Global bounds for the distance to solutions of co-coercive variational inequalities

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Abstract. We establish two global bounds measuring the distance from any vector to the solution set of the co-coercive variational inequality. To prove our results, we use the fact that the co-coercivity condition is sufficient for the (strong) monotonicity of (perturbed) fixed point and normal maps associated with variational inequalities.

Key words: Variational inequalities; co-coercive maps; normal/fixed point maps; global bounds.

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1 Introduction

Throughout the paper, $\|\cdot\|$ denotes the 2-norm (Euclidean norm) of the vector in R^n. Let $f: R^n \to R^n$ be a continuous function and K be a closed convex set in R^n. The finite-dimensional variational inequality, denoted by VI(K, f), is to find an element $x^* \in K$ such that

$$(x - x^*)^T f(x^*) \geq 0 \quad \text{for all} \quad x \in K.$$

The above problem can be reformulated as the following fixed point equation and normal equation (see e.g. [2, 11]):

$$\pi_\alpha(x) = x - \Pi_K(x - \alpha f(x)) = 0, \tag{1}$$

$$\Phi_\alpha(x) = f(\Pi_K(x)) + \alpha(x - \Pi_K(x)) = 0, \tag{2}$$

where α is an arbitrary positive scalar and $\Pi_K(\cdot)$ is the projection operator on the convex set K, i.e., $\Pi_K(x) = \min_{z\in K} \|z - x\|$. Equation (1) is called the fixed point equation, and (2) is called the normal equation. It is well-known that x^* is a solution to VI(K, f) if and only if it is a solution to equation (1), i.e., $\pi_\alpha(x^*) = 0$, and that if x^* is a solution to VI(K, f), then $x^* - \frac{1}{\alpha} f(x^*)$ is a solution to equation (2); Conversely, if $\Phi_\alpha(u^*) = 0$, then $\Pi_K(u^*)$ is a solution to VI(K, f).

The following definition has been extensively used in the literature.

Definition 1.1 *(i) A function $f: R^n \to R^n$ is said to be monotone if $(x-y)^T(f(x)-f(y)) \geq 0$ for all $x, y \in R^n$.*

(ii) A mapping f is said to be strongly monotone if there is a scalar $c > 0$ such that $(x - y)^T(f(x) - f(y)) \geq c\|x - y\|^2$ for all $x, y \in R^n$.

(iii) f is said to be co-coercive in R^n if there exists a constant $\beta > 0$ such that

$$(x - y)^T(f(x) - f(y)) \geq \beta \parallel f(x) - f(y) \parallel^2 \tag{3}$$

for all $x, y \in R^n$.

The co-coercivity condition was used in several papers such as [3, 4, 5, 9, 11, 12]. It is evident that any co-coercive map is monotone and Lipschitz continuous (with constant $L = 1/\beta$) but it is not necessary to be strongly monotone (for example, the constant mapping). Zhao and Li [11] showed that the co-coercivity of f can be related to the monotonicity of $\pi_\alpha(x)$ and $\Phi_\alpha(x)$ and the strong monotonicity of their Tikhonov-type perturbed forms. They proved under suitable choice of α that the fixed point map and normal map are monotone if f satisfies the co-coercivity condition (3). They also proved the strong monotonicity of the perturbed forms of $\pi_\alpha(x)$ and $\Phi_\alpha(x)$ under co-coercivity or monotonicity assumption on f. For the later use, we state their results as follows.

Theorem 1.1 [11] *Let K be an arbitrary closed convex set in R^n. Let $f: R^n \to R^n$ be a continuous co-coercive map with modulus $\beta > 0$, i.e., condition (3) is satisfied. Then the following results hold:*

(a) *For any fixed α satisfying $0 < \alpha \leq 4\beta$, the fixed point map $\pi_\alpha(x)$ defined by (1) is monotone in x.*

(b) *If scalars α and ε are chosen such that $0 < \alpha < 4\beta$ and $0 < \varepsilon < 2(1/\alpha - 1/(4\beta))$, then the perturbed map $\pi_{\alpha,\varepsilon}(x) = x - \Pi_K(x - \alpha(f(x) + \varepsilon x))$ is strongly monotone (in x).*

Theorem 1.2 [11] *Let K be an arbitrary closed convex set in R^n. Let $f: R^n \to R^n$ be a continuous co-coercive map with modulus $\beta > 0$. Then the following results hold:*

