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Geometric dual formulation for first-derivative-based univariate cubic L_1 splines

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Abstract With the objective of generating "shape-preserving" smooth interpolating curves that represent data with abrupt changes in magnitude and/or knot spacing, we study a class of first-derivative-based $\mathcal{C}^1$-smooth univariate cubic L_1 splines. An L_1 spline minimizes the L_1 norm of the difference between the first-order derivative of the spline and the local divided difference of the data. Calculating the coefficients of an L_1 spline is a nonsmooth non-linear convex program. Via Fenchel's conjugate transformation, the geometric dual program is a smooth convex program with a linear objective function and convex cubic constraints. The dual-to-primal transformation is accomplished by solving a linear program.

Keywords Conjugate function · Convex program · Cubic L_1 spline · Shape-preserving interpolation · Piecewise polynomial

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1 Introduction

Let $\{x_i : i = 0, 1, \ldots, n\}$ be a set of strictly increasing knots covering the real interval $[a, b]$, that is,

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b. \tag{1}$$

Let $\{z_i \in R : i = 0, 1, \ldots, n\}$ be a set of corresponding values. We seek a piecewise polynomial function $\mathcal{Z}(x)$ with some desired properties that interpolates the data set (x_i, z_i), $i = 0, 1, \ldots, n$ [3,4,10,22,25,29]. In particular, we are interested in piecewise polynomial interpolants that "preserve the shape" of the data. One simple and frequently used procedure is piecewise linear interpolation. Such interpolation preserves shape well in the sense that there is no extraneous nonphysical oscillation. In particular, when the z_i's are convex (concave or monotone), the piecewise linear interpolant is also convex (concave or monotone, respectively). However, piecewise linear interpolants are generically nonsmooth at the knots and may not be sufficiently accurate unless the knots are closely spaced. Cubic piecewise polynomials are alternatives that have been widely investigated.

The traditional approach in designing a piecewise cubic polynomial interpolant $\mathcal{Z}(x)$ is to minimize the L_2 norm of the second derivative, that is, to minimize

$$\int_a^b (z''(x))^2 \, \mathrm{d}x,$$

over the space of functions with second derivatives in $L_2(a, b)$ subject to the interpolation conditions $z(x_i) = z_i$, $i = 0, 1, ..., n$ [2,3,13,25,27]. (Boundary conditions, such as $z''(a) = z''(b) = 0$, could be added with no major change to the theory presented below.) These splines exhibit many desirable theoretical approximation properties but have extraneous oscillation and do not preserve the shape of the data well. For this reason, they are used less widely than linear splines. Designing cubic splines that preserve the shape of data well is an important topic of research. One direction of research has focused on replacing the L_2 norm by the L_p norm [1–3,9,12,14,16,20,24,25]. Cubic L_1 splines, that is, cubic splines that minimize the L_1 norm of the second derivative, have attracted attention recently because they do not have the nonphysical oscillation endemic in traditional spines, preserve the shape of the data well and can be calculated by efficient algorithms [5,6,8,16,18]. In the literature, all of the cubic L_1 splines have obtained by minimizing the L_1 norm of the second derivative of the spline except for those in [19], which are obtained by minimizing the L_1 norm of the difference between the first-order derivative of the spline and the local divided difference of the data.

Motivated by the success of the computational experiments in [19], we investigate in this paper a theoretic model for the new class of C^1-smooth cubic L_1 splines, which we call "first-derivative-based L_1 splines." Calculating the coefficients of first-derivative-based L_1 splines turns out to be a non-differentiable convex programming problem. Traditional non-smooth optimization methods [28] can be applied, but such methods cannot easily be formulated to take advantage of the structure inherent in first-derivative-based L_1 splines and are, therefore, neither theoretically nor computationally efficient. Following [5], we investigate the possibility of using differentiable convex optimization techniques to handle this problem by studying its geometric dual model obtained through Fenchel's conjugate transform [23,26]. We show that the dual problem is a smooth convex program with a linear objective function and cubic constraints. This allows one to apply efficient interior-point algorithms for smooth convex programming [21] or to design new algorithms related to [6]. Once a dual solution is found, we show that the primal solution can be obtained by solving a sparse linear programming problem [11].

The rest of this paper is organized as follows. In Sect. 2, we formulate an optimization model with a non-differentiable convex objective function for first-derivative-based univariate cubic L_1 spline. In Sect. 3, we derive the conjugate function of this non-differentiable objective function. In Sect. 4, we formulate the geometric dual program for the original problem. We also show how to construct a primal optimal solution via a linear programming for dual-to-primal conversion. Concluding remarks are given in the last section.

2 Optimization model

The "data spacing" (knot spacing) and the (local) "data slope" (slope of the linear spline) are

$$h_i = x_{i+1} - x_i \qquad \text{and} \qquad \Delta z_i = \frac{z_{i+1} - z_i}{h_i}, \quad i = 0, 1, \ldots, n-1,$$

respectively. Our objective is to find a $\mathcal{C}^1$-smooth piecewise cubic polynomial $\mathcal{Z}(x)$ that interpolates the data (x_i, z_i), $i = 0, 1 \ldots, n$, and minimizes the L_1 distance between its first-order derivative and the data slope. More specifically, on each subinterval $[x_i, x_{i+1}]$, we seek a cubic Hermite polynomial

$$\mathcal{Z}_i(x) = p_i + q_i(x - x_i) + \frac{u_i}{2}(x - x_i)^2 + \frac{v_i}{6}(x - x_i)^3,$$

that joins together $\mathcal{C}^1$-smoothly with the cubic Hermite polynomial(s) in the adjacent interval(s) on the right and left and minimizes the difference between

$$\mathcal{Z}_i'(x) = q_i + u_i(x - x_i) + \frac{v_i}{2}(x - x_i)^2$$

and $\Delta z_i, i = 0, 1, \ldots, n$, over $[a, b]$. We denote the vector

$$(p_0, q_0, u_0, v_0,\ p_1, q_1, u_1, v_1,\ \ldots,\ p_{n-1}, q_{n-1}, u_{n-1}, v_{n-1},\ p_n)^T$$

by $\mathcal{X} \in R^{4n+1}$. Denote the indicator function of a convex set D by $\delta(x|D)$: $\delta(x|D) = 0$ if $x \in D$; $\delta(x|D) = \infty$ otherwise. Let $C_i = \{z_i\}$, $i = 0, 1, \ldots, n$. With this notation, finding the cubic L_1 spline is equivalent to solving the following optimization problem:

$$\begin{aligned} \min_{\mathcal{X} \in R^{4n+1}} \ \mathcal{F}(\mathcal{X}) &:= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} \left| \mathcal{Z}_i'(x) - \Delta z_i \right| dx + \sum_{i=0}^{n} \delta(p_i | C_i) \\ \text{Subject to } & p_i + h_i q_i + \frac{h_i^2}{2} u_i + \frac{h_i^3}{6} v_i = p_{i+1},\ i = 0, 1, \ldots, n-1, \\ & q_i + h_i u_i + \frac{h_i^2}{2} v_i = q_{i+1},\ i = 0, 1, \ldots, n-2. \end{aligned} \tag{2}$$

The two groups of constraints require that two adjacent local cubic polynomials have the same function value and first-order derivatives at common knots. The interpolation requirements $p_i = z_i$, $i = 0, 1, \ldots, n$ is incorporated into the objective function using the indicator function. This will make it easier for us to derive its dual problem. No explicit boundary conditions for the spline at x_0 and x_n are included in the above model, although they could easily be incorporated as we will mention in the conclusion.

For convenience, we let

$$f_i(q_i, u_i, v_i) := \int_{x_i}^{x_{i+1}} \left| \mathcal{Z}_i'(x) - \Delta z_i \right| \mathrm{d}x$$
$$= h_i \int_0^1 \left| q_i - \Delta z_i + u_i h_i \eta + \frac{v_i h_i^2}{2} \eta^2 \right| \mathrm{d}\eta,$$

where $\eta := (x - x_i)/h_i$. Then the objective function of (2) becomes

$$\mathcal{F}(\mathcal{X}) = \sum_{i=0}^{n} \delta(p_i | C_i) + \sum_{i=0}^{n-1} f_i(q_i, u_i, v_i).$$

For $v_i \neq 0$, we define

$$\chi_i(\eta) := \frac{q_i - \Delta z_i}{v_i h_i^2} + \frac{u_i}{v_i h_i} \eta + \frac{\eta^2}{2}. \tag{3}$$

Let $\eta_1^{(i)}$ and $\eta_2^{(i)}$ denote the two roots (if they exist) of the equation $\chi_i(\eta) = 0$ in variable η, that is,

$$\eta_1^{(i)} = -\frac{u_i}{v_i h_i} - \sqrt{\left(\frac{u_i}{v_i h_i}\right)^2 - 2\left(\frac{q_i - \Delta z_i}{v_i h_i^2}\right)}, \tag{4}$$

$$\eta_2^{(i)} = -\frac{u_i}{v_i h_i} + \sqrt{\left(\frac{u_i}{v_i h_i}\right)^2 - 2\left(\frac{q_i - \Delta z_i}{v_i h_i^2}\right)}. \tag{5}$$

Proposition 2.1 *The function $f_i(q_i,u_i,v_i)$ can be explicitly expressed as*

$$f_i=\begin{cases} |v_i|\Big\{\frac{6h_i(q_i-\Delta z_i)+3u_ih_i^2+v_ih_i^3}{6v_i}-\frac{2u_i}{v_i}\left[\frac{1}{3}\left(\frac{u_i}{v_i}\right)^2-\frac{q_i-\Delta z_i}{v_i}\right] \\ +\frac{2}{3}\left[\left(\frac{u_i}{v_i}\right)^2-2\left(\frac{q_i-\Delta z_i}{v_i}\right)\right]^{\frac{3}{2}}\Big\}, & \text{if } v_i\neq 0, \eta_1^{(i)}\leq 0\leq \eta_2^{(i)}\leq 1,\\ |v_i|\Big\{-\frac{6h_i(q_i-\Delta z_i)+3u_ih_i^2+v_ih_i^3}{6v_i}+\frac{2u_i}{v_i}\left[\frac{1}{3}\left(\frac{u_i}{v_i}\right)^2-\frac{q_i-\Delta z_i}{v_i}\right] \\ +\frac{2}{3}\left[\left(\frac{u_i}{v_i}\right)^2-2\left(\frac{q_i-\Delta z_i}{v_i}\right)\right]^{\frac{3}{2}}\Big\}, & \text{if } v_i\neq 0, 0\leq\eta_1^{(i)}\leq 1\leq \eta_2^{(i)},\\ |v_i|\left(\frac{6h_i(q_i-\Delta z_i)+3u_ih_i^2+v_ih_i^3}{6v_i}+\frac{4}{3}\left[\left(\frac{u_i}{v_i}\right)^2-2\left(\frac{q_i-\Delta z_i}{v_i}\right)\right]^{\frac{3}{2}}\right), & \text{if } v_i\neq 0, 0\leq\eta_1^{(i)}\leq \eta_2^{(i)}\leq 1,\\ -|v_i|\left(\frac{6h_i(q_i-\Delta z_i)+3u_ih_i^2+v_ih_i^3}{6v_i}\right), & \text{if } v_i\neq 0, \eta_1^{(i)}\leq 0< 1\leq \eta_2^{(i)},\\ |v_i|\left(\frac{6h_i(q_i-\Delta z_i)+3u_ih_i^2+v_ih_i^3}{6v_i}\right), & \text{if } v_i\neq 0, \text{otherwise},\\ |u_i|\left[\left(\frac{q_i-\Delta z_i}{u_i}\right)^2+\frac{h_i(q_i-\Delta z_i)}{u_i}+\frac{h_i^2}{2}\right], & \text{if } v_i=0, u_i\neq 0, \frac{q_i-\Delta z_i}{u_ih_i}\\ & \in[-1,0],\\ |u_i|\left(\frac{h_i^2}{2}+\frac{h_i(q_i-\Delta z_i)}{u_i}\right), & \text{if } v_i=0, u_i\neq 0, \frac{q_i-\Delta z_i}{u_ih_i}\geq 0,\\ -|u_i|\left(\frac{h_i^2}{2}+\frac{h_i(q_i-\Delta z_i)}{u_i}\right), & \text{if } v_i=0, u_i\neq 0, \frac{q_i-\Delta z_i}{u_ih_i}\leq -1,\\ h_i\,|q_i-\Delta z_i|\,, & \text{if } v_i=0, u_i=0. \end{cases}$$

Proof We first consider the case with $v_i=0$. In this situation, we have the following four subcases:

(a) $u_i=0$. It is evident that $f_i=h_i|q_i-\Delta z_i|$.

(b) $u_i\neq 0$ and $\frac{q_i-\Delta z_i}{u_ih_i}\geq 0$. Then

$$f_i=h_i|u_ih_i|\int_0^1\left(\eta+\frac{q_i-\Delta z_i}{u_ih_i}\right)\mathrm{d}\eta=|u_i|\left(\frac{h_i^2}{2}+\frac{h_i(q_i-\Delta z_i)}{u_i}\right).$$

(c) $u_i\neq 0$ and $\frac{q_i-\Delta z_i}{u_ih_i}\leq -1$. Then

$$f_i=h_i|u_ih_i|\int_0^1-\left(\eta+\frac{q_i-\Delta z_i}{u_ih_i}\right)\mathrm{d}\eta=-|u_i|\left(\frac{h_i^2}{2}+\frac{h_i(q_i-\Delta z_i)}{u_i}\right).$$

(d) $u_i\neq 0$ and $-1\leq\frac{q_i-\Delta z_i}{u_ih_i}\leq 0$. Then we have

$$\begin{aligned} f_i&=h_i^2|u_i|\left[\int_0^{\left|\frac{q_i-\Delta z_i}{u_ih_i}\right|}-\left(\eta+\frac{q_i-\Delta z_i}{u_ih_i}\right)\mathrm{d}\eta+\int_{\left|\frac{q_i-\Delta z_i}{u_ih_i}\right|}^1\left(\eta+\frac{q_i-\Delta z_i}{u_ih_i}\right)\mathrm{d}\eta\right]\\ &=|u_i|\left[\left(\frac{q_i-\Delta z_i}{u_i}\right)^2+\frac{h_i(q_i-\Delta z_i)}{u_i}+\frac{h_i^2}{2}\right]. \end{aligned}$$

Next consider the case with $v_i \neq 0$. Note that

$$f_i = h_i^3|v_i| \int_0^1 \left| \frac{q_i - \Delta z_i}{v_i h_i^2} + \frac{u_i}{v_i h_i}\eta + \frac{\eta^2}{2} \right| d\eta = h_i^3 |v_i| \int_0^1 |\chi_i(\eta)|\, d\eta,$$

where $\chi_i(\eta)$ is defined by (3). Denote

$$g_i(\eta) = \frac{q_i - \Delta z_i}{v_i h_i^2}\eta + \frac{u_i}{v_i h_i}\frac{\eta^2}{2} + \frac{\eta^3}{6}.$$

