r\}_{r>0}$, which satisfies the following two properties (according to Definition 2.3).

(a) $\{x^r\} \subset \mathbb{R}^n_+, \|x^r\| \to \infty$,

(b) For each r, there exists a scalar $\mu_r > 0$ such that

$$f_i(x^r) = -\mu_r[\beta x_i^r + \tfrac{1}{2}(1-\beta)d_i] \quad \text{if } x_i^r > 0, \tag{25}$$

$$f_i(x^r) \geqslant -\tfrac{1}{2}\mu_r(1-\beta)d_i \quad \text{if } x_i^r = 0. \tag{26}$$

Thus, we have

$$\begin{aligned}&(x_i^r - u_i)(f_i(x^r) - f_i(u))\\ &\quad = -(x_i^r - u_i)[\mu_r(\beta x_i^r + \tfrac{1}{2}(1-\beta)d_i) + f_i(u)] \quad \text{if } x_i^r > 0,\end{aligned} \tag{27}$$

$$(x_i^r - u_i)(f_i(x^r) - f_i(u)) \leqslant u_i[\tfrac{1}{2}(1-\beta)d_i + f_i(u)] \quad \text{if } x_i^r = 0. \tag{28}$$

Therefore, for each $i \in \{1, 2, \ldots, n\}$, we have

$$(x_i^r - u_i)(f_i(x^r) - f_i(u)) \leqslant u_i[\mu_i(\beta x_i^r + \tfrac{1}{2}(1-\beta)d_i) + f_i(u)]. \tag{29}$$

Since $\{x^r\} \subset \mathbb{R}^n_+$ and $\|x^r\| \to \infty$, there must be an index q such that $x_q^r \to \infty$ as $r \to \infty$, and notice that

$$\mu_r(\beta x_q^r + \tfrac{1}{2}(1-\beta)d_q) + f_q(u) > 0,$$

it follows from Eq. (27) that

$$(x_q^r - u_q)(f_q(x^r) - f_q(u)) \to -\infty \quad \text{as } r \to \infty. \tag{30}$$

If there exists r_0 such that

$$\max_{1\leqslant i\leqslant n}(x_i^r - u_i)(f_i(x^r) - f_i(u)) < 0$$

for all $r \geqslant r_0$, we will elicit a contradiction. Indeed, we have

$$f(u)^{\mathrm{T}}(x^r - u) = \sum_{i\in\{i:x_i^r>0\}} f_i(u)(x_i^r - u_i) + \sum_{i\in\{i:x_i^r=0\}} -u_i f_i(u).$$

The right-hand side of the above tends to ∞ since $x_q^r \to \infty$, so that we have

$$f(u)^{\mathrm{T}}(x^r - u) - \tau \max_{1\leqslant i\leqslant n}(x_i - y_i)(f_i(x) - f_i(y)) \geqslant f(u)^{\mathrm{T}}(x^r - u) > 0$$

for sufficiently large r. Since f is a quasi-P_*^M-map, the above inequality implies that

$$f(x^r)^{\mathrm{T}}(x^r - u) \geqslant 0. \tag{31}$$

However, by Eqs. (25) and (26)

$$\begin{aligned} f(x^r)^{\mathrm{T}}&(x^r - u) \\ &= \sum_{i\in\{i:x_i^r>0\}} -(x_i^r - u_i)\mu_r\left[\beta x_i^r + \frac{1}{2}(1-\beta)d_i\right] + \sum_{i\in\{i:x_i^r=0\}} -u_i f_i(x^r) \\ &\leqslant -\mu_r\left[\sum_{i\in\{i:x_i^r>0\}}(x_i^r - u_i)\left(\beta x_i^r + \frac{1}{2}(1-\beta)d_i\right) - \sum_{i\in\{i:x_i^r=0\}}\frac{1}{2}u_i(1-\beta)d_i\right] < 0. \end{aligned} \tag{32}$$

The last inequality follows from that $\mu_r > 0$ and $x_q^r \to \infty$. The above inequality is in contradiction with Eq. (31).