(a) *For any constant α such that $\alpha > 1/(4\beta)$, the normal map $\Phi_\alpha(x)$ given by (2) is monotone in x.*

(b) *If $0 < \varepsilon < \alpha$ and $\alpha > 1/(4\beta)$, then the perturbed normal map $\Phi_{\alpha,\varepsilon}(x) = f(\Pi_K(x)) + \varepsilon\Pi_K(x) + \alpha(x - \Pi_K(x))$ is strongly monotone in x.*

Error bounds have played a very important role not only in theoretical analysis but also in convergence analysis of iterative algorithms for VI(K, f). A comprehensive, state-of-art survey of the theory and rich applications of error bounds can be found in [1] and [6] and the references therein. In this paper, we use the perturbed fixed point or normal map to establish the global bounds of the solution of VI(K, f). Our methods are different from those approaches used in [1]. Specifically, the following two basic results are employed to show our global bounds for solutions of variational inequalities. To our best knowledge, such results are the first time being used to establish the global bounds for variational inequalities. The first one is the Williamson's geometric estimation of fixed points of Lipschitz contraction maps.

Lemma 1.1 [10] *Let $T : D \subset R^n \to R^n$ be a Lipschitz mapping of D into R^n with constant $L \in (0,1)$, i.e., $\|T(x) - T(y)\| \leq L\|x - y\|$ for all x, y in D. Let $x \in D$ and suppose $x \neq T(x)$. Then the fixed-point of T is contained in the closed ball $B(e, \kappa)$ centered at $e \in R^n$ with radius κ, where*

$$e = (1 - 1/(1 - L^2))x + T(x)/(1 - L^2)$$

and

$$\kappa = L\|x - T(x)\|/(1 - L^2).$$

The next result is the upper-semicontinuity theorem concerning weakly univalent maps established by Ravindran and Gowda [8].

Lemma 1.2 [8] *Let $g : R^n \to R^n$ be weakly univalent, that is, g is continuous and there exists one-to-one continuous function $g_k : R^n \to R^n$ such that $g_k \to g$ uniformly on every bounded subset of R^n. Suppose that $g^{-1}(0) = \{x \in R^n : g(x) = 0\}$ is nonempty and compact. Then for any given $\gamma > 0$, there exists a scalar $\delta > 0$ such that for any weakly univalent function $h : R^n \to R^n$ with*

$$\sup_{\bar{\Omega}} \|h(x) - g(x)\| < \delta,$$

we have

$$\emptyset \neq h^{-1}(0) \subseteq g^{-1}(0) + \gamma B,$$

where B denotes the open unit ball in R^n and $\bar{\Omega}$ the closure of $\Omega = g^{-1}(0) + \gamma B$.

2 Global bounds for solutions of VI(K, f)

For any set $D \subseteq R^n$, we denote by $\text{dist}(x, D)$ the distance from the vector x to D, i.e.,

$$\text{dist}(x, D) = \{\min \|x - y\| : y \in D\}.$$

We denote the solution set of a variational inequality by $\text{SOL}(K, f)$.

We now prove the global estimation for the solution of a co-coercive VI(K, f) by using the perturbed fixed point map and the normal map.

Theorem 2.1 *Let f be a co-coercive map with modulus $\beta > 0$ on R^n. Suppose that the solution set of VI(K, f) is nonempty and bounded. Let α be a constant satisfying $0 < \alpha < \beta$. Then there exists a constant $\delta > 0$ such that for any ε satisfying*

$$0 < \varepsilon < \min\{\frac{\delta}{\alpha M^*}, \frac{1}{2\alpha}, \frac{2}{\alpha} - \frac{2}{\beta}\},$$

the following estimation holds

$$dist(x, SOL(K, f)) \leq \frac{\|\pi_{\alpha,\varepsilon}(x)\|}{1 - r} + \alpha$$

for all $x \in R^n$ where $M^* = \sup_{x \in \bar{\Omega}} \|x\|$, $\Omega = SOL(K, f) + \alpha B$ and