Clearly, we have $g_i'(\eta) = \chi_i(\eta)$ and $g_i(0) = 0$. There are two situations to consider. Situation (A): $\left(\frac{u_i}{v_i h_i}\right)^2 - 2\left(\frac{q_i - \Delta z_i}{v_i h_i^2}\right) \leq 0$.Then $\chi_i(\eta) \geq 0$ and we have

$$f_i = h_i^3|v_i| \int_0^1 \chi_i(\eta) d\eta = h_i^3|v_i| g_i(1) = |v_i| \left(\frac{h_i(q_i - \Delta z_i)}{v_i} + \frac{u_i h_i^2}{2v_i} + \frac{h_i^3}{6} \right).$$

Situation (B): $\left(\frac{u_i}{v_i h_i}\right)^2 - 2\left(\frac{q_i - \Delta z_i}{v_i h_i^2}\right) \geq 0$. Then the quadratic function $\chi_i(\eta) = 0$ has roots given by (4) and (5). From $\chi_i(\eta_1^{(i)}) = \chi_i(\eta_2^{(i)}) = 0$, we see that

$$\frac{(\eta_1^{(i)})^2}{2} = -\left(\frac{q_i - \Delta z_i}{v_i h_i^2} + \frac{u_i}{v_i h_i}\eta_1^{(i)} \right), \quad \frac{(\eta_2^{(i)})^2}{2} = -\left(\frac{q_i - \Delta z_i}{v_i h_i^2} + \frac{u_i}{v_i h_i}\eta_2^{(i)} \right).$$

Therefore,

$$\begin{aligned}
g_i(\eta_1^{(i)}) &= \eta_1^{(i)} \left(\frac{q_i - \Delta z_i}{v_i h_i^2} + \frac{u_i}{v_i h_i}\frac{\eta_1^{(i)}}{2} + \frac{1}{3}\left(-\frac{q_i - \Delta z_i}{v_i h_i^2} - \frac{u_i}{v_i h_i}\eta_1^{(i)} \right) \right) \\
&= \frac{2\eta_1^{(i)}}{3}\left(\frac{q_i - \Delta z_i}{v_i h_i^2} \right) + \frac{u_i}{v_i h_i}\frac{(\eta_1^{(i)})^2}{6} \\
&= \frac{2\eta_1^{(i)}}{3}\left(\frac{q_i - \Delta z_i}{v_i h_i^2} \right) + \frac{1}{3}\left(\frac{u_i}{v_i h_i} \right)\left(-\frac{q_i - \Delta z_i}{v_i h_i^2} - \frac{u_i}{v_i h_i}\eta_1^{(i)} \right) \\
&= \eta_1^{(i)} \left[\frac{2}{3}\left(\frac{q_i - \Delta z_i}{v_i h_i^2} \right) - \frac{1}{3}\left(\frac{u_i}{v_i h_i} \right)^2 \right] - \frac{1}{3}\left(\frac{u_i}{v_i h_i} \right)\left(\frac{q_i - \Delta z_i}{v_i h_i^2} \right).
\end{aligned}$$

Similarly,

$$g_i(\eta_2^{(i)}) = \eta_2^{(i)} \left[\frac{2}{3}\left(\frac{q_i - \Delta z_i}{v_i h_i^2} \right) - \frac{1}{3}\left(\frac{u_i}{v_i h_i} \right)^2 \right] - \frac{1}{3}\left(\frac{u_i}{v_i h_i} \right)\left(\frac{q_i - \Delta z_i}{v_i h_i^2} \right).$$

Substituting $\eta_1^{(i)}$ and $\eta_2^{(i)}$ into the above two equations, we have

$$g_i(\eta_1^{(i)}) = \frac{u_i}{v_i h_i}\left[\frac{1}{3}\left(\frac{u_i}{v_i h_i} \right)^2 - \frac{q_i - \Delta z_i}{v_i h_i^2} \right] + \frac{1}{3}\left[\left(\frac{u_i}{v_i h_i} \right)^2 - 2\left(\frac{q_i - \Delta z_i}{v_i h_i^2} \right) \right]^{3/2},$$

$$g_i(\eta_2^{(i)}) = \frac{u_i}{v_i h_i}\left[\frac{1}{3}\left(\frac{u_i}{v_i h_i} \right)^2 - \frac{q_i - \Delta z_i}{v_i h_i^2} \right] - \frac{1}{3}\left[\left(\frac{u_i}{v_i h_i} \right)^2 - 2\left(\frac{q_i - \Delta z_i}{v_i h_i^2} \right) \right]^{3/2}.$$

We now have four subcases to consider: *Subcase (B$_1$):* $\eta_1^{(i)}, \eta_2^{(i)} \notin (0, 1)$. This leads to three possibilities: $\eta_1^{(i)} \leq \eta_2^{(i)} \leq 0$, $1 \leq \eta_1^{(i)} \leq \eta_2^{(i)}$, and $\eta_1^{(i)} \leq 0 < 1 \leq \eta_2^{(i)}$. For each of these possibilities, $\chi_i(\eta)$ retains the same sign on $[0,1]$. For the first two possibilities, we have $\chi_i(\eta) \geq 0$ on $[0,1]$. Therefore,

$$f_i = h_i^3|v_i| \int_0^1 \chi_i(\eta)\mathrm{d}\eta = |v_i| \left(\frac{h_i(q_i - \Delta z_i)}{v_i} + \frac{u_i h_i^2}{2v_i} + \frac{h_i^3}{6} \right).$$

For the third possibility with $\eta_1^{(i)} \leq 0 < 1 \leq \eta_2^{(i)}$, we see that $\chi_i(\eta) \leq 0$ on $[0,1]$. Thus

$$f_i = -|v_i| \left(\frac{h_i(q_i - \Delta z_i)}{v_i} + \frac{u_i h_i^2}{2v_i} + \frac{h_i^3}{6} \right).$$

Subcase (B$_2$): $\eta_1^{(i)} \leq 0 \leq \eta_2^{(i)} \leq 1$. In this situation, we have

$$\begin{aligned}
f_i &= h_i^3|v_i| \left(\int_0^{\eta_2^{(i)}} |\chi_i(\eta)|\mathrm{d}\eta + \int_{\eta_2^{(i)}}^1 |\chi_i(\eta)|\mathrm{d}\eta \right) \\
&= h_i^3|v_i| \left(\int_0^{\eta_2^{(i)}} -\chi_i(\eta)\mathrm{d}\eta + \int_{\eta_2^{(i)}}^1 \chi_i(\eta)\mathrm{d}\eta \right) \\
&= h_i^3|v_i|(g_i(1) - 2g_i(\eta_2^{(i)})) \\
&= |v_i| \left\{ \frac{6h_i(q_i - \Delta z_i) + 3u_i h_i^2 + h_i^3 v_i}{6v_i} - \frac{2u_i}{v_i} \left[\frac{1}{3} \left(\frac{u_i}{v_i} \right)^2 - \frac{q_i - \Delta z_i}{v_i} \right] \right. \\
&\quad \left. + \frac{2}{3} \left[\left(\frac{u_i}{v_i} \right)^2 - 2 \left(\frac{q_i - \Delta z_i}{v_i} \right) \right]^{3/2} \right\}.
\end{aligned}$$

Subcase (B$_3$): $0 \leq \eta_1^{(i)} \leq 1 \leq \eta_2^{(i)}$. Then we have

$$\begin{aligned}
f_i &= h_i^3|v_i| \left(\int_0^{\eta_1^{(i)}} \chi_i(\eta)\mathrm{d}\eta - \int_{\eta_1^{(i)}}^1 \chi_i(\eta)\mathrm{d}\eta \right) = h_i^3|v_i|(2g_i(\eta_1^{(i)}) - g_i(1)) \\
&= |v_i| \left\{ \frac{2u_i}{v_i} \left[\frac{1}{3} \left(\frac{u_i}{v_i} \right)^2 - \frac{q_i - \Delta z_i}{v_i} \right] + \frac{2}{3} \left[\left(\frac{u_i}{v_i} \right)^2 - 2 \left(\frac{q_i - \Delta z_i}{v_i} \right) \right]^{3/2} \right. \\
&\quad \left. - \frac{6h_i(q_i - \Delta z_i) + 3u_i h_i^2 + v_i h_i^3}{6v_i} \right\}.
\end{aligned}$$

Subcase (B$_4$): $0 \leq \eta_1^{(i)} \leq \eta_2^{(i)} \leq 1$. Then we have

$$\begin{aligned}
f_i &= h_i^3|v_i| \left(\int_0^{\eta_1^{(i)}} \chi_i(\eta)\mathrm{d}\eta - \int_{\eta_1^{(i)}}^{\eta_2^{(i)}} \chi_i(\eta)\mathrm{d}\eta + \int_{\eta_2^{(i)}}^1 \chi_i(\eta)\mathrm{d}\eta \right) \\
&= h_i^3|v_i|(g_i(1) + 2g_i(\eta_1^{(i)}) - 2g_i(\eta_2^{(i)})) \\
&= |v_i| \left(\frac{6h_i(q_i - \Delta z_i) + 3u_i h_i^2 + v_i h_i^3}{6v_i} + \frac{4}{3} \left[\left(\frac{u_i}{v_i} \right)^2 - 2 \left(\frac{q_i - \Delta z_i}{v_i} \right) \right]^{3/2} \right).
\end{aligned}$$

The proof is complete. □

It is important to note that the function f_i in Proposition 2.1 is non-differentiable but convex. The convexity of $f_i(q_i, u_i, v_i)$ is a consequence of the following lemma when $m = 3,\ c = -\Delta z_i,\ \phi_1 = 1,\ \phi_2 = h_i\eta$ and $\phi_3 = \frac{h_i^2}{2}\eta^2$.

Lemma 2.1 *Let $\phi_i(\eta)\colon\ R \to\ R, i = 1, \ldots, m$, be univariate continuous functions and let $\psi\colon\ R^m \to\ R_+$ be the function defined by*

$$\psi(x_1, x_2, ..., x_m) = \int_a^b \left| c + \sum_{i=1}^m x_i\phi_i(\eta) \right| \mathrm{d}\eta,$$

where $a < b$ and c is a constant. Then ψ is a convex function.

Proof Let $x = (x_1, x_2, ..., x_m)$ and $y = (y_1, y_2, ..., y_m)$. For any $\lambda \in [0, 1]$

$$\begin{aligned}
&\psi(\lambda x + (1-\lambda)y)\\
&= \int_a^b \left| c + \sum_{i=1}^m [\lambda x_i + (1-\lambda)y_i]\phi_i(\eta) \right| \mathrm{d}\eta\\
&= \int_a^b \left| \lambda\left(c + \sum_{i=1}^m x_i\phi_i(\eta) \right) + (1-\lambda)\left(c + \sum_{i=1}^m y_i\phi_i(\eta) \right) \right| \mathrm{d}\eta\\
&\le \int_a^b \left| \lambda\left(c + \sum_{i=1}^m x_i\phi_i(\eta) \right) \right| \mathrm{d}\eta + \int_a^b \left| (1-\lambda)\left(c + \sum_{i=1}^m y_i\phi_i(\eta) \right) \right| \mathrm{d}\eta\\
&= \lambda\psi(x) + (1-\lambda)\psi(y).
\end{aligned}$$

Hence, ψ is a convex function. □

The following facts will be used in the next section.

Lemma 2.2

(a) The condition of "$v_i \neq 0$ and $0 \le \eta_1^{(i)} \le \eta_2^{(i)} \le 1$" is equivalent to

$$\left(\frac{u_i}{v_i}\right)^2 \ge 2\left(\frac{q_i - \Delta z_i}{v_i}\right) \ge 2\max\left\{0, -\left(\frac{h_i^2}{2} + \frac{u_i h_i}{v_i}\right)\right\} \text{ and } \frac{u_i}{v_i} \in [-h_i, 0]. \quad (6)$$

(b) The condition "$v_i \neq 0$ and $\eta_1^{(i)} \le 0 \le \eta_2^{(i)} \le 1$" is equivalent to

$$0 \ge \frac{q_i - \Delta z_i}{v_i} \ge -\left(\frac{h_i^2}{2} + \frac{u_i h_i}{v_i}\right).$$

(c) The condition of "$v_i \neq 0$ and $0 \le \eta_1^{(i)} \le 1 \le \eta_2^{(i)}$" is equivalent to

$$0 \le \frac{q_i - \Delta z_i}{v_i} \le -\left(\frac{h_i^2}{2} + \frac{u_i h_i}{v_i}\right).$$

Proof (1) Note that $\eta_1^{(i)} \ge 0$ if and only if

$$\sqrt{\left(\frac{u_i}{v_i h_i}\right)^2 - 2\left(\frac{q_i - \Delta z_i}{v_i h_i^2}\right)} \le -\frac{u_i}{v_i h_i} \quad (7)$$

and $\eta_2^{(i)} \le 1$ if and only if

$$\sqrt{\left(\frac{u_i}{v_i h_i}\right)^2 - 2\left(\frac{q_i - \Delta z_i}{v_i h_i^2}\right)} \le 1 + \frac{u_i}{v_i h_i}. \tag{8}$$

The two inequalities imply that

$$\left(\frac{u_i}{v_i h_i}\right)^2 - 2\left(\frac{q_i - \Delta z_i}{v_i h_i^2}\right) \ge 0, \quad \frac{u_i}{v_i h_i} \in [-1, 0].$$

Squaring both sides of (7) and (8) yields

$$\frac{q_i - \Delta z_i}{v_i h_i^2} \ge 0, \quad \frac{q_i - \Delta z_i}{v_i h_i^2} \ge -\left(\frac{1}{2} + \frac{u_i}{v_i h_i}\right).$$

The combination of these inequalities leads to (6). The converse is also true. In fact, it is easy to see that (6) implies both (7) and (8), and hence (6) is equivalent to $0 \le \eta_1^{(i)} \le \eta_2^{(i)} \le 1$.
(b) Note that

$$\begin{aligned} \eta_1^{(i)} \le 0 \text{ and } \eta_2^{(i)} \ge 0 &\Leftrightarrow \sqrt{\left(\frac{u_i}{v_i h_i}\right)^2 - 2\left(\frac{q_i - \Delta z_i}{v_i h_i^2}\right)} \ge \left|\frac{u_i}{v_i h_i}\right| \\ &\Leftrightarrow \frac{q_i - \Delta z_i}{v_i h_i^2} \le 0 \end{aligned} \tag{9}$$

and $\eta_2^{(i)} \le 1$ is equivalent to (8). Squaring both sides of (8) and using (9), we have

$$0 \ge \frac{q_i - \Delta z_i}{v_i h_i^2} \ge -\left(\frac{1}{2} + \frac{u_i}{v_i h_i}\right). \tag{10}$$

Conversely, (10) implies that

$$\left(\frac{u_i}{v_i h_i}\right)^2 - 2\left(\frac{q_i - \Delta z_i}{v_i h_i^2}\right) \ge 0 \text{ and } \frac{u_i}{v_i h_i} \ge -\frac{1}{2} > -1.$$

It is not difficult to verify that this inequality and (10) imply that $\eta_1^{(i)} \le 0 \le \eta_2^{(i)} \le 1$.
(c) Statement (c) can be proved in a manner similar to that in which (b) was proved. □

3 Derivation of the conjugate function of $\mathcal{F}(\mathcal{X})$

The existing algorithms for non-differentiable optimization are in general not as efficient as those for smooth optimization. This motivates us to study the dual problem of (2). We will show that the geometric dual problem of (2) is a smooth convex programming problem. Hence, we have a wider set of algorithms, including interior-point algorithms, to solve the dual problem. Moreover, a dual model may provide a convenient framework for the theoretical analysis and algorithmic design. We devote this section to deriving the conjugate function of the objective function $\mathcal{F}(\mathcal{X})$. As we will see, this is not an easy task even though the original function is convex.