In what follows, we consider the case: there exists a subsequence $\{r_j\} \to \infty$ as $j \to \infty$ such that

$$\max_{1\leqslant i\leqslant n}(x_i^{r_j} - u_i)\ (f_i(x^{r_j}) - f_i(u)) \geqslant 0$$

for all $r_j\,(j = 1, 2, \ldots)$. We also elicit a contradiction. To accomplish this, we consider two possible subcases.

Case 1: The sequence $\{\mu_r\}_{r>0}$ is bounded. Clearly, there must be a subsequence of $\{r_j\}$, denoted also by $\{r_j\}$, such that for some fixed index m we have

$$(x_m^{r_j} - u_m)(f_m(x^{r_j}) - f_m(u)) = \max_{1\leqslant i\leqslant n}(x_i^{r_j} - u_i)(f_i(x^{r_j}) - f_i(u)) \geqslant 0. \tag{33}$$

Notice that if $x_i^{r_j} > u_i$, Eq. (27) implies that

$$(x_i^{r_j} - u_i)(f_i(x^{r_j}) - f_i(u)) < 0.$$

Thus, from Eq. (33) we have that $0 \leqslant x_m^{r_j} \leqslant u_m$, which combines with Eq. (29) to yield

$$(x_m^{r_j} - u_m)(f_m(x^{r_j}) - f_m(u)) \leqslant u_m\left[\mu_{r_j}\left(\beta u_m + \frac{1}{2}(1-\beta)d_m\right) + f_m(u)\right]. \tag{34}$$

Combining Eqs. (33) and (34) leads to

$$\begin{aligned} f(u)^{\mathrm{T}}&(x^{r_j} - u) - \tau\max_{1\leqslant i\leqslant n}(x^{r_j} - u_i)(f_i(x^{r_j}) - f_i(u)) \\ &\geqslant f(u)^{\mathrm{T}}(x^{r_j} - u) - \tau u_m\left[\mu_{r_j}\left(\beta u_m + \frac{1}{2}(1-\beta)d_m\right) + f_m(u)\right] \\ &= \sum_{i\in\{i:x_i^r>0\}} f_i(u)^{\mathrm{T}}(x_i^{r_j} - u_i) - \sum_{i\in\{i:x_i^r=0\}} u_i f_i(u) \\ &\quad - \tau u_m\left[\mu_{r_j}\left(\beta u_m + \frac{1}{2}(1-\beta)d_m\right) + f_m(u)\right] > 0. \end{aligned} \tag{35}$$

The above last inequality follows from that $x_q^{r_j} \to \infty$ and the boundedness of $\{\mu_{r_j}\}$.

Since f is a quasi-P_*^M-map, it follows from Eq. (35) that

$$f(x^{r_j})^{\mathrm{T}}(x^{r_j} - u) \geqslant 0. \tag{36}$$

By the same argument as Eq. (32), we can show that

$$f(x^{r_j})^{\mathrm{T}}(x^{r_j} - u) < 0$$

holds for sufficiently large r_j, which is in contradiction with Eq. (36).

Case 2: The sequence $\{\mu_r\}_{r>0}$ is unbounded. Without loss of generality, we assume that $\mu_r \to \infty$ as $r \to \infty$. Let $\{r_j\}$ and m be the same as Case 1. It is easy to see that Eqs. (33) and (34) remain valid. Thus, by Eqs. (33), (34) and (25), we have