$$r = \sqrt{(1 - \alpha\varepsilon)^2 + 2\alpha^2\varepsilon/\beta} < 1.$$

Proof: Given $\alpha > 0$, it well-known that the set

$$\pi_\alpha^{-1}(0) = \{x \in R^n : \pi_\alpha(x) = 0\}$$

coincides with the solution of VI(K, f), i.e.,

$$\pi_\alpha^{-1}(0) = \text{SOL}(K, f).$$

By the assumption, $\pi_\alpha^{-1}(0)$ is bounded. Let α be a fixed constant satisfying $0 < \alpha < \beta$. By Theorem 1.1 (a), $\pi_\alpha(x)$ is monotone, and hence weakly univalent. It follows from Lemma 1.2 that there exists a constant $\delta > 0$ such that for any weakly univalent function $h : R^n \to R^n$ satisfying

$$\sup_{x\in\bar{\Omega}} \|h(x) - \pi_\alpha(x)\| < \delta, \qquad (4)$$

where $\bar{\Omega}$ is the closure of the set $\Omega := \pi_\alpha^{-1}(0) + \alpha B$, we have

$$\emptyset \neq h^{-1}(0) \subseteq \pi_\alpha^{-1}(0) + \alpha B = \mathrm{SOL}(K, f) + \alpha B. \qquad (5)$$

Denote $M^* = \sup_{x\in\bar{\Omega}} \|x\|$. Consider the perturbed map $\pi_{\alpha,\varepsilon}(x)$, where

$$0 < \varepsilon < \min\{\frac{\delta}{\alpha M^*}, \frac{1}{2\alpha}, \frac{2}{\alpha} - \frac{2}{\beta}\},$$

which implies that $\varepsilon < 2(1/\alpha - 1/(4\beta))$. It follows from the Theorem 1.1 (b) that $\pi_{\alpha,\varepsilon}(x)$ is strongly monotone. Thus the set $\pi_{\alpha,\varepsilon}^{-1}(0)$ has a unique element, denoted by $x_\alpha(\varepsilon)$. Notice that

$$\begin{aligned}
&\sup_{x\in\bar{\Omega}} \|\pi_{\alpha,\varepsilon}(x) - \pi_\alpha(x)\| \\
&= \sup_{x\in\bar{\Omega}} \|\Pi_K(x - \alpha(f(x) + \varepsilon x) - \Pi_K(x - \alpha(f(x))\| \\
&\le \sup_{x\in\bar{\Omega}} \|x - \alpha(f(x) + \varepsilon x) - (x - \alpha f(x))\| \\
&= \sup_{x\in\bar{\Omega}} \alpha\varepsilon\|x\| \\
&= \alpha\varepsilon M^* \\
&< \delta.
\end{aligned}$$

Since any monotone map is weakly univalent, substituting $\pi_{\alpha,\varepsilon}(x)$ for $h(x)$ in (4) and (5), we have

$$\begin{aligned}
\{x_\alpha(\varepsilon)\} &= \pi_{\alpha,\varepsilon}^{-1}(0) \subseteq \pi_\alpha^{-1}(0) + \alpha B \\
&= \mathrm{SOL}(K, f) + \alpha B,
\end{aligned}$$

which implies that

$$\mathrm{dist}(x_\alpha(\varepsilon), \mathrm{SOL}(K, f)) \le \alpha. \qquad (6)$$

We now give an estimation for the term $\|x - x_\alpha(\varepsilon)\|$. Since f is co-coercive, for any $x \in R^n$ we have

$$\begin{aligned}
& \|\Pi_K(x - \alpha(f(x) + \varepsilon x)) - \Pi_K(y - \alpha(f(y) + \varepsilon y))\|^2 \\
\le\ & \|x - \alpha(f(x) + \varepsilon x) - (y - \alpha(f(y) + \varepsilon y))\|^2 \\
=\ & (1 - \alpha\varepsilon)^2\|x - y\|^2 - 2\alpha(1 - \alpha\varepsilon)(x - y)^T(f(x) - f(y)) \\
& +\alpha^2\|f(x) - f(y)\|^2 \\
\le\ & (1 - \alpha\varepsilon)^2\|x - y\|^2 - 2\alpha(1 - \alpha\varepsilon)\beta\|f(x) - f(y)\|^2 \\
& +\alpha^2\|f(x) - f(y)\|^2 \quad (\text{ since } \varepsilon < 1/(2\alpha) < 1/\alpha) \\
=\ & (1 - \alpha\varepsilon)^2\|x - y\|^2 + 2\alpha^2\varepsilon\beta\|f(x) - f(y)\|^2 \\
& +(\alpha^2 - 2\alpha\beta)\|f(x) - f(y)\|^2 \\
\le\ & (1 - \alpha\varepsilon)^2\|x - y\|^2 + 2\alpha^2\varepsilon\beta\|f(x) - f(y)\|^2 \\
& (\text{since } \ 0 < \alpha < \beta) \\
\le\ & ((1 - \alpha\varepsilon)^2 + 2\alpha^2\varepsilon/\beta)\|x - y\|^2.
\end{aligned}$$