We will use notion and terminology of [26]. For a given convex function $\psi(x): R^m \to (-\infty, \infty]$, we denote by $\mathrm{dom}(\psi)$ the effective domain of ψ, that is, $\mathrm{dom}(\psi) = \{x : \psi(x) < \infty\}$. The conjugate function, also called the Fenchel conjugate transformation, of ψ is defined as

$$\psi^*(y) = \sup_{x \in R^m} (x^T y - \psi(x)).$$

In particular, the conjugate function of the indicator function $\delta(\cdot|D)$ of a convex set D is the support function of D, that is, $\delta^*(x^*|D) = \sup_{x \in D} x^T x^*$. For the single-point set $C_i = \{z_i\}$, we have

$$\delta^*(x^*|C_i) = z_i x^* \;\text{ and }\; \mathrm{dom}(\delta^*(\cdot|C_i)) = R. \tag{11}$$

Denote by $\mathcal{W} \in R^{4n+1}$ the vector

$$\left(w_0^{(0)}, w_1^{(0)}, w_2^{(0)}, w_3^{(0)}, w_0^{(1)}, w_1^{(1)}, w_2^{(1)}, w_3^{(1)}, ..., w_0^{(n-1)}, w_1^{(n-1)}, w_2^{(n-1)}, w_3^{(n-1)}, w_0^{(n)}\right).$$

The conjugate of $\mathcal{F}(\mathcal{X})$ can be expressed as follows.

$$\begin{aligned}
&\mathcal{F}^*(\mathcal{W}) \\
&= \sup_{\mathcal{X} \in R^{4n+1}} \left(\mathcal{W}^T \mathcal{X} - \mathcal{F}(\mathcal{X})\right) \\
&= \sup_{\mathcal{X} \in R^{4n+1}} \left(\mathcal{W}^T \mathcal{X} - \sum_{i=0}^{n} \delta(p_i|C_i) - \sum_{i=0}^{n-1} f_i(q_i, u_i, v_i)\right) \\
&= \sup_{\mathcal{X} \in R^{4n+1}} \left(\sum_{i=0}^{n} (w_0^{(i)} p_i - \delta(p_i|C_i)) + \sum_{i=0}^{n-1} \left(w_1^{(i)} q_i + w_2^{(i)} u_i + w_3^{(i)} v_i - f_i(q_i, u_i, v_i)\right)\right) \\
&= \sum_{i=0}^{n} \sup_{p_i \in R} (w_0^{(i)} p_i - \delta(p_i|C_i)) + \sum_{i=0}^{n-1} \sup_{(q_i, u_i, v_i) \in R^3} \left(w_1^{(i)} q_i + w_2^{(i)} u_i + w_3^{(i)} v_i - f_i(q_i, u_i, v_i)\right) \\
&= \sum_{i=0}^{n} \delta^*(w_0^{(i)}|C_i) + \sum_{i=0}^{n-1} f_i^*(w_1^{(i)}, w_2^{(i)}, w_3^{(i)}) \\
&= \sum_{i=0}^{n} z_i w_0^{(i)} + \sum_{i=0}^{n-1} f_i^*(w_1^{(i)}, w_2^{(i)}, w_3^{(i)}).
\end{aligned} \tag{12}$$

Thus, it suffices to calculate the conjugate function of $f_i(q_i, u_i, v_i)$, $i = 0, ..., n-1$. In this section, for convenience and without loss of generality, we omit the index i of all variables and constants. For instance, we use h for knot space h_i, (w_1, w_2, w_3) for vector $(w_1^{(i)}, w_2^{(i)}, w_3^{(i)})$, (q, u, v) for vector (q_i, u_i, v_i), and so on. We also write $w = (w_1, w_2, w_3)^T$ for simplicity.

We note that the domain of f_i (i.e., equal to R^3) can be partitioned into the following sub-regions:

$$\begin{aligned}
\Gamma_1 &= \left\{(q,u,v) : v \neq 0,\ \left(\frac{u}{v}\right)^2 \leq 2\left(\frac{q-\Delta z}{v}\right)\right\},\\
\Gamma_2 &= \left\{(q,u,v) : v \neq 0, \left(\frac{u}{v}\right)^2 \geq \frac{2(q-\Delta z)}{v} \geq 2\max\left\{0, -\frac{h^2}{2} - \frac{uh}{v}\right\}, \frac{u}{v} \geq 0\right\},\\
\Gamma_3 &= \left\{(q,u,v) : v \neq 0, \left(\frac{u}{v}\right)^2 \geq \frac{2(q-\Delta z)}{v} \geq 2\max\left\{0, -\frac{h^2}{2} - \frac{uh}{v}\right\}, \frac{u}{v} \leq -h\right\},\\
\Gamma_4 &= \left\{(q,u,v) : v \neq 0, \left(\frac{u}{v}\right)^2 \geq \frac{2(q-\Delta z)}{v} \geq 2\max\left\{0, -\frac{h^2}{2} - \frac{uh}{v}\right\}, \frac{u}{v} \in [-h,0]\right\},\\
\Gamma_5 &= \left\{(q,u,v) : v \neq 0,\ 0 \geq \frac{q-\Delta z}{v} \geq -\frac{h^2}{2} - \frac{uh}{v}\right\},\\
\Gamma_6 &= \left\{(q,u,v) : v \neq 0,\ 0 \leq \frac{q-\Delta z}{v} \leq -\frac{h^2}{2} - \frac{uh}{v})\right\},\\
\Gamma_7 &= \left\{(q,u,v) : v \neq 0,\ \frac{q-\Delta z}{v} \leq \min\left\{0, -\frac{h^2}{2} - \frac{uh}{v}\right\}\right\},\\
\Gamma_8 &= \{(q,u,v) : v = 0, u = 0\},\\
\Gamma_9 &= \left\{(q,u,v) : v = 0, u \neq 0, \frac{q-\Delta z}{u} \in [-h,0]\right\},\\
\Gamma_{10} &= \left\{(q,u,v) : v = 0, u \neq 0, \frac{q-\Delta z}{u} \geq 0\right\},\\
\Gamma_{11} &= \left\{(q,u,v) : v = 0, u \neq 0, \frac{q-\Delta z}{u} \leq -h\right\}.
\end{aligned}$$

It is easy to see that $\cup_{i=1}^{11}\Gamma_i = \text{dom}\,(f_i) = R^3$ and

$$\Gamma_5 \cup \Gamma_6 = \left\{(q,u,v) : v \neq 0, \max\left\{0, -\frac{h^2}{2} - \frac{uh}{v}\right\} \geq \frac{q-\Delta z}{v} \geq \min\left\{0, -\frac{h^2}{2} - \frac{uh}{v}\right\}\right\}.$$

By Lemma 2.2, we know that $0 \leq \eta_1^{(i)} \leq \eta_2^{(i)} \leq 1$ corresponds to Γ_4, $\eta_1^{(i)} \leq 0 \leq \eta_2^{(i)} \leq 1$ corresponds to Γ_5 and $0 \leq \eta_1^{(i)} \leq 1 \leq \eta_2^{(i)}$ corresponds to Γ_6. Denote

$$\mathcal{G}_j(w_1,w_2,w_3) = \sup_{(q,u,v)\in\Gamma_j} \left(w_1 q + w_2 u + w_3 v - f_i(q,u,v)\right),\ j = 1, ..., 11.$$

Then,

$$\begin{aligned}
f_i^*(w_1,w_2,w_3) &= \sup_{(q,u,v)\in R^3} \left(w_1 q + w_2 u + w_3 v - f_i(q,u,v)\right)\\
&= \sup_{(q,u,v)\in\bigcup_{j=1}^{11}\Gamma_j} \left(w_1 q + w_2 u + w_3 v - f_i(q,u,v)\right)\\
&= \max_{1\leq j\leq 11}\ \sup_{(q,u,v)\in\Gamma_j} \left(w_1 q + w_2 u + w_3 v - f_i(q,u,v)\right)\\
&= \max_{1\leq j\leq 11} \mathcal{G}_j(w_1,w_2,w_3) \qquad (13)
\end{aligned}$$

and

$$\mathrm{dom}(f_i^*) = \bigcap_{j=1}^{11} \mathrm{dom}(\mathcal{G}_j). \tag{14}$$

Since the value and domain of the conjugate function $f_i^*(w_1, w_2, w_3)$ are completely determined by those of $\mathcal{G}_j(w_1, w_2, w_3)$, it is sufficient to calculate the value and domain of $\mathcal{G}_j, j = 1, ..., 11$.

From Proposition 2.1, we see that the most complicated branches of the function f_i occur at regions Γ_5 and Γ_6. Here we focus on the calculation of the value and domain of $\mathcal{G}_5(w_1, w_2, w_3)$. The value and domain for $\mathcal{G}_6$ can be obtained analogously. The derivations required for other cases with $v \neq 0$ are easier than that for $\mathcal{G}_5(w_1, w_2, w_3)$ and can be carried out with similar techniques.

The function f_i over Γ_5 is given by

$$f_i = |v|\left\{\frac{6h(q-\Delta z) + 3uh^2 + vh^3}{6v} - \frac{2u}{v}\left[\frac{1}{3}\left(\frac{u}{v}\right)^2 - \frac{q-\Delta z}{v}\right]\right.$$
$$\left. + \frac{2}{3}\left[\left(\frac{u}{v}\right)^2 - \frac{2(q-\Delta z)}{v}\right]^{3/2}\right\}.$$

Let

$$\alpha = \frac{q-\Delta z}{v} \quad \text{and} \quad t = \frac{u}{v}. \tag{15}$$

Note that α and t can be independent of a change in v, provided q and u change appropriately. Using (15), f_i reduces to

$$f_i = |v|\left\{\alpha h + \frac{h^2}{2}t + \frac{h^3}{6} - \frac{2}{3}t^3 + 2t\alpha + \frac{2}{3}(t^2 - 2\alpha)^{3/2}\right\}$$

and Γ_5 reduces to $\left\{(\alpha, t, v) : v \neq 0,\ 0 \geq \alpha \geq -\left(\frac{h^2}{2} + th\right)\right\}$.

Let $S = \left\{(\alpha, t) :\ 0 \geq \alpha \geq -\left(\frac{h^2}{2} + th\right)\right\}$. We have

$$\begin{aligned}
& w_1 q + w_2 u + w_3 v - f_i \\
&= w_1 \Delta z + w_1 \alpha v + w_2 t v + w_3 v - f_i \\
&= \begin{cases} w_1 \Delta z + \left\{(w_1 - h)\alpha + \left(w_2 - \frac{h^2}{2}\right)t + w_3 - \frac{h^3}{6} - 2t\alpha + \frac{2}{3}t^3 - \frac{2}{3}(t^2 - 2\alpha)^{\frac{3}{2}}\right\} v, \\ \text{if } v > 0 \\ w_1 \Delta z + \left\{(w_1 + h)\alpha + \left(w_2 + \frac{h^2}{2}\right)t + w_3 + \frac{h^3}{6} + 2t\alpha - \frac{2}{3}t^3 + \frac{2}{3}(t^2 - 2\alpha)^{\frac{3}{2}}\right\} v, \\ \text{if } v < 0 \end{cases} \\
&= w_1 \Delta z + \psi_w(\alpha, t, v),
\end{aligned}$$

where w represents (w_1, w_2, w_3) and

$$\psi_w(\alpha, t, v) := \begin{cases} g_w(\alpha, t)v, & \text{if } v > 0, \\ y_w(\alpha, t)v, & \text{if } v < 0, \end{cases} \tag{16}$$

$$g_w(\alpha, t) := (w_1 - h)\alpha + \left(w_2 - \frac{h^2}{2}\right)t + w_3 - \frac{h^3}{6} - 2t\alpha + \frac{2}{3}t^3 - \frac{2}{3}(t^2 - 2\alpha)^{3/2},$$

$$y_w(\alpha,t) := (w_1+h)\alpha + \left(w_2+\frac{h^2}{2}\right)t + w_3 + \frac{h^3}{6} + 2t\alpha - \frac{2}{3}t^3 + \frac{2}{3}(t^2-2\alpha)^{3/2}.$$

It is easy to check that $g_w(\alpha,t)$ is concave on S, while $y_w(\alpha,t)$ is convex on S. Note that $v \neq 0$ can be taken as any value in R. Hence, $\mathcal{G}_5$ is finite if and only if $g_w(\alpha,t) \leq 0$ and $y_w(\alpha,t) \geq 0$ for all $(\alpha,t) \in S$, that is,

$$\sup_{(\alpha,t)\in S} g_w(\alpha,t) \leq 0 \quad \text{and} \quad \inf_{(\alpha,t)\in S} y_w(\alpha,t) \geq 0, \tag{17}$$

which leads to

$$\sup_{v\neq 0,(\alpha,t)\in S} \psi_w(\alpha,t,v) = 0.$$

Then, whenever $\mathcal{G}_5(w_1,w_2,w_3)$ is finite, we have

$$\begin{aligned}\mathcal{G}_5(w_1,w_2,w_3) &= \sup_{v\neq 0,(\alpha,t)\in S} \Delta z w_1 + \psi_w(\alpha,t,v)\\ &= \Delta z w_1 + \sup_{v\neq 0,(\alpha,t)\in S} \psi_w(\alpha,t,v) \qquad (18)\\ &= \Delta z w_1.\end{aligned}$$

This provides the value of $\mathcal{G}_5$. It remains to determine the effective domain of $\mathcal{G}_5$, that is, the set of all points $w = (w_1,w_2,w_3)$ satisfying (17).

To derive conditions that w must satisfy, we first assume that (17) holds.