$$\begin{aligned}
& f(x^{r_j})^{\mathrm{T}}(u - x^{r_j}) - \tau \max_{1 \leqslant i \leqslant n} (x_i^{r_j} - u_i)(f_i(x^{r_j}) - f_i(u)) \\
& \geqslant f(x^{r_j})^{\mathrm{T}}(u - x^{r_j}) - \tau u_m \left[\mu_{r_j} \left(\beta u_m + \frac{1}{2}(1-\beta) d_m \right) + f_m(u) \right] \\
& \geqslant \sum_{i \in \{i : x_i^{r_j} > 0\}} \mu_{r_j} \left(\beta x_i^{r_j} + \frac{1}{2}(1-\beta) d_i \right)(x_i^{r_j} - u_i) + \left(\sum_{i \in \{i : x_i^{r_j} = 0\}} - \frac{1}{2} \mu_{r_j} u_i d_i (1-\beta) \right) \\
& \quad - \tau \mu_{r_j} \left(\beta u_m^2 + \frac{1}{2}(1-\beta) d_m u_m \right) - \tau u_m f_m(u) \\
& = \mu r_j \left[-\tau \left(\beta u_m^2 + \frac{1}{2}(1-\beta) d_m u_m \right) - \sum_{i \in \{i : x_i^{r_j} = 0\}} \frac{1}{2} u_i d_i (1-\beta) \right. \\
& \quad \left. + \sum_{i \in \{i : x_i^{r_j} > 0\}} \left(\beta x_i^{r_j} + \frac{1}{2}(1-\beta) d_i \right)(x^{r_j} - u_i) \right] - \tau u_m f_m(u).
\end{aligned}$$

Since $\mu_{r_j} \to \infty$ and $x_q^{r_j} \to \infty$, the first term of the above last equality tends to ∞ as $r_j \to \infty$, thus for sufficiently large r_j

$$f(x^{r_j})^{\mathrm{T}}(u - x^{r_j}) - \tau \max_{1 \leqslant i \leqslant n} (x_i^{r_j} - u_i)(f_i(x^{r_j}) - f_i(u)) > 0. \tag{37}$$

By the quasi-P_*^M-property of f, Eq. (37) implies that

$$f(u)^{\mathrm{T}}(u - x^{r_j}) \geqslant 0 \tag{38}$$

holds for sufficiently large r_j.

On the other hand, since $\{x^{r_j}\} \subset \mathbb{R}_+^n$ and there exists at least one component $x_q^{r_j} \to +\infty$, the following inequality

$$f(u)^{\mathrm{T}}(u - x^{r_j}) < 0$$

holds for sufficiently large r_j, which is in contradiction with Eq. (38). The proof is complete. $\square$

Let $\beta = 1$ and $\beta = 0$, respectively, we have the following immediate consequence of Theorem 3.1.

Corollary 3.1. *Under strictly feasible condition, if f is a continuous quasi-P_*^M-mapping, then there exist no exceptional family of elements and D-orientation sequence for the function f, and hence the corresponding nonlinear complementarity problem has a solution.*

In the setting of nonlinear complementarity problems, Theorem 3.1 generalized the results of Karamardian [11] and Cottle and Yao [3] concerning pseudo-monotone maps. It also generalized the results of Hajisavvas and Schaible involving quasi-monotone maps, which can be stated as follows.

Corollary 3.2 (Ref. [4]). *Under strictly feasible condition, if f is a quasi-monotone function, then the corresponding nonlinear complementarity problem has a solution.*

The following result is also straightforward from Theorem 3.1.

Corollary 3.3 (Ref. [22]). *Under strictly feasible condition, if f is a nonlinear P_*-map, then the nonlinear complementarity problem has no exceptional family of elements for f, and hence the nonlinear complementarity problem has a solution.*

It should be pointed out that the strictly feasibility condition "$u \geqslant 0$, $f(u) > 0$" cannot be relaxed to the feasible condition, that is, "$u \geqslant 0$, $f(u) \geqslant 0$". Because quasi-P_*^M-maps includes as a special case the nonlinear monotone maps. Megiddo [15] gave an example to show that a nonlinear monotone complementarity problem satisfying feasible condition may have no solution.

4. Conclusions

- We introduce the concept of exceptional family for generalized variational inequality. This concept encompasses several previous concepts such as exceptional family of elements, exceptional sequence and D-orientation sequence for a continuous function.
- An alternative theorem is established for generalized variational inequality.
- We define the class of nonlinear quasi-P_*^M-maps, which is broader than quasi-monotone and P_*-maps.

- We show that there exists a solution for the quasi-P_*^M- complementarity problem if the strictly feasible condition holds. This result relaxed several existence conditions in the literature.

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