The last inequality follows from $\|f(x) - f(y)\| \le (1/\beta)\|x - y\|$ since f is a co-coercive map with modulus $\beta > 0$. Under our choice of ε, we can easily verify that

$$r = \sqrt{(1 - \alpha\varepsilon)^2 + 2\alpha^2\varepsilon/\beta} \in (0, 1). \qquad (7)$$

Therefore, the mapping $p^\varepsilon(x) =: \Pi_K(x - \alpha(f(x) + \varepsilon x))$ is a contraction map. It is evident that $x_\alpha(\varepsilon)$ is the unique fixed point of $p^\varepsilon(x)$. By Lemma 1.1, we have

$$\{x_\alpha(\varepsilon)\} \subset B\left(x - \frac{x - p^\varepsilon(x)}{1 - r^2}, \frac{r\|x - p^\varepsilon(x)\|}{1 - r^2}\right)$$

for all $x \in R^n$. Therefore,

$$\left\|x_\alpha(\varepsilon) - \left[x - \frac{x - p^\varepsilon(x)}{1 - r^2}\right]\right\| \le \frac{r\|x - p^\varepsilon(x)\|}{1 - r^2}$$

for all $x \in R^n$. Hence,

$$\frac{\|x - p^\varepsilon(x)\|}{1 + r} \le \|x - x_\alpha(\varepsilon)\| \le \frac{\|x - p^\varepsilon(x)\|}{1 - r}.$$

Since $\pi_{\alpha,\varepsilon}(x) = x - p^\varepsilon(x)$, particularly, we have $\|x - x_\alpha(\varepsilon)\| \le \|\pi_{\alpha,\varepsilon}(x)\|/(1 - r)$. Therefore, by (6) and the above, we have

$$\begin{aligned}
&\mathrm{dist}(x, \mathrm{SOL}(K, f)) \\
&\le \|x - x_\alpha(\varepsilon)\| + \mathrm{dist}(x_\alpha(\varepsilon), \mathrm{SOL}(K, f)) \\
&\le \|\pi_{\alpha,\varepsilon}(x)\|/(1 - r) + \alpha
\end{aligned}$$

for any $x \in R^n$, where r is given by (7). □

The next global estimation is by means of a perturbed normal map. Its proof is similar to that of Theorem 2.1.

Theorem 2.2 *Let f be co-coercive on R^n with modulus $\beta > 0$. Suppose that the solution* *set of VI(K,f) is nonempty and bounded. Let α be a constant satisfying*

$$0<\varepsilon<\min\{\frac{\delta}{C^*},2\alpha-\frac{2}{\beta},\frac{\alpha}{2}\}.$$

Then following estimation holds for all $x\in R^n$,

$$dist(x,SOL(K,f))\le\|\Phi_{\alpha,\varepsilon}(x)\|/(1-r)+\alpha$$

where $C^*=\sup_{x\in\bar{\Omega}}\|\Pi_K(x)\|$, $\Omega:=SOL(K,f)+\alpha B$ and $r=\sqrt{(1-\varepsilon/\alpha)^2+2\varepsilon/\alpha^2\beta}\in(0,1)$.

Proof: Notice that zeros of $\Phi_\alpha(x)$ are in one-to-one correspondence with the solutions of VI(K,f). Namely, if x^* solves VI(K,f), then $x^*-\frac{1}{\alpha}f(x^*)\in\Phi_\alpha^{-1}(0)$; Conversely, if $u^*\in\Phi_\alpha^{-1}(0)$, then $x^*=\Pi_K(u^*)$ is a solution to VI(K,f). Since f and $\Pi_K(\cdot)$ are continuous, the set of zeros of $\Phi_\alpha(x)$ is bounded if and only if SOL(K,f) is bounded. Thus by our assumption, $\Phi_\alpha^{-1}(0)$ is nonempty and bounded.