Lemma 3.1 *Let $\mathcal{B}^{(1)}$ be the boundary of S with $\alpha = 0$, that is, $\mathcal{B}^{(1)} = \left\{(0,t) : -\frac{h^2}{2} - ht \leq 0\right\}$ $= \left\{(0,t) : t \in \left[-\frac{h}{2},\infty\right)\right\}$. Then $g_w(\alpha,t) \leq 0$ on $\mathcal{B}^{(1)}$ if and only if w belongs to the following set:*

$$\begin{aligned}\mathcal{D}_1 := &\left\{w : |w_2| \leq \frac{h^2}{2}, w_3 \leq \frac{h^3}{6} - \frac{1}{3}\left(\frac{h^2}{2} - w_2\right)^{3/2}\right\}\\ &\cup \left\{w : w_2 \leq -\frac{h^2}{2}, w_3 - \frac{h}{2}w_2 \leq \frac{h^3}{12}\right\}.\end{aligned}$$

Proof On $\mathcal{B}^{(1)}$, $g_w(\alpha,t)$ reduces to

$$\nu_w(t) := g_w(0,t) = \left(w_2 - \frac{h^2}{2}\right)t + w_3 - \frac{h^3}{6} - \frac{2}{3}(|t|^3 - t^3).$$

We first consider the subinterval $\left[-\frac{h}{2},0\right]$ over which the function reduces to

$$\nu_w(t) := \left(w_2 - \frac{h^2}{2}\right)t + w_3 - \frac{h^3}{6} + \frac{4}{3}t^3.$$

Since ν_w is a univariate concave function on $\left[-\frac{h}{2},0\right]$, there are three possible cases:

(a) The first derivative $\nu_w'(t) \geq 0$ on $\left[-\frac{h}{2},0\right]$. In this case, $t^* = 0$ is a maximum point with the maximum value $\nu_w(t^*) = \nu_w(0) = w_3 - \frac{h^3}{6}$. Since $\nu_w'(t) = w_2 - \frac{h^2}{2} + 4t^2$, the condition $\nu_w'(t) \geq 0$ on $\left[-\frac{h}{2}, 0\right]$ is equivalent to $w_2 \geq \frac{h^2}{2}$. Thus, $\nu_w(t) \leq 0$ on $\left[-\frac{h}{2}, 0\right]$ if and only if w satisfies the following condition:

$$\left\{w \in R^3 : w_2 \geq \frac{h^2}{2}, w_3 \leq \frac{h^3}{6}\right\}.$$

(b) $\nu_w'(t) \leq 0$ on $\left[-\frac{h}{2}, 0\right]$. $t^* = -\frac{h}{2}$ is a maximum point with the maximum value $\nu_w(t^*) = \nu_w(\frac{-h}{2}) = w_3 - \frac{h}{2}w_2 - \frac{h^3}{12}$. Hence, $\nu_w(t) \leq 0$ on $\left[-\frac{h}{2}, 0\right]$ if and only if w satisfies the following condition:

$$\left\{w \in R^3 : w_2 \leq -\frac{h^2}{2}, w_3 - \frac{h}{2}w_2 \leq \frac{h^3}{12}\right\}.$$

(c) $\nu'(t^*) = 0$ for some $t^* \in \left[-\frac{h}{2}, 0\right]$. Then, $4(t^*)^2 = \frac{h^2}{2} - w_2$, which implies that $|w_2| \leq \frac{h^2}{2}$. Consequently, $-t^* = \frac{1}{2}\sqrt{\frac{h^2}{2} - w_2}$ and the maximum value of ν is given by

$$\nu_w(t^*) = w_3 - \frac{h^3}{6} + \frac{1}{3}\left(\frac{h^2}{2} - w_2\right)^{3/2}.$$

In this case, $\nu_w(t) \leq 0$ on $\left[-\frac{h}{2}, 0\right]$ if and only if w satisfies the following condition:

$$\left\{w \in R^3 : |w_2| \leq \frac{h^2}{2}, w_3 \leq \frac{h^3}{6} - \frac{1}{3}\left(\frac{h^2}{2} - w_2\right)^{3/2}\right\}.$$

Summarizing the three cases, we know that $\nu_w(t) \leq 0$ on $\left[-\frac{h}{2}, 0\right]$ if and only if

$$\begin{aligned} w \in \hat{D} := &\left\{w : w_2 \geq \frac{h^2}{2}, w_3 \leq \frac{h^3}{6}\right\} \cup \left\{w : w_2 \leq -\frac{h^2}{2}, w_3 - \frac{h}{2}w_2 \leq \frac{h^3}{12}\right\} \\ &\cup \left\{w : \ |w_2| \leq \frac{h^2}{2}, \ w_3 \leq \frac{h^3}{6} - \frac{1}{3}\left(\frac{h^2}{2} - w_2\right)^{3/2}\right\}. \end{aligned}$$

Now consider the function over $[0, \infty)$. We see that $\nu_w(t) = \left(w_2 - \frac{h^2}{2}\right)t + w_3 - \frac{h^3}{6}$. Clearly, $\nu_w(t) \leq 0$ for $t \in [0, \infty)$ if and only if

$$w \in \tilde{D} := \left\{w \in R^3 : w_2 \leq \frac{h^2}{2}, \ w_3 \leq \frac{h^3}{6}\right\}.$$

Therefore, on the whole interval $\left[-\frac{h}{2}, \infty\right)$, the function $\nu_w(t) \leq 0$ if and only if $w \in \mathcal{D}_1 := \hat{D} \cap \tilde{D}$, which is the desired result. □

Lemma 3.2 *Let* $\mathcal{B}^{(2)} := \{(\alpha, t) : 0 \geq \alpha = -(\frac{h^2}{2} + ht)\}$, *that is, the boundary of* S *with* $\alpha = -\left(\frac{h^2}{2} + ht\right)$. *Then* $g_w(\alpha, t) \leq 0$ *over* $\mathcal{B}^{(2)}$ *if and only if* w *belongs to the following set:*

$$\mathcal{D}_2 := \left\{w \in R^3 : \ w_2 - w_1 h \leq \frac{h^2}{2}, \ w_3 - \frac{h}{2}w_2 \leq \frac{h^3}{12}\right\}.$$

Proof Substitute $\alpha = -\left(\frac{h^2}{2} + ht\right)$ into $g_w(\alpha, t)$ and note that $t + h \geq t + \frac{h}{2} \geq 0$ for $(\alpha, t) \in S$. It is easy to see that

$$
\begin{aligned}
& g_w(\alpha, t) \\
& = -(w_1 - h)\left(\frac{h^2}{2} + ht\right) + \left(w_2 - \frac{h^2}{2}\right)t + w_3 - \frac{h^3}{6} + 2t\left(\frac{h^2}{2} + ht\right) \\
& \quad + \frac{2}{3}t^3 - \frac{2}{3}(t+h)^3 \\
& = w_3 - \frac{h^2}{2}w_1 - \frac{h^3}{3} + \left(w_2 - \frac{h^2}{2} - w_1 h\right)t \\
& = w_3 - \frac{h}{2}w_2 - \frac{h^3}{12} + \left(w_2 - \frac{h^2}{2} - w_1 h\right)\left(t + \frac{h}{2}\right).
\end{aligned}
$$

It follows that $g_w(\alpha, t) \leq 0$ on $\mathcal{B}^{(2)}$ if and only if w satisfies the following conditions:

$$
w_2 - w_1 h \leq \frac{h^2}{2} \text{ and } w_3 - \frac{h}{2}w_2 \leq \frac{h^3}{12}. \qquad \square
$$

Now we see that, if $g_w(\alpha, t) \leq 0$ on S, then $g_w(\alpha, t) \leq 0$ holds on the boundary of S. The following result follows immediately from Lemmas 3.1 and 3.2.

Proposition 3.1 *If $g_w(\alpha, t)) \leq 0$ on S, then w in R^3 belongs to the following set:*

$$
\begin{aligned}
\mathcal{D}_1 \cap \mathcal{D}_2 = & \left\{ w: \ w_2 \leq -\frac{h^2}{2}, \ w_2 - w_1 h \leq \frac{h^2}{2}, \ w_3 - \frac{h}{2}w_2 \leq \frac{h^3}{12} \right\} \\
\cup & \left\{ w: |w_2| \leq \frac{h^2}{2}, w_2 - w_1 h \leq \frac{h^2}{2}, w_3 - \frac{h}{2}w_2 \leq \frac{h^3}{12}, w_3 \leq \frac{h^3}{6} - \frac{1}{3}\left(\frac{h^2}{2} - w_2\right)^{3/2} \right\}.
\end{aligned}
$$

Next we consider the conditions that w must satisfy if the function $y_w(\alpha, t) \geq 0$ on S. The analysis is analogous to that of $g_w(\alpha, t)$. We keep the notation $\mathcal{B}^{(1)}$ and $\mathcal{B}^{(2)}$ to denote the boundaries of S as before.

Lemma 3.3 *$y_w(\alpha, t) \geq 0$ on $\mathcal{B}^{(1)}$ if and only if $w \in R^3$ belongs to the following set:*

$$
\begin{aligned}
\mathcal{Q}_1 := & \left\{ w: w_2 \geq \frac{h^2}{2}, w_3 - \frac{h}{2}w_2 \geq -\frac{h^3}{12} \right\} \\
& \cup \left\{ w: |w_2| \leq \frac{h^2}{2}, w_3 \geq -\frac{h^3}{6} + \frac{1}{3}\left(\frac{h^2}{2} + w_2\right)^{3/2} \right\}.
\end{aligned} \tag{19}
$$

Proof Note that on $\mathcal{B}^{(1)}$, $y_w(\alpha, t)$ reduces to

$$
\mu_w(t) := y_w(0, t) = \left(w_2 + \frac{h^2}{2}\right)t + w_3 + \frac{h^3}{6} + \frac{2}{3}(|t|^3 - t^3).
$$

On $\left[-\frac{h}{2}, 0\right]$, the function further reduces to

$$
\mu_w(t) = \left(w_2 + \frac{h^2}{2}\right)t + w_3 + \frac{h^3}{6} - \frac{4}{3}t^3.
$$

Since $\mu_w'(t) = w_2 + \frac{h^2}{2} - 4t^2$ and $\mu_w(t)$ is convex on $\left[-\frac{h}{2}, 0\right]$, there are three possible cases for locating the minimum point.

(a) $\mu'_w(t) \leq 0$ on $\left[-\frac{h}{2}, 0\right]$. The minimum value of μ_w is attained at $t^* = 0$ with $\mu_w(t^*) = w_3 + \frac{h^3}{6}$. In this case, $\mu_w(t) \geq 0$ on $\left[-\frac{h}{2}, 0\right]$ if and only if w belongs to the set $\{w : w_2 \leq -\frac{h^2}{2}, w_3 \geq -\frac{h^3}{6}\}$.

(b) $\mu'_w(t) \geq 0$ on $\left[-\frac{h}{2}, 0\right]$. The minimum point is $t^* = -\frac{h}{2}$ with $\mu_w(t^*) = w_3 - \frac{h}{2}w_2 + \frac{h^3}{12}$. Hence, $\mu_w(t) \geq 0$ on $\left[-\frac{h}{2}, 0\right]$ if and only if w belongs to the set $\left\{w : w_2 \geq \frac{h^2}{2}, w_3 - \frac{h}{2}w_2 + \frac{h^3}{12} \geq 0\right\}$.

(c) $\mu'_w(t^*) = 0$ for some $t^* \in \left[-\frac{h}{2}, 0\right]$. In this case, $4(t^*)^2 = \frac{h^2}{2} + w_2$, which implies that $|w_2| \leq \frac{h^2}{2}$. The minimum value of μ is given by

$$\mu_w(t^*) = w_3 + \frac{h^3}{6} - \frac{1}{3}\left(\frac{h^2}{2} + w_2\right)^{3/2}.$$

Hence $\mu_w(t) \geq 0$ on $\left[-\frac{h}{2}, 0\right]$ if and only if w belongs to the following set

$$\left\{w \in R^3 : |w_2| \leq \frac{h^2}{2}, w_3 + \frac{h^3}{6} - \frac{1}{3}\left(\frac{h^2}{2} + w_2\right)^{3/2} \geq 0\right\}.$$

In summary, we know that $\mu_w(t) \geq 0$ on $\left[-\frac{h}{2}, 0\right]$ if and only if

$$w \in \mathcal{P}_1 := \left\{w : w_2 \leq -\frac{h^2}{2}, w_3 \geq -\frac{h^3}{6}\right\} \cup \left\{w : w_2 \geq \frac{h^2}{2}, w_3 - \frac{h}{2}w_2 \geq -\frac{h^3}{12}\right\}$$
$$\cup \left\{w : |w_2| \leq \frac{h^2}{2}, w_3 \geq -\frac{h^3}{6} + \frac{1}{3}\left(\frac{h^2}{2} + w_2\right)^{\frac{3}{2}}\right\}.$$

On the other hand, for $t \in [0, \infty)$, the function $\mu_w(t) = \left(w_2 + \frac{h^2}{2}\right)t + w_3 + \frac{h^3}{6}$. Therefore, $\mu_w(t) \geq 0$ on $[0, \infty)$ if and only if

$$w \in \mathcal{P}_2 := \left\{w \in R^3 : w_2 \geq -\frac{h^2}{2}, w_3 \geq -\frac{h^3}{6}\right\}.$$

Therefore, $y_w(\alpha, t) \geq 0$ on $\mathcal{B}^{(1)}$ if and only if $w \in \mathcal{P}_1 \cap \mathcal{P}_2$,which is equal to (19). □

Lemma 3.4 *$y_w(\alpha, t) \geq 0$ on $\mathcal{B}^{(2)}$ if and only if*

$$w \in \mathcal{Q}_2 := \left\{w \in R^3 : w_3 - \frac{h}{2}w_2 + \frac{h^3}{12} \geq 0,\ w_2 + \frac{h^2}{2} - w_1 h \geq 0\right\}.$$

Proof Substituting $\alpha = -\left(\frac{h^2}{2} + ht\right)$ into $y(\alpha, t)$ and noting that $t + h \geq t + \frac{h}{2} \geq 0$ for$(\alpha, t) \in S_i$, we have

$$\begin{aligned} y_w(\alpha, t) &= -(w_1 + h)\left(\frac{h^2}{2} + ht\right) + \left(w_2 + \frac{h^2}{2}\right)t + w_3 + \frac{h^3}{6} - 2t\left(\frac{h^2}{2} + ht\right) - \frac{2}{3}t^3 + \frac{2}{3}(t + h)^3 \\ &= w_3 - \frac{h}{2}w_2 + \frac{h^3}{12} + \left(w_2 + \frac{h^2}{2} - w_1 h\right)\left(t + \frac{h}{2}\right). \end{aligned}$$

Consequently, $y_w(\alpha, t) \geq 0$ on $\mathcal{B}^{(2)}$ if and only if w satisfies that $w_3 - \frac{h}{2}w_2 \geq -\frac{h^3}{12}$ and $w_2 - w_1 h \geq -\frac{h^2}{2}$. □

Adding up Lemmas 3.3 and 3.4, we have the following result.