Let $\alpha>1/\beta$. By Theorem 1.2 (a), we see that $\Phi_\alpha(x)$ is monotone, and hence weakly univalent. From Lemma 1.2, for such a given scalar α, there exists a corresponding constant $\delta>0$ such that for any weakly univalent function $h(x)$ with

$$\sup_{x\in\bar{\Omega}}\|h(x)-\Phi_\alpha(x)\|<\delta \tag{8}$$

where $\bar{\Omega}$ is the closure of the set $\Omega=\Phi_\alpha^{-1}(0)+\alpha B$, we have

$$\emptyset\neq h^{-1}(0)\subseteq\Phi_\alpha^{-1}(0)+\alpha B. \tag{9}$$

Let $\varepsilon>0$ be given such that

$$0<\varepsilon<\min\{\frac{\delta}{C^*},2\alpha-\frac{2}{\beta},\frac{\alpha}{2}\},$$

where $C^*=\sup_{x\in\bar{\Omega}}\|\Pi_K(x)\|$. Then we have

$$\begin{aligned}&\sup_{x\in\bar{\Omega}}\|\Phi_{\alpha,\varepsilon}(x)-\Phi_\alpha(x)\|\\&\le\sup_{x\in\bar{\Omega}}\|f(\Pi_K(x))+\varepsilon\Pi_K(x)+\alpha(x-\Pi_K(x))\\&-(f(\Pi_K(x))+\alpha(x-\Pi_K(x)))\|\\&=\varepsilon\sup_{x\in\bar{\Omega}}\|\Pi_K(x)\|\\&=\varepsilon C^*<\delta.\end{aligned}$$

which implies that $h(x):=\Phi_{\alpha,\varepsilon}(x)$ satisfies (8). Therefore, it follows from (9) that $\Phi_{\alpha,\varepsilon}^{-1}(0)\subseteq\Phi_\alpha^{-1}(0)+\alpha B$. By Theorem 1.2 (b), $\Phi_{\alpha,\varepsilon}(x)$ is strongly monotone, and hence $\Phi_{\alpha,\varepsilon}^{-1}(0)$ contains a unique element denoted by $x_\alpha(\varepsilon)$. Thus,

$$\text{dist}(x_\alpha(\varepsilon),\Phi_\alpha^{-1}(0))\le\alpha. \tag{10}$$

On the other hand, by co-coercive property of f, we have

$$\begin{aligned}&\left\|x-\frac{1}{\alpha}\Phi_{\alpha,\varepsilon}(x)-(y-\frac{1}{\alpha}\Phi_{\alpha,\varepsilon}(y))\right\|^2\\=&\ \|-\frac{1}{\alpha}f(\Pi_K(x))+(1-\frac{\varepsilon}{\alpha})\Pi_K(x)\\&-\left[-\frac{1}{\alpha}f(\Pi_K(y))+(1-\frac{\varepsilon}{\alpha})\Pi_K(y)\right]\|^2\\=&\ \frac{1}{\alpha^2}\|f(\Pi_K(x))-f(\Pi_K(y))\|^2\\&+(\frac{\alpha-\varepsilon}{\alpha})^2\|\Pi_K(x)-\Pi_K(y)\|^2-\frac{2}{\alpha}(1-\frac{\varepsilon}{\alpha})\\&\bullet(f(\Pi_K(x))-f(\Pi_K(y)))^T(\Pi_K(x)-\Pi_K(y))\\\le&\ \left[\frac{1}{\alpha^2}-\frac{2\beta}{\alpha}(1-\frac{\varepsilon}{\alpha})\right]\|f(\Pi_K(x))-f(\Pi_K(y))\|^2\\&+(1-\frac{\varepsilon}{\alpha})^2\|\Pi_K(x)-\Pi_K(y)\|^2\\\le&\ \frac{2\varepsilon\beta}{\alpha^2}\|f(\Pi_K(x))-f(\Pi_K(y))\|^2\\&+(1-\frac{\varepsilon}{\alpha})^2\|\Pi_K(x)-\Pi_K(y)\|^2\\\le&\ \left[(1-\frac{\varepsilon}{\alpha})^2+\frac{2\varepsilon}{\alpha^2\beta}\right]\|\Pi_K(x)-\Pi_K(y)\|^2\\\le&\ \left[(1-\frac{\varepsilon}{\alpha})^2+\frac{2\varepsilon}{\alpha^2\beta}\right]\|x-y\|^2.\end{aligned}$$