Proposition 3.2 *If $y_w(\alpha,t) \geq 0$ on S, then*

$$w \in \mathcal{Q}_1 \cap \mathcal{Q}_2 = \left\{ w : w_2 \geq \frac{h^2}{2},\ w_2 - w_1 h \geq -\frac{h^2}{2}, w_3 - \frac{h}{2} w_2 \geq -\frac{h^3}{12} \right\}$$
$$\cup \left\{ w : |w_2| \leq \frac{h^2}{2}, w_3 - \frac{h}{2} w_2 \geq -\frac{h^3}{12}, w_2 - w_1 h \geq -\frac{h^2}{2}, w_3 \geq -\frac{h^3}{6} + \frac{1}{3}\left(\frac{h^2}{2} + w_2\right)^{3/2} \right\}.$$

With Propositions 3.1 and 3.2, we can conclude that condition (17) implies that w is in

$$\Omega := (\mathcal{D}_1 \cap \mathcal{D}_2) \cap (\mathcal{Q}_1 \cap \mathcal{Q}_2). \tag{20}$$

We want to show that this set is the effective domain of the conjugate function $\mathcal{G}_5$. It is sufficient to prove that, if $w \in \Omega$, then condition (17) holds. This is not straightforward. We need the following technical lemma first:

Lemma 3.5
(a) The condition

$$\inf_{(\alpha,t)\in S} \left(\left| \frac{\partial g_w}{\partial \alpha} \right| + \left| \frac{\partial g_w}{\partial t} \right| \right) = 0 \tag{21}$$

holds if and only if

$$w \in \Omega_g^* := \left\{ w \in R^3 :\ w_1 \in [-h,h],\ w_2 + \frac{1}{4}(w_1 - h)^2 = \frac{h^2}{2} \right\}. \tag{22}$$

(b) The condition

$$\inf_{(\alpha,t)\in S} \left(\left| \frac{\partial y_w}{\partial \alpha} \right| + \left| \frac{\partial y_w}{\partial t} \right| \right) = 0 \tag{23}$$

holds if and only if

$$w \in \Omega_y^* := \left\{ w \in R^3 :\ w_1 \in [-h,h],\ \frac{1}{4}(w_1 + h)^2 - w_2 = \frac{h^2}{2} \right\}. \tag{24}$$

Moreover, $g_w(\alpha,t)$ satisfies condition (21) *and* $\sup_{(\alpha,t)\in S} g_w(\alpha,t) \leq 0$ *if and only if*

$$w \in \widetilde{\Omega}_g := \Omega_g^* \cap \left\{ w :\ w_3 \leq \frac{h^3}{6} - \frac{1}{3}\left(\frac{h^2}{2} - w_2\right)^{3/2} \right\}$$
$$= \left\{ w : w_1 \in [-h,h], w_2 + \frac{1}{4}(w_1 - h)^2 = \frac{h^2}{2},\ w_3 \leq \frac{h^3}{6} - \frac{1}{3}\left(\frac{h^2}{2} - w_2\right)^{3/2} \right\}.$$

Similarly, y_w satisfies condition (23) and $\inf_{(\alpha,t)\in S} y_w(\alpha,t) \geq 0$ *if and only if*

$$w \in \widetilde{\Omega}_y := \Omega_y^* \cap \left\{ w :\ w_3 \geq -\frac{h^3}{6} + \frac{1}{3}\left(\frac{h^2}{2} + w_2\right)^{3/2} \right\}$$
$$= \left\{ w : w_1 \in [-h,h], \frac{1}{4}(w_1 + h)^2 - w_2 = \frac{h^2}{2}, w_3 \geq -\frac{h^3}{6} + \frac{1}{3}\left(\frac{h^2}{2} + w_2\right)^{3/2} \right\}.$$

Proof (a)Assume that (21) holds. Then there exists a sequence $\{(\alpha_k, t_k) \in S\}$ such that

$$0 = \lim_{k\to\infty} \frac{\partial g_w}{\partial \alpha}\Big|_{(\alpha_k,t_k)} = \lim_{k\to\infty} \left(w_1 - h - 2t_k + 2\sqrt{(t_k)^2 - 2\alpha_k}\right),$$

$$0 = \lim_{k\to\infty} \frac{\partial g_w}{\partial t}\Big|_{(\alpha_k,t_k)} = \lim_{k\to\infty} \left(w_2 - \frac{h^2}{2} - 2\alpha_k + 2(t_k)^2 - 2t_k\sqrt{(t_k)^2 - 2\alpha_k}\right).$$

Therefore, for each k, there exist two sequences $\varepsilon_k^{(1)}, \varepsilon_k^{(2)} \to 0$ such that

$$2\sqrt{(t_k)^2 - 2\alpha_k} = 2t_k - \left(w_1 - h - \varepsilon_k^{(1)}\right), \tag{25}$$

$$-2t_k\sqrt{(t_k)^2 - 2\alpha_k} = -2(t_k)^2 + 2\alpha_k + \frac{h^2}{2} - w_2 + \varepsilon_k^{(2)}. \tag{26}$$

Squaring both sides of (25) leads to

$$\begin{aligned} 2\alpha_k &= \left(w_1 - h - \varepsilon_k^{(1)}\right) t_k - \frac{1}{4}\left(w_1 - h - \varepsilon_k^{(1)}\right)^2 \\ &= \left(w_1 - h - \varepsilon_k^{(1)}\right)\left(t_k - \frac{1}{4}\left(w_1 - h - \varepsilon_k^{(1)}\right)\right). \end{aligned} \tag{27}$$

On the other hand, multiplying both sides of (25) by t_k and adding to (26) lead to

$$2\alpha_k = \left(w_1 - h - \varepsilon_k^{(1)}\right) t_k + w_2 - \frac{h^2}{2} - \varepsilon_k^{(2)}. \tag{28}$$

It follows from (27) and (28) that

$$w_2 - \frac{h^2}{2} - \varepsilon_k^{(2)} = -\frac{1}{4}\left(w_1 - h - \varepsilon_k^{(1)}\right)^2 \quad \text{for } k \geq 1.$$

Since $\varepsilon_k^{(1)}, \varepsilon_k^{(2)} \to 0$ as $k \to \infty$, the above equality implies that

$$\frac{1}{4}(w_1 - h)^2 = \frac{h^2}{2} - w_2.$$

We now prove that $-h \leq w_1 \leq h$. Assume that $w_1 > h$. Since $\alpha_k \leq 0$, when k becomes sufficiently large, it follows from (27) that

$$t_k \leq \frac{1}{4}\left(w_1 - h - \varepsilon_k^{(1)}\right).$$

However, (25) implies that $\frac{1}{2}(w_1 - h - \varepsilon_k^{(1)}) \leq t_k$. This causes a contradiction. Next, assume that $w_1 < -h$. It follows from (27) that

$$t_k \geq \frac{1}{4}(w_1 - h - \varepsilon_k^{(1)}). \tag{29}$$

Since $(\alpha_k, t_k) \in S$, we have $\alpha_k + ht_k \geq -\frac{h^2}{2}$. From the first equation of (27), we have

$$-h^2 \leq 2(\alpha_k + ht_k) = t_k(w_1 + h - \varepsilon_k^{(1)}) - \frac{1}{4}(w_1 - h - \varepsilon_k^{(1)})^2.$$

Since $w_1 + h + \varepsilon_k^{(1)} < 0$, we see that

$$t_k \leq \frac{\frac{1}{4}\left(w_1 - h - \varepsilon_k^{(1)}\right)^2 - h^2}{w_1 + h - \varepsilon_k^{(1)}} = \frac{(w_1 - h - \varepsilon_k^{(1)})^2 - 4h^2}{4(w_1 + h - \varepsilon_k^{(1)})} < \frac{1}{4}(w_1 - h - \varepsilon_k^{(1)}).$$

This contradicts (29). Hence, w must be in the set of (22).

Conversely, we show that, for each w belonging to the set of (22), the concave function g_w has a finite stationary point in the set S and, consequently, condition (21) holds. Let $\alpha^* = 0$ and $t^* = \frac{1}{4}(w_1 - h)$. It is easy to verify that (α^*, t^*) is in S and that it is a stationary point of $g_w(\alpha, t)$, that is, $\nabla g_w(\alpha^*, t^*) = 0$. Moreover, since $h - w_1 = 2\sqrt{\frac{h^2}{2} - w_2}$, the maximum value of the concave function g_w over S is given by

$$g_w(\alpha^*, t^*) = w_3 - \frac{h^3}{6} + \frac{1}{3}\left(\frac{h^2}{2} - w_2\right)^{3/2}. \tag{30}$$

(b) We prove the result for function $y_w(\alpha, t)$ first. The proof is similar to that of $g_w(\alpha, t)$. Let $\{(\alpha_k, t_k) \in S\}$ be the sequence such that

$$0 = \lim_{k\to\infty} \frac{\partial y_w}{\partial \alpha}\Big|_{(\alpha_k, t_k)} = \left(\lim_{k\to\infty} w_1 + h + 2t_k - 2\sqrt{(t_k)^2 - 2\alpha_k}\right),$$

$$0 = \lim_{k\to\infty} \frac{\partial y_w}{\partial t}\Big|_{(\alpha_k, t_k)} = \lim_{k\to\infty} \left(w_2 + \frac{h^2}{2} + 2\alpha_k - 2(t_k)^2 + 2t_k\sqrt{(t_k)^2 - 2\alpha_k}\right).$$

We introduce two auxiliary variables $\varepsilon_k^{(1)}, \varepsilon_k^{(2)}$, both tending to zero as $k \to \infty$, such that

$$2\sqrt{(t_k)^2 - 2\alpha_k} = 2t_k + w_1 + h - \epsilon_k^{(1)}, \tag{31}$$

$$-2t_k\sqrt{(t_k)^2 - 2\alpha_k} = w_2 + \frac{h^2}{2} + 2\alpha_k - 2(t_k)^2 - \varepsilon_k^{(2)}. \tag{32}$$

Squaring both sides of (31) leads to

$$\begin{aligned} 2\alpha_k &= -(w_1 + h - \epsilon_k^{(1)})t_k - \frac{1}{4}(w_1 + h - \epsilon_k^{(1)})^2 \\ &= (w_1 + h - \epsilon_k^{(1)})\left(-t_k - \frac{1}{4}(w_1 + h - \epsilon_k^{(1)})\right). \end{aligned} \tag{33}$$

Multiplying both sides of (31) by t_k and adding to (32) yields

$$2\alpha_k = -(w_1 + h - \epsilon_k^{(1)})t_k - (w_2 + \frac{h^2}{2} - \epsilon_k^{(2)}).$$

The above two relations require w to satisfy

$$\frac{1}{4}(w_1 + h - \epsilon_k^{(1)})^2 = \frac{h^2}{2} + w_2 - \epsilon_k^{(2)}.$$

Letting $k \to \infty$, we have

$$\frac{1}{4}(w_1 + h)^2 = \frac{h^2}{2} + w_2.$$

Next, we prove that $-h \le w_1 \le h$. Assume $w_1 < -h$. Since $\alpha_k \le 0$ and $\varepsilon_k^{(1)} \to 0$, it follows from the second equation of (33) that

$$t_k \le -\frac{1}{4}(w_1 + h - \epsilon_k^{(1)})$$

for k sufficiently large. By (31), however, we have $t_k \ge -\frac{1}{2}(w_1 + h - \epsilon_k^{(1)}) > -\frac{1}{4}(w_1 + h - \epsilon_k^{(1)})$. This causes a contradiction. Now assume that $w_1 > h$. For k sufficiently large, it follows from (33) that

$$t_k \ge -\frac{1}{4}(w_1 + h - \epsilon_k^{(1)}).$$

Since $(\alpha_k, t_k) \in S$, we have $\alpha_k + ht \ge -\frac{h^2}{2}$. Following (33), we see that

$$-h^2 \le 2(\alpha_k + ht_k) = t_k(h - w_1 + \epsilon_k^{(1)}) - \frac{1}{4}(w_1 + h + \epsilon_k^{(1)})^2.$$

Since $w_1 > h$, the above inequality implies that

$$t_k \le \frac{\frac{1}{4}(w_1 + h - \epsilon_k^{(1)})^2 - h^2}{h - w_1 - \epsilon_k^{(1)}} = \frac{(w_1 + h - \epsilon_k^{(1)})^2 - 4h^2}{4(h - w_1 - \epsilon_k^{(1)})} < -\frac{1}{4}(w_1 + h - \epsilon_k^{(1)}).$$

This again causes a contradiction. As a result, the vector w must be in the set of (24).

Conversely, we show that, for each w in the set of (24), the convex function y_w has a stationary point in S. Let $\alpha^* = -\frac{(w_1+h)^2}{4}$ and $t^* = \frac{1}{4}(w_1 + h)$. Note that, for $w_1 \in [-h, h]$, we have $\alpha^* + ht^* \ge -\frac{h^2}{2}$. Hence, $(\alpha^*, t^*) \in S$ and it is easy to verify that (α^*, t^*) is a stationary point of $y_w(\alpha, t)$, that is, $\nabla y_w(\alpha^*, t^*) = 0$. Consequently, (23) holds and the corresponding minimum value of the convex function y_w is given by

$$y(\alpha^*, t^*) = w_3 + \frac{h^3}{6} - \frac{1}{3}\left(\frac{h^2}{2} + w_2\right)^{3/2}. \tag{34}$$

The second part of the lemma follows from what we have proved for (a), (b), (30), and (34). □

Remark 3.1 When a function satisfies a condition like (21), we say that the function has an asymptotic stationary point on S. A finite stationary point is clearly an asymptotic stationary point. The converse, however, is not true for a general function. From the proof of Lemma 3.5, we have actually shown that, if $w \in \Omega_g^*$, then a concave g_w has a finite stationary point on the set S. In view of part (a) of Lemma 3.5, we can conclude that the following three statements are equivalent: (a) g_w has an asymptotic stationary point on S, (b) g_w has a finite stationary point on S, and (c) $w \in \Omega_g^*$. Similarly, for y_w, the following three conditions are equivalent: (a′) y_w has an asymptotic stationary point on S, (b′) y_w has a finite stationary point on S, and (c′) $w \in \Omega_y^*$.

The main result for the region Γ_5 is stated as follows.

Theorem 3.1 *$\mathcal{G}_5(w_1 w_2, w_3) = \Delta z w_1$ with an effective domain Ω defined by* (20)*, that is,*

$$dom(\mathcal{G}_5) = \Big\{w : \, |w_2| \le \frac{h^2}{2}, w_3 - \frac{h}{2}w_2 \in \left[-\frac{h^3}{12}, \frac{h^3}{12}\right], w_2 - w_1 h \in \left[-\frac{h^2}{2}, \frac{h^2}{2}\right],$$
$$-\frac{h^3}{6} + \frac{1}{3}\left(\frac{h^2}{2} + w_2\right)^{\frac{3}{2}} \le w_3 \le \frac{h^3}{6} - \frac{1}{3}\left(\frac{h^2}{2} - w_2\right)^{\frac{3}{2}}\Big\}.$$

Proof As pointed out previously, $w \in \mathrm{dom}(\mathcal{G}_5)$ is equivalent to (17). We have already proved that (17) implies $w \in \Omega$. Hence $\mathrm{dom}(\mathcal{G}_5) \subseteq \Omega$ and we need only to prove that the converse is also true.

Let w be any vector in Ω. We will show that (17) holds. We give here a detailed proof for $\sup_{(\alpha,t)\in S} g_w(\alpha, t) \le 0$. A similar proof can be constructed to show that $\inf_{(\alpha,t)\in S} y_w(\alpha, t) \ge 0$.

We first prove that, for any $w \in \Omega$, the function g_w is bounded above on S. (Similarly, we can prove that y_w is bounded below on S). The gradient of the function g_w w.r.t. (α, t) is given by

$$\nabla g_w(\alpha,t) = \begin{pmatrix} \frac{\partial g_w}{\partial \alpha} \\ \frac{\partial g_w}{\partial t} \end{pmatrix} = \begin{pmatrix} w_1 - h - 2t + 2\sqrt{t^2 - 2\alpha} \\ w_2 - \frac{h^2}{2} - 2\alpha + 2t^2 - 2t\sqrt{t^2 - 2\alpha} \end{pmatrix}$$
$$= \begin{pmatrix} w_1 - h - \frac{8\alpha}{2t + 2\sqrt{t^2 - 2\alpha}} \\ w_2 - \frac{h^2}{2} - 2\alpha + \frac{8\alpha t^2}{2t^2 + 2t\sqrt{t^2 - 2\alpha}} \end{pmatrix}$$
$$= \begin{pmatrix} w_1 - h - \frac{4\left(\frac{\alpha}{t}\right)}{1 + \sqrt{1 - 2\left(\frac{\alpha}{t^2}\right)}} \\ w_2 - \frac{h^2}{2} - 2\alpha + \frac{4\alpha}{1 + \sqrt{1 - 2\left(\frac{\alpha}{t^2}\right)}} \end{pmatrix}.$$

For any $(\alpha, t) \in S$, we have

$$0 \ge \frac{\alpha}{t} \ge -\left(\frac{h^2}{2t} + h\right), \quad 0 \ge \frac{\alpha}{t^2} \ge -\left(\frac{h^2}{2t^2} + \frac{h}{t}\right).$$

This indicates that $\lim_{t\to\infty} \frac{\alpha}{t^2} = 0$ and $\left|\frac{\alpha}{t}\right|$ is bounded, provided $t \ge t_0 > 0$ where t_0 is a fixed positive scalar. It is easy to see that

$$\lim_{t\to\infty} \frac{\partial g_w}{\partial t} = w_2 - \frac{h^2}{2} \le 0.$$

There are three possible cases.

Case 1 There exist constants $\epsilon > 0$ and $t_0 > 0$ such that $\frac{\partial g_w}{\partial \alpha} \ge \epsilon$ for any $(\alpha, t) \in S$ with $t \ge t_0$. Then $g_w(\alpha, t)$ is an increasing function w.r.t. α, and, hence, $g_w(\alpha, t) \le g_w(0, t)$ for all $(\alpha, t) \in S$ with $t \ge t_0$. By Lemma 3.1, $g_w(0, t) \le 0$, since $w \in \Omega \subset \mathcal{D}_1$. Therefore, g_w has an upper bound on S when $t \ge t_0$. On the other hand, since the set $T := \{(\alpha, t) : (\alpha, t) \in S, \ t \le t_0\}$ is bounded, by continuity, g_w is bounded on T. Therefore, g_w is bounded above on S.

Case 2 There exist constants $\epsilon > 0$ and $t_0 > 0$ such that $\frac{\partial g_w}{\partial \alpha} \le -\epsilon$ for any $(\alpha, t) \in S$ with $t \ge t_0$. Then $g_w(\alpha, t)$ is a decreasing function w.r.t. α. Noting that $0 \ge \alpha \ge -\frac{h^2}{2} - ht$, for any $(\alpha, t) \in S$ with $t \ge t_0$, we know that $g_w(\alpha, t) \le g_w\left(-\frac{h^2}{2} - ht, t\right)$. The right-hand side of this inequality is the value of g_w on the boundary $\alpha = -\frac{h^2}{2} - ht$ of S. Using Lemma 3.2, for any $w \in \Omega \subset \mathcal{D}_2$, we have $g_w(-\frac{h^2}{2} - ht, t) \leq 0$. Let T be defined as in case (1). By continuity, g_w is bounded on T. Thus, g_w is bounded above on S.

Case 3 There are no positive scalars ϵ and t_0 such that either case 1 or 2 happens. Since $\nabla g_w(\alpha, t)$ is continuous on the convex set S, it is equivalent to say that

$$\inf_{(\alpha,t)\in S, t\geq t_0} \left| \frac{\partial g_w}{\partial \alpha} \right| = 0 \quad \forall t_0 > 0. \tag{35}$$

In this case, if $w_2 = \frac{h^2}{2}$, then $\lim_{t\to\infty} \frac{\partial g_w}{\partial t} = 0$. Hence, (21) holds and, by (1) of Lemma 3.5, we have $w \in \Omega_g^*$. Thus, $w \in \Omega \cap \Omega_g^* \subseteq \widetilde{\Omega}_g$. Lemma 3.5 further implies that $g_w(\alpha, t) \leq 0$ on S.

Assume that $w_2 < \frac{h^2}{2}$. In this case $\frac{\partial g_w}{\partial t} < 0$ for sufficiently large t. Take a point $(\hat{\alpha}, \hat{t}) \in S$ with $\hat{t} > 0$ such that

$$\left| \frac{\partial g_w}{\partial \alpha}\Big|_{(\hat{\alpha},\hat{t})} \right| \leq \hat{\epsilon}, \qquad \frac{\partial g_w}{\partial t}\Big|_{(\hat{\alpha},\hat{t})} + \hat{\epsilon} h < 0. \tag{36}$$

By (35) and the fact of $\frac{\partial g_w}{\partial t} < 0$ for sufficiently large t, there exist $(\hat{\alpha}, \hat{t})$ and $\hat{\epsilon} > 0$ satisfying (36), provided that $\hat{t}$ is sufficiently large. By the concavity of g_w on S, for all $(\alpha, t) \in S$ with $t \geq \hat{t}$, we have

$$\begin{aligned}
g_w(\alpha,t) &\leq g_w(\hat{\alpha},\hat{t}) + \nabla g_w(\hat{\alpha},\hat{t})^T \begin{pmatrix} \alpha - \hat{\alpha} \\ t - \hat{t} \end{pmatrix} \\
&\leq g_w(\hat{\alpha},\hat{t}) - \hat{\alpha}\frac{\partial g_w}{\partial \alpha}\Big|_{(\hat{\alpha},\hat{t})} - \hat{t}\frac{\partial g_w}{\partial t}\Big|_{(\hat{\alpha},\hat{t})} + \left|\frac{\partial g_w}{\partial \alpha}\Big|_{(\hat{\alpha},\hat{t})}\right| |\alpha| + t\frac{\partial g_w}{\partial t}\Big|_{(\hat{\alpha},\hat{t})} \\
&\leq g_w(\hat{\alpha},\hat{t}) - \hat{\alpha}\frac{\partial g_w}{\partial \alpha}\Big|_{(\hat{\alpha},\hat{t})} - \hat{t}\frac{\partial g_w}{\partial t}\Big|_{(\hat{\alpha},\hat{t})} + \hat{\epsilon}(\frac{h^2}{2} + ht) + t\frac{\partial g_w}{\partial t}\Big|_{(\hat{\alpha},\hat{t})} \\
&\leq g_w(\hat{\alpha},\hat{t}) - \hat{\alpha}\frac{\partial g_w}{\partial \alpha}\Big|_{(\hat{\alpha},\hat{t})} - \hat{t}\frac{\partial g_w}{\partial t}\Big|_{(\hat{\alpha},\hat{t})} + \hat{\epsilon}\frac{h^2}{2} + \left(\frac{\partial g_w}{\partial t}\Big|_{(\hat{\alpha},\hat{t})} + \hat{\epsilon}h\right)t \\
&\leq g_w(\hat{\alpha},\hat{t}) - \hat{\alpha}\frac{\partial g_w}{\partial \alpha}\Big|_{(\hat{\alpha},\hat{t})} - \hat{t}\frac{\partial g_w}{\partial t}\Big|_{(\hat{\alpha},\hat{t})} + \hat{\epsilon}\frac{h^2}{2}.
\end{aligned}$$

Thus, g_w is bounded above on $\{(\alpha,t) : (\alpha,t) \in S, t \geq \hat{t}\}$. Note that $\{(\alpha,t) : (\alpha,t) \in S, t \leq \hat{t}\}$ is also bounded. Therefore, g_w is bounded below on S.

In summary, we have proved that, for any given $w \in \Omega$, the function g_w is bounded from above on S. Similarly, we can prove that y_w is bounded from below on S.

We now prove that, for any $w \in \Omega$, condition (17) holds. For $w \in \widetilde{\Omega}_g \cap \Omega$, by Lemma 3.5, we know $\sup_{(\alpha,t)\in S} g_w(\alpha,t) \leq 0$. Assume that $w \in \Omega \backslash \widetilde{\Omega}_g$. Since $w \in \Omega$ requires that

$$w_3 \leq \frac{h^3}{6} - \frac{1}{3}\left(\frac{h^2}{2} - w_2\right)^{3/2},$$

through the construction of $\widetilde{\Omega}_g$, we know the condition $w \in \Omega \backslash \widetilde{\Omega}_g$ implies that $w \notin \Omega_g^*$, when Ω_g^* is given by (22). By (1) of Lemma 3.5, the condition $w \in \Omega \backslash \widetilde{\Omega}_g$ implies that

$$\inf_{(\alpha,t)\in S} \left(\left|\frac{\partial g_w}{\partial \alpha}\right| + \left|\frac{\partial g_w}{\partial t}\right| \right) > 0.$$

This means that there is neither a finite nor an asymptotic stationary point of g_w on the convex set S. We deduce that the finite supremum is achieved on the boundary of the set S, that is,

$$\sup_{(\alpha,t)\in S} g_w(\alpha,t) = \sup_{(\alpha,t)\in\mathcal{B}^{(1)}\cup\mathcal{B}^{(2)}} g_w(\alpha,t) \le 0.$$

The inequality follows from Lemmas 3.1 and 3.2. The fact that $\inf_{(\alpha,t)\in S} y_w(\alpha,t) \ge 0$ can be proved in the same way. Therefore, $w \in \Omega$ implies that (17) holds. $\square$

So far we have taken care of region Γ_5. Applying similar analysis, we can establish results analogous to Theorem 3.1 for the remaining cases with $v \neq 0$, that is, for regions Γ_i, $i = 1,2,3,4,6,7$. The general procedure goes like this. First, set α and t as (15). Then define the set S and the functions $\psi_w(\alpha,t,v)$, $g_w(\alpha,t)$ and $y_w(\alpha,t)$ for each region. Since (18) holds for each j, $j = 1,2,3,4,6,7$, we have

$$\mathcal{G}_j(w_1,w_2,w_3) = \Delta z w_1 \quad \forall\ w \in \mathrm{dom}(\mathcal{G}_j).$$

Note that, on Γ_1, Γ_2, and Γ_3, the function $f_i(q,u,v)$ is the same, that is, for $i = 1,2,3$,

$$f_i(q,u,v) = |v| \left\{ \frac{6h(q-\Delta z) + 3uh^2 + vh^3}{6v} \right\}.$$

Since $(q,u,v) \in \Gamma_7$ implies that $\eta_1^{(i)} \le 0 \le 1 \le \eta_2^{(i)}$, on Γ_7, we have

$$f_i(q,u,v) = -|v| \left\{ \frac{6h(q-\Delta z) + 3uh^2 + vh^3}{6v} \right\}.$$

We list the effective domains of the functions $\mathcal{G}_j(w_1,w_2,w_3)$, $j = 1,2,3,4,6,7$, below without repeating the lengthy but similar proofs.

$$\mathrm{dom}(\mathcal{G}_1) = \left\{ w : |w_1| \le h,\ \left(w_2 - \frac{h^2}{2}\right)^2 - 2(w_1 - h)\left(w_3 - \frac{h^3}{6}\right) \le 0, \right.$$
$$\left. \left(w_2 + \frac{h^2}{2}\right)^2 - 2(w_1 + h)\left(w_3 + \frac{h^3}{6}\right) \le 0 \right\}$$

$$\mathrm{dom}(\mathcal{G}_2) = \left\{ w :\ |w_1| \le h,\ |w_2| \le \frac{h^2}{2},\ |w_3| \le \frac{h^3}{6} \right\},$$

$$\mathrm{dom}(\mathcal{G}_3) = \left\{ w :\ |w_1| \le h,\ \ w_2 - w_1 h \in \left[-\frac{h^2}{2}, \frac{h^2}{2} \right], \right.$$
$$\left. w_2 h - \frac{h^2}{2} w_1 - \frac{h^3}{6} \le w_3 \le w_2 h - \frac{h^2}{2} w_1 + \frac{h^3}{6} \right\},$$

$$\begin{aligned}
\mathrm{dom}(\mathcal{G}_4) = \Bigg\{ & w: \, |w_2| \le \frac{h^2}{2}, \; w_2 - w_1 h \in \left[-\frac{h^2}{2}, \frac{h^2}{2}\right], \\
& -\frac{h^3}{6} + \frac{1}{3}\left(\frac{h^2}{2} + w_2\right)^{3/2} \le w_3 \le \frac{h^3}{6} - \frac{1}{3}\left(\frac{h^2}{2} - w_2\right)^{3/2}, \\
& \frac{h}{2}w_2 + \frac{h^3}{12} - \frac{h}{2}\rho^* + \frac{1}{3}\left(\rho^*\right)^{3/2} \le w_3 \le \frac{h}{2}w_2 - \frac{h^3}{12} + \frac{h}{2}\rho - \frac{1}{3}\rho^{3/2}, \\
& (w_2 - \frac{h^2}{2})^2 \le 2(w_3 - \frac{h^3}{6})(w_1 - h), \; (\frac{h^2}{2} + w_2)^2 \le 2(w_3 + \frac{h^3}{6})(w_1 + h) \Bigg\}, \\
\mathrm{dom}(\mathcal{G}_6) = \Bigg\{ & w \in R^3: \, |w_2| \le \frac{h^2}{2}, \; w_2 - w_1 h \in \left[-\frac{h^2}{2}, \frac{h^2}{2}\right], \\
& \frac{h}{2}w_2 + \frac{h^3}{12} - \frac{h}{2}\rho^* + \frac{1}{3}\left(\rho^*\right)^{3/2} \le w_3 \le \frac{h}{2}w_2 - \frac{h^3}{12} + \frac{h}{2}\rho - \frac{1}{3}\rho^{3/2} \Bigg\}, \\
\mathrm{dom}(\mathcal{G}_7) = \Bigg\{ & w: |w_1| \le h, |w_2| \le \frac{h^2}{2}, w_2 - w_1 h \in \left[-\frac{h^2}{2}, \frac{h^2}{2}\right], \\
& w_3 - \frac{h}{2}w_2 \in \left[-\frac{h^3}{12}, \frac{h^3}{12}\right] \Bigg\},
\end{aligned}$$

where $\rho^* = \frac{h^2}{2} - w_2 + w_1 h$, and $\rho = w_2 - w_1 h + \frac{h^2}{2}$.