The second inequality follows from the fact

$$1/\alpha^2-2\beta/\alpha\le0.$$

The third inequality follows from that

$$\|f(\Pi_K(x))-f(\Pi_K(y))\|\le\frac{1}{\beta}\|\Pi_K(x)-\Pi_K(y)\|.$$

Under our choice of α and ε, it is easy to see that $r=:\sqrt{(1-\varepsilon/\alpha)^2+2\varepsilon/\alpha^2\beta}\in(0,1)$. Thus, the mapping $x-\frac{1}{\alpha}\Phi_{\alpha,\varepsilon}(x)$ is a Lipschitz continuous map with the constant less than 1, i.e., a contraction map. Notice that the (unique) solution $x_\alpha(\varepsilon)$ of $\Phi_{\alpha,\varepsilon}(x)=0$ is just the fixed point of the map $x-\frac{1}{\alpha}\Phi_{\alpha,\varepsilon}(x)$. By Lemma 1.1, we have for any $x\in R^n$ that $\{x_\alpha(\varepsilon)\}$ is contained in the following ball

$$B\left(\frac{-r^2x}{1-r^2}+\frac{x-\frac{1}{\alpha}\Phi_{\alpha,\varepsilon}(x)}{1-r^2},\ \frac{r\|\frac{1}{\alpha}\Phi_{\alpha,\varepsilon}(x)\|}{1-r^2}\right).$$

Thus for any $x \in R^n$, we have

$$\frac{\|\Phi_{\alpha,\varepsilon}(x)\|}{\alpha(1+r)} \le \|x - x_\alpha(\varepsilon)\| \le \frac{\|\Phi_{\alpha,\varepsilon}(x)\|}{\alpha(1-r)}.$$

Therefor, by (10) and the above inequality, we have

$$\begin{aligned}&\mathrm{dist}(x, \Phi_\alpha^{-1}(0))\\ &\le \|x - x_\alpha(\varepsilon)\| + \mathrm{dist}(x_\alpha(\varepsilon), \Phi_\alpha^{-1}(0))\\ &\le \frac{\|\Phi_{\alpha,\varepsilon}(x)\|}{\alpha(1-r)} + \alpha\end{aligned}$$

for any $x \in R^n$. The proof is completed. □

We have shown two global bounds for the solution of VI(K, f) by using the perturbed fixed point map and normal map. In particular, if f is strongly monotone and Lipschitz continuous, i.e., there exist constants $c > 0$ and $L > 0$ such that $(f(x) - f(y))^T(x - y) \ge c\|x - y\|^2$ and $L\|x - y\| \ge \|f(x) - f(y)\|$ for any distinct x, y, then we see that $(f(x) - f(y))^T(x - y) \ge (c/L^2)\|f(x) - f(y)\|^2$. Thus, strongly monotone and Lipschitz maps are co-coercive. Since in this case the corresponding VI(K, f) has a unique solution (see [2]), we see that results of Theorems 2.1 and 2.2 hold trivially when the functions are strongly monotone and Lipschitz continuous. However, in this special case, we can further improve this result such that $\|\pi_\alpha(x)\|$ and $\|\Phi_\alpha(x)\|$ become the global error bounds for the solution of VI(K, f) under suitable choice of α. The global error bound using the residual function $\|\pi_\alpha(x)\|$ was proved by Pang [7]. See also Proposition 6.3.1 in [1]. So, we do not discuss the error bounds by using $\|\pi_\alpha(x)\|$. Here, we prove a global error bound by using the residual function $\|\Phi_\alpha(x)\|$.