Finally, we consider the situation for regions $\Gamma_8, \Gamma_9, \Gamma_{10}$, and Γ_{11}. Because $v = 0$ holds for each case, the analysis becomes simple.

(a) For $u = 0$. The region is Γ_8, on which the corresponding f_i reduces to $h_i|q - \Delta z|$. Hence,

$$\begin{aligned}
\mathcal{G}_8(w_1, w_2, w_3) &= \sup_{(v,u)=0, q \in R} w_1 q + w_2 u + w_3 v - f_i \\
&= \sup_{q \in R} \Delta z w_1 + w_1(q - \Delta z) - h_i|q - \Delta z| \\
&= \Delta z w_1 + \sup_{q \in R} \left(w_1(q - \Delta z) - h_i|q - \Delta z|\right).
\end{aligned}$$

Then it is easy to verify that $\mathcal{G}_8$ is finite if and only if $|w_1| \le h$. Consequently, $\mathrm{dom}(\mathcal{G}_8) = \{w \in R^3 : |w_1| \le h\}$ and $\mathcal{G}_8(w_1, w_2, w_3) = \Delta z w_1$ for $w \in \mathrm{dom}(\mathcal{G}_8)$.

(b) For $u \ne 0$ and $\frac{q - \Delta z}{u} \in [-h, 0]$. In this case,

$$f_i(q, u, 0) = |u| \left(\left(\frac{q - \Delta z}{u}\right)^2 + \left(\frac{q - \Delta z}{u}\right) h + \frac{h^2}{2} \right) = |u| \left(\beta^2 + \beta h + \frac{h^2}{2}\right),$$

where $\beta = \frac{q - \Delta z}{u}$. Therefore,

$$\begin{aligned}
w_1 q + w_2 u - f_i &= w_1 \Delta z + w_1 \beta u + w_2 u - |u| \left(\beta^2 + \beta h + \frac{h^2}{2}\right) \\
&= w_1 \Delta z + \psi_w(\beta, u),
\end{aligned}$$

where $\beta \in [-h, 0]$ and

$$\psi_w(\beta, u) = \begin{cases} g_w(\beta) u, & \text{if } u > 0, \\ y_w(\beta) u, & \text{if } u < 0, \end{cases} \tag{37}$$

$$g_w(\beta) = (w_1 - h)\beta - \beta^2 + w_2 - \frac{h^2}{2},$$

$$y_w(\beta) = (w_1 + h)\beta + \beta^2 + w_2 + \frac{h^2}{2}.$$

It is easy to see that $g_w(\beta) \le 0$ on $[-h, 0]$ if and only if

$$w \in \left\{ w : |w_1| \le h,\ w_2 - \frac{h^2}{2} + \frac{(w_1 - h)^2}{4} \le 0 \right\}$$

and $y_w \ge 0$ on $[-h, 0]$ if and only if

$$w \in \left\{ w : |w_1| \le h,\ -(w_2 + \frac{h^2}{2}) + \frac{(w_1 + h)^2}{4} \le 0 \right\}.$$

Note that

$$\mathcal{G}_9(w_1, w_2, w_3) = \Delta z w_1 + \sup_{u \in R, \beta \in [-h,0]} \psi_w(\beta, u).$$

Hence,

$$\mathrm{dom}(\mathcal{G}_9) = \left\{ w \in R^3 : |w_1| \le h, -\frac{h^2}{2} + \frac{(w_1 + h)^2}{4} \le w_2 \le \frac{h^2}{2} - \frac{(w_1 - h)^2}{4} \right\},$$

on which $\mathcal{G}_9(w_1, w_2, w_3) = \Delta z w_1$.

(c) For $\beta = \frac{q - \Delta z}{u} \in [0, \infty)$. We have

$$f_i(q, u, 0) = |u| \left(\left(\frac{q - \Delta z}{u} \right) h + \frac{h^2}{2} \right) = |u| \left(\beta h + \frac{h^2}{2} \right)$$

and

$$w_1 q + w_2 u - f_i = w_1 \Delta z + w_1 \beta u + w_2 u - |u| \left(\beta h + \frac{h^2}{2} \right) = w_1 \Delta z + \psi_w(\beta, u),$$

where $\psi_w(\beta, u)$ is defined by (37). However, $g_w(\beta)$ and $y_w(\beta)$ are given by

$$g_w(\beta) = (w_1 - h)\beta + w_2 - \frac{h^2}{2}, \qquad y_w(\beta) = (w_1 + h)\beta + w_2 + \frac{h^2}{2}.$$

Clearly, on $[0, \infty)$, $g_w \le 0$ and $y_w \ge 0$ if and only if $|w_1| \le h$ and $|w_2| \le \frac{h^2}{2}$. Therefore,

$$\mathrm{dom}(\mathcal{G}_{10}) = \left\{ w \in R^3 :\ |w_1| \le h,\ |w_2| \le \frac{h^2}{2} \right\}$$

on which $\mathcal{G}_{10}(w_1, w_2, w_3) = \Delta z w_1$.

(d) For $\beta = \frac{q - \Delta z}{u} \in (-\infty, -h]$. Proceeding as in (c), we have

$$\mathrm{dom}\ (\mathcal{G}_{11}) = \left\{ w \in R^3 : |w_1| \le h, w_2 - w_1 h \in \left[-\frac{h^2}{2}, \frac{h^2}{2} \right] \right\}$$

on which $\mathcal{G}_{11}(w_1, w_2, w_3) = \Delta z w_1$.

Using (13) and (14), we have the following main result of this section:

Theorem 3.2 *The conjugate function $f_i^*(w_1, w_2, w_3)$ in (13) has a value of $\Delta z w_1$ over its effective domain defined by the following inequalities:*

$$w_2 - w_1 h \in \left[-\frac{h^2}{2}, \frac{h^2}{2}\right], \tag{38}$$

$$w_3 - w_2 h + \frac{h^2}{2} w_1 \in \left[-\frac{h^3}{6}, \frac{h^3}{6}\right], \tag{39}$$

$$\left(w_2 - \frac{h^2}{2}\right)^2 - 2(w_1 - h)\left(w_3 - \frac{h^3}{6}\right) \le 0, \tag{40}$$

$$\left(w_2 + \frac{h^2}{2}\right)^2 - 2(w_1 + h)\left(w_3 + \frac{h^3}{6}\right) \le 0, \tag{41}$$

$$-\frac{h^2}{2} + \frac{(w_1 + h)^2}{4} \le w_2 \le \frac{h^2}{2} - \frac{(w_1 - h)^2}{4}, \tag{42}$$

$$-\frac{h^3}{6} + \frac{1}{3}\left(\frac{h^2}{2} + w_2\right)^{3/2} \le w_3 \le \frac{h^3}{6} - \frac{1}{3}\left(\frac{h^2}{2} - w_2\right)^{3/2}, \tag{43}$$

$$\frac{h}{2} w_2 + \frac{h^3}{12} - \frac{h}{2}\rho^* + \frac{1}{3}\left(\rho^*\right)^{3/2} \le w_3 \le \frac{h}{2} w_2 - \frac{h^3}{12} + \frac{h}{2}\rho - \frac{1}{3}\rho^{3/2}, \tag{44}$$

where $\rho^* = \frac{h^2}{2} - w_2 + w_1 h$, and $\rho = w_2 - w_1 h + \frac{h^2}{2}$.

Remark 3.2 Taking the intersection operation on the right-hand side of (14), we see that the following inequalities are redundant and can be removed:

$$|w_1| \le h, \quad |w_2| \le \frac{h^2}{2}, \quad |w_3| \le \frac{h^3}{6}, \quad w_3 - \frac{h}{2} w_2 \in \left[-\frac{h^3}{12}, \frac{h^3}{12}\right]. \tag{45}$$

Inequalities (42) imply that $|w_2| \le \frac{h^2}{2}$. With (38), this further implies that $|w_1| \le h$. With (43), this implies that $|w_3| \le \frac{h^3}{6}$. Also, note that inequality $w_3 - \frac{h}{2} w_2 \in \left[-\frac{h^3}{12}, \frac{h^3}{12}\right]$ is redundant. In fact, from (43), we have

$$w_3 - \frac{h}{2} w_2 \ge \vartheta(w_2) := -\frac{h^3}{6} + \frac{1}{3}\left(\frac{h^2}{2} + w_2\right)^{3/2} - \frac{h}{2} w_2,$$

$$w_3 - \frac{h}{2} w_2 \le \varrho(w_2) := \frac{h^3}{6} - \frac{1}{3}\left(\frac{h^2}{2} + w_2\right)^{3/2} - \frac{h}{2} w_2.$$

It is not difficult to show that under (42) we have $|w_2| \le \frac{h^2}{2}$ and the convex function $\vartheta(w_2) - h^3/12$ has a lower bound of $-h^3/12$, the concave function $\varrho(w_2)$ has an upper bound of $h^3/12$.

Remark 3.3 Now we claim that the set formed by inequalities (38) through (44) is convex. It is evident that all the constraint functions stated in Theorem 3.2 are convex except (40) and (41). However, we can prove that the set formed by inequalities (40) and (41) remains convex. This follows from the fact that $\{(a,b,c) : a^2 - 2bc \leq 0, a \geq 0, b \geq 0, c \geq 0\}$ is a convex set, because $a + b \geq 2\sqrt{ab}$ if $(a,b) \geq 0$. An alternative explanation for this fact will be given in the next section.

4 Dual model and dual-to-primal conversion

To establish the geometric dual model for problem (2), we need the so-called Fenchel Dual Theorem [26]. Let $f\colon R^m \to (-\infty, +\infty]$ be a closed proper convex function on R^m with the conjugate function f^* and let K be a subspace in R^n with the orthogonal complement $K^\perp$. The problem $\inf_{y \in K^\perp} f^*(y)$ is called the geometric (or Fenchel's) dual problem of the primal problem $\inf_{x \in K} f(x)$. The following key results hold for this setting:

Theorem 4.1 [26] *Let f and K be defined as above. Then*

$$\inf_{x \in K} f(x) = - \inf_{y \in K^\perp} f^*(y)$$

if either (a) ri (dom (f)) $\cap K \neq \emptyset$ *or (b)* $ri(dom(f^*)) \cap K^\perp \neq \emptyset$ *holds.*

Moreover, under (a) the infimum of f^* over $K^\perp$ is attained and under (b) the infimum of f over K is attained. In general, x^* and y^* satisfy

$$f(x^*) = \inf_{x \in K} f(x) = - \inf_{y \in K^\perp} f^*(y) = -f^*(y^*),$$

if and only if

$$y^* \in \partial f(x^*), \;\; x^* \in K, \;\; y^* \in K^\perp.$$

Following Theorem 3.2 and (12), we have

$$\mathcal{F}^*(\mathcal{W}) = \sum_{i=0}^{n} z_i w_0^{(i)} + \sum_{i=0}^{n-1} \Delta z_i w_1^{(i)} + \delta(\mathcal{W} | \mathrm{dom}(\mathcal{F}^*)). \tag{46}$$

From definition,

$$f_i^*(w_1^{(i)}, w_2^{(i)}, w_3^{(i)}) \geq \sup_{(q_i,u_i,v_i)=0} (w_1^{(i)} q_i + w_2^{(i)} u_i + w_3^{(i)} v_i - f_i(q_i,u_i,v_i)) = -|\Delta z_i|.$$

Hence, f_i^* is bounded below for each i. It follows from (12) that $\mathcal{F}^*(\mathcal{W}) < \infty$ if and only if $f_i^*(w_1^{(i)}, w_2^{(i)}, w_3^{(i)}) < \infty$ for each i. By (12) and (11), we have

$$\begin{aligned} \mathrm{dom}(\mathcal{F}^*) &= \left(\prod_{i=0}^{n-1} \mathrm{dom}(\delta^*(\cdot | C_i)) \times \mathrm{dom}(f_i^*) \right) \times \mathrm{dom}(\delta^*(\cdot | C_n)) \\ &= \left(\prod_{i=0}^{n-1} R \times \mathrm{dom}(f_i^*) \right) \times R, \end{aligned} \tag{47}$$

which is a closed convex set. By Theorem 23.8 in [26],

$$\begin{aligned}
\partial\mathcal{F}^*(\mathcal{W}) &= \partial\left(\sum_{i=0}^{n} z_i w_0^{(i)} + \sum_{i=0}^{n-1} \Delta z_i w_1^{(i)} + \delta(\mathcal{W}|\mathrm{dom}(\mathcal{F}^*))\right) \\
&= \partial\left(\sum_{i=0}^{n} z_i w_0^{(i)} + \sum_{i=0}^{n-1} \Delta z_i w_1^{(i)}\right) + \partial\left(\delta(\mathcal{W}|\mathrm{dom}(\mathcal{F}^*))\right) \\
&= \mathrm{vec}(z, \Delta z) + \mathcal{N}_{\mathrm{dom}(\mathcal{F}^*)}(\mathcal{W}), \qquad (48)
\end{aligned}$$

where vec $(z, \Delta z)$ is a vector of constants given by

$$\mathrm{vec}\ (z, \Delta z) = \left(z_0, \Delta z_0, 0, 0, z_1, \Delta z_1, 0, 0, ..., z_{n-1}, \Delta z_{n-1}, 0, 0, z_n\right)^T$$

and $\mathcal{N}_{\mathrm{dom}(\mathcal{F}^*)}(\mathcal{W})$ denotes the normal cone of $\mathrm{dom}(\mathcal{F}^*)$ at the point $\mathcal{W}$.