Theorem 2.3 *Let f be strongly monotone on R^n with modulus $c > 0$ and let f be Lipschitz continuous on R^n with constant $L > 0$. Let α be a fixed scalar such that $\alpha > \max\{2c, L^2/2c\}$. Denote by x^* the (unique) solution of normal equation $\Phi_\alpha(x) = 0$. Then*

$$\frac{\|\Phi_\alpha(x)\|}{\alpha(1+r)} \le \|x - x^*\| \le \frac{\|\Phi_\alpha(x)\|}{\alpha(1-r)}$$

for all $x \in R^n$, where

$$r = \sqrt{1 - \frac{2c}{\alpha}\left(1 - \frac{L^2}{2c\alpha}\right)}.$$

Proof: We show that under the assumption the mapping $x - \frac{1}{\alpha}\Phi_\alpha(x)$ is Lipschitz continuous on R^n and the Lipschitz constant is less than 1. Indeed,

$$\begin{aligned}&\left\|x - \frac{1}{\alpha}\Phi_\alpha(x) - \left(y - \frac{1}{\alpha}\Phi_\alpha(y)\right)\right\|^2\\ =\ &\left\|-\frac{f(\Pi_K(x))}{\alpha} + \Pi_K(x) + \frac{f(\Pi_K(y))}{\alpha} - \Pi_K(y)\right\|^2\\ =\ &\frac{1}{\alpha^2}\|f(\Pi_K(x)) - f(\Pi_K(y))\|^2 + \|\Pi_K(x) - \Pi_K(y)\|^2\\ &-\frac{2}{\alpha}(f(\Pi_K(x)) - f(\Pi_K(y)))^T(\Pi_K(x) - \Pi_K(y))\\ \le\ &\frac{L^2}{\alpha^2}\|\Pi_K(x) - \Pi_K(y)\|^2 + \|\Pi_K(x) - \Pi_K(y)\|^2\\ &-\frac{2c}{\alpha}\|\Pi_K(x) - \Pi_K(y)\|^2\\ =\ &\left(1 - \frac{2c}{\alpha} + \frac{L^2}{\alpha^2}\right)\|\Pi_K(x) - \Pi_K(y)\|^2\\ \le\ &\left(1 - \frac{2c}{\alpha}(1 - \frac{L^2}{2c\alpha})\right)\|x - y\|^2.\end{aligned}$$

The first inequality follows from the Lipschitz continuity and strong monotonicity of f. The last inequality follows from the non-expansiveness of the projection operator. Since

$$\alpha > \max\{2c, L^2/2c\},$$

we have

$$r = \sqrt{1 - \frac{2c}{\alpha}(1 - \frac{L^2}{2c\alpha})} \in (0, 1). \qquad (11)$$

Therefore, $x - \frac{1}{\alpha}\Phi_\alpha(x)$ is a contraction map with the constant r given as the above. It is evident that the fixed point of $x - \frac{1}{\alpha}\Phi_\alpha(x)$ coincides with the solution of $\Phi_\alpha(x) = 0$. Denote it by x^*. By Lemma 1.1, for any $x \in R^n$ we see that $\{x^*\}$ is contained in the following ball

$$B\left(\frac{-r^2x}{1 - r^2} + \frac{x - \frac{1}{\alpha}\Phi_\alpha(x)}{1 - r^2},\ \frac{r\|\frac{1}{\alpha}\Phi_\alpha(x)\|}{1 - r^2}\right)$$

Therefore, we have

$$\frac{\|\Phi_\alpha(x)\|}{\alpha(1+r)} \le \|x - x^*\| \le \frac{\|\Phi_\alpha(x)\|}{\alpha(1-r)}$$

for all $x \in R^n$, where r is given by (11). □

Remark: We have shown in the proof of Theorem 2.3 that the map $x - \frac{1}{\alpha}\Phi_\alpha(x)$ is a contraction map if $\alpha > \max\{2c, L^2/2c\}$. By Banach's fixed point theorem, the Picard-type iterative scheme

$$x^{k+1} = x^k - \frac{1}{\alpha}\Phi_\alpha(x^k) = \Pi_K(x^k) - \frac{1}{\alpha}f(\Pi_K(x^k))$$

converges to the unique solution x^*, i.e., $\Phi_\alpha(x^*) = 0$, and hence $y^* = \Pi_K(x^*)$ is the unique solution to VI(K, f). Moreover,

$$\|x^{k+1} - x^*\| \le r^k \|x^1 - x^0\|/(1-r),$$

where r is given by (11). Hence, the iterative scheme

$$y^{k+1} = \Pi_K(\Pi_K(y^k) - \frac{1}{\alpha}f(\Pi_K(y^k))),$$

$$\alpha > \max\{2c, L^2/2c\}$$

converges R-linearly to the unique solution of strongly monotone and Lipschitz continuous variational inequality problems.

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