Recall that in the primal problem the subspace

$$K = \{\mathcal{X} \in R^{4n+1} : A\mathcal{X} = 0\},$$

where A is a $(2n-1) \times (4n+1)$ matrix of the coefficients in the constraints, that is,

$$\begin{bmatrix}
1 & h_0 & \frac{h_0^2}{2} & \frac{h_0^3}{6} & -1 & & & & & & & & & & \\
& 1 & h_0 & \frac{h_0^2}{2} & 0 & -1 & & & & & & & & & \\
& & & & 1 & h_1 & \frac{h_1}{2} & \frac{h_1^3}{6} & -1 & & & & & & \\
& & & & & 1 & h_1 & \frac{h_1}{2} & 0 & -1 & & & & & \\
& & & & & & & & & \cdots & & & & & \\
& & & & & & & & & & \cdots & 1 & h_{n-1} & \frac{h_{n-1}^2}{2} & \frac{h_{n-1}^2}{6} & -1
\end{bmatrix}.$$

Hence, $K^\perp$ is the range space $R(A^T)$, that is, a linear subspace generated by the rows of A, and the geometric dual problem of (2) is

$$\inf_{\mathcal{W}\in R^{4n+1}} \{\mathcal{F}^*(\mathcal{W}) : \mathcal{W} \in K^\perp = R(A^T)\}.$$

By (46), (47), and Theorem 3.2 , the dual problem becomes

$$\begin{aligned}
\min_{\mathcal{W}\in R^{4n+1}} \quad & \sum_{i=0}^{n} z_i w_0^{(i)} + \sum_{i=0}^{n-1} \Delta z_i w_1^{(i)} \\
\text{s. t.} \quad & w_2^{(i)} - w_1^{(i)} h_i \in \left[-\frac{h_i^2}{2}, \frac{h_i^2}{2}\right], \quad i = 0, \ldots, n-1, \\
& w_3^{(i)} - w_2^{(i)} h_i + \frac{h_i^2}{2} w_1^{(i)} \in \left[-\frac{h_i^3}{6}, \frac{h_i^3}{6}\right], \quad i = 0, \ldots, n-1, \\
& \left(w_2^{(i)} - \frac{h_i^2}{2}\right)^2 - 2(w_1^{(i)} - h_i)\left(w_3^{(i)} - \frac{h_i^3}{6}\right) \le 0, \quad i = 0, \ldots, n-1, \qquad (49)
\end{aligned}$$

$$
\begin{aligned}
&\left(w_2^{(i)}+\frac{h_i^2}{2}\right)^2-2(w_1^{(i)}+h_i)\left(w_3^{(i)}+\frac{h_i^3}{6}\right)\le 0,\quad i=0,\ldots,n-1,\\
&-\frac{h_i^2}{2}+\frac{(w_1^{(i)}-h_i)^2}{4}\le w_2^{(i)}\le\frac{h_i^2}{2}-\frac{(w_1^{(i)}-h_i)^2}{4},\quad i=0,\ldots,n-1,\\
&-\frac{h_i^3}{6}+\frac{1}{3}\left(\frac{h_i^2}{2}+w_2^{(i)}\right)^{3/2}\le w_3^{(i)}\le\frac{h_i^3}{6}-\frac{1}{3}\left(\frac{h_i^2}{2}-w_2^{(i)}\right)^{3/2},\quad i=0,\ldots,n-1,\\
&\frac{h_i}{2}w_2+\frac{h_i^3}{12}-\frac{h_i}{2}\rho^*+\frac{1}{3}\left(\rho^*\right)^{3/2}\le w_i^{(3)}\le\frac{h_i}{2}w_2^{(i)}-\frac{h_i^3}{12}+\frac{h_i}{2}\rho-\frac{1}{3}\rho^{3/2},\quad i=0,\ldots,n,\\
&\left(w_0^{(0)},w_1^{(0)},w_2^{(0)},w_3^{(0)},w_0^{(1)},w_1^{(1)},w_2^{(1)},w_3^{(1)},\ldots,w_0^{(n-1)},w_1^{(n-1)},w_2^{(n-1)},w_3^{(n-1)},w_0^{(n)}\right)^T\\
&\in\mathcal{R}(A^T), \qquad (50)
\end{aligned}
$$

where

$$
\rho^*=\frac{h_i^2}{2}-w_2^{(i)}+w_1^{(i)}h_i,\quad \rho=w_2^{(i)}-h_iw_1^{(i)}+\frac{h_i^2}{2}. \tag{51}
$$

As pointed out in Remark 3.3, the feasible region of the above problem is convex, although constraint functions (49) and (50) are not convex. To see this more clearly, we replace these two inequalities by semi-definite constraints. In fact, noting that (45) is implied by other constraints, (49) and (50) can be replaced by the following semi-definite constraints without changing the problem:

$$
\begin{bmatrix} 2(h_i-w_1^{(i)}) & \frac{h_i^2}{2}-w_2^{(i)} \\ \frac{h_i^2}{2}-w_2^{(i)} & \frac{h_i^3}{6}-w_3^{(i)} \end{bmatrix}\succeq O,\qquad
\begin{bmatrix} 2(w_1^{(i)}+h_i) & w_2^{(i)}+\frac{h_i^2}{2} \\ w_2^{(i)}+\frac{h_i^2}{2} & w_3^{(i)}+\frac{h_i^3}{6} \end{bmatrix}\succeq O,
$$

where " $\succeq O$" means the square matrix is positive semi-definite. Consequently, we have

$$
\begin{bmatrix} 2(h_i-w_1^{(i)}) & \frac{h_i^2}{2}-w_2^{(i)} & 0 & 0 \\ \frac{h_i^2}{2}-w_2^{(i)} & \frac{h_i^3}{6}-w_3^{(i)} & 0 & 0 \\ 0 & 0 & 2(w_1^{(i)}+h_i) & w_2^{(i)}+\frac{h_i^2}{2} \\ 0 & 0 & w_2^{(i)}+\frac{h_i^2}{2} & w_3^{(i)}+\frac{h_i^3}{6} \end{bmatrix}\succeq O.
$$

Therefore, the dual problem can also be viewed as a convex program with semi-definite constraints.

It is not difficult to see that condition (a) of Theorem 4.1 holds for the primal problem. Hence, the dual attains its infimum. On the other hand, it is evident that $0\in\mathrm{ri}(\mathrm{dom}(\mathcal{F}^*))$, that is, condition (b) of Theorem 4.1 is valid. Therefore, the primal problem (2) also attains its infimum.

The last constraint of the dual problem indicates that the variable $\mathcal{W}$ can be represented as a linear combination of the rows of A. Denoting by $\mathcal{Y}$ the combined coefficients, that is, $\mathcal{W}=A^T\mathcal{Y}$, and substituting it into other constraints, we can eliminate variable $\mathcal{W}$ and formulate the dual problem in terms of $\mathcal{Y}$, which has $2n-1$ instead of $4n+1$ variables. However, such elimination may make other constraints more complicated. When n becomes large, elimination could be wise, but not for problems with small n. Since we have a smooth convex dual problem, most nonlinear programming algorithms including the interior point algorithms become applicable.

Our remaining task is to construct a dual-to-primal conversion mechanism. We show that a primal optimal solution can be calculated by solving a sparse linear program determined by a dual optimal solution $\mathcal{W}^*$. To do this, we introduce the notation

$$
\begin{aligned}
&C^l_{1i}(w) = -\tfrac{h_i^2}{2} - w_2^{(i)} + w_i^{(1)} h_i, && C^r_{1i}(w) = -\tfrac{h_i^2}{2} + w_2^{(i)} - w_i^{(1)} h_i,\\
&C^l_{2i}(w) = -\tfrac{h_i^3}{6} + \tfrac{1}{3}\left(\tfrac{h_i^2}{2} + w_2^{(i)}\right)^{3/2} - w_3^{(i)}, && C^r_{2i}(w) = w_3^{(i)} - \tfrac{h_i^3}{6} + \tfrac{1}{3}\left(\tfrac{h_i^2}{2} - w_2^{(i)}\right)^{3/2},\\
&C^l_{3i}(w) = -\tfrac{h_i^2}{2} + \tfrac{(w_1^{(i)}-h_i)^2}{4} - w_2^{(i)}, && C^r_{3i}(w) = w_2^{(i)} - \tfrac{h_i^2}{2} + \tfrac{(w_1^{(i)}-h_i)^2}{4},\\
&C^l_{4i}(w) = -\tfrac{h_i^3}{6} - w_3^{(i)} + w_2^{(i)} h_i - \tfrac{h_i^2}{2} w_1^{(i)}, && C^r_{4i}(w) = w_3^{(i)} - w_2^{(i)} h_i + \tfrac{h_i^2}{2} w_1^{(i)} - \tfrac{h_i^3}{6},\\
&C^l_{5i}(w) = (w_2^{(i)} - \tfrac{h_i^2}{2})^2 - 2(w_1^{(i)} - h_i)(w_3^{(i)} - \tfrac{h_i^3}{6}), && C^r_{5i}(w) = (w_2^{(i)} + \tfrac{h_i^2}{2})^2 - 2(w_1^{(i)} + h_i)(w_3^{(i)} + \tfrac{h_i^3}{6}),\\
&C^l_{6i}(w) = \tfrac{h_i}{2} w_2^{(i)} + \tfrac{h_i^3}{12} - \tfrac{h_i}{2}\rho^* + \tfrac{1}{3}\left(\rho^*\right)^{\frac{3}{2}} - w_3^{(i)}, && C^r_{6i}(w) = w_3^{(i)} - \tfrac{h_i}{2} w_2^{(i)} + \tfrac{h_i^3}{12} - \tfrac{h_i}{2}\rho + \tfrac{1}{3}\rho^{\frac{3}{2}},
\end{aligned}
$$

where $i = 0, \dots, n-1$, and ρ^*, ρ are given by (51).

Let $\mathcal{W}^*$ be a dual optimal solution. By Theorem 4.1, $\mathcal{X}^*$ is a primal optimal solution if and only if it satisfies the following conditions:

$$\mathcal{W}^* \in \partial\mathcal{F}(\mathcal{X}^*), \quad \mathcal{X}^* \in \{\mathcal{X} : A\mathcal{X} = 0\} \quad \text{and} \quad \mathcal{W}^* \in R(A^T).$$

Since the third condition is automatically satisfied for a dual solution, we can reduce the conditions to

$$\mathcal{W}^* \in \partial\mathcal{F}(\mathcal{X}^*) \quad \text{and} \quad \mathcal{X}^* \in \{\mathcal{X} : A\mathcal{X} = 0\}. \tag{52}$$

Note that $\mathcal{F}(\mathcal{X})$ is a closed proper convex function. By Theorem 23.5 of [26], we have

$$\mathcal{W} \in \partial\mathcal{F}(\mathcal{X}) \quad \text{if and only if } \mathcal{X} \in \partial\mathcal{F}^*(\mathcal{W}).$$

Hence, (52) becomes

$$\mathcal{X}^* \in \{\mathcal{X} : A\mathcal{X} = 0\} \cap \partial\mathcal{F}^*(\mathcal{W}^*).$$

It follows from (48) that $\mathcal{X}^*$ is a primal optimal solution if and only if

$$A\mathcal{X}^* = 0 \quad \text{and} \quad \mathcal{X}^* \in \mathrm{vec}(z, \Delta z) + \mathcal{N}_{\mathrm{dom}\,(\mathcal{F}^*)}(\mathcal{W}^*). \tag{53}$$

Once a dual optimal solution $\mathcal{W}^*$ is obtained, the index set (corresponding to binding constraints) defined below becomes completely known:

$$
\begin{aligned}
I(\mathcal{W}^*) := &\{(l,k,i) : \; C^l_{ki}(\mathcal{W}^*) = 0, \;\; k = 1, \dots, 6; \;\; i = 0, \dots, n-1\}\\
&\cup \{(r,k,i) : \; C^r_{ki}(\mathcal{W}^*) = 0, \;\; k = 1, \dots, 6; \;\; i = 0, \dots, n-1\}.
\end{aligned}
$$

Let

$$\rho^l_{ki} = \begin{cases} 1, \text{ if } (l,k,i) \in I(\mathcal{W}^*), \\ 0, \text{ otherwise.} \end{cases} \qquad \rho^r_{ki} = \begin{cases} 1, \text{ if } (r,k,i) \in I(\mathcal{W}^*), \\ 0, \text{ otherwise.} \end{cases}$$

The normal cone $\mathcal{N}_{\mathrm{dom}(\mathcal{F}^*)}(\mathcal{W}^*)$ is generated by the gradients of those binding constraints at $\mathcal{W}^*$. That is,

$$\mathcal{N}_{\mathrm{dom}(\mathcal{F}^*)}(\mathcal{W}^*) = \Big\{ \sum_{k=1}^{6}\sum_{i=0}^{n-1} \left(\lambda_{ki}^{l}\rho_{ki}^{l}\nabla C_{ki}^{l}(\mathcal{W}^*) + \lambda_{ki}^{r}\rho_{ki}^{r}\nabla C_{ki}^{r}(\mathcal{W}^*)\right) : \lambda_{ki}^{l} \geq 0,$$
$$\lambda_{ki}^{r} \geq 0,\ k = 1,\ldots,6; i = 0,\ldots,n-1 \Big\}.$$

Any solution to (53) is an optimal primal solution. However, we may seek a solution with certain properties, such as a solution with minimal L_1 norm, L_2 norm or L_∞ norm. If we take the L_1 norm, then we have the following problem:

$$\begin{aligned} &\min \|\mathcal{X}\|_1 \\ &\text{Subject to } A\mathcal{X} = 0, \\ &\qquad \mathcal{X} = \mathrm{vec}(z, \Delta z) + \sum_{k=1}^{6}\sum_{i=0}^{n-1} \left(\lambda_{ki}^{l}\rho_{ki}^{l}\nabla C_{ki}^{l}(\mathcal{W}^*) + \lambda_{ki}^{r}\rho_{ki}^{r}\nabla C_{ki}^{r}(\mathcal{W}^*)\right), \\ &\qquad \lambda_{ki}^{l} \geq 0, \quad \lambda_{ki}^{r} \geq 0, \quad k = 1,\ldots,6, \quad i = 0,\ldots,n-1. \end{aligned} \tag{54}$$

Let $\mathcal{X}_+ = \max\{\mathcal{X}, 0\}$ and $\mathcal{X}_- = \min\{\mathcal{X}, 0\}$, where the $\min(\cdot)$ and $\max(\cdot)$ operations are performed componentwise. Then

$$\mathcal{X} = \mathcal{X}_+ - \mathcal{X}_- \qquad \text{and} \qquad |\mathcal{X}\|_1 = e^T(\mathcal{X}_+ + \mathcal{X}_-).$$

This says that problem (54) is actually a linear program. We can be solved it efficiently using either interior-point or simplex based methods [11]. Note that the gradients $\nabla C_{ki}^{l}(\mathcal{W}^*)$ and $\nabla C_{ki}^{r}(\mathcal{W}^*)$ are $(4n-1)$-dimensional vectors with only three nonzero components. Hence, problem (54) is in general sparse.

If we choose the L_2 norm, then the resulting problem becomes a quadratic programming problem that can be handled by many available algorithms.

5 Concluding remarks

Recent computational results have indicated that first-derivative-based L_1 splines, that is, C^1-smooth cubic splines obtained by minimizing the L_1 norm of the difference between the first-order derivative of the spline and the divided differences of the data, have excellent shape preservation capability. Finding the coefficients of a first-derivative-based L_1 spline is a non-differentiable convex programming problem. We have derived a geometric dual that is a smooth convex program with a linear objective function and cubic constraints. This dual program is theoretically more convenient than the primal program and will allow use of more efficient algorithms, such as interior-point algorithms, to calculate the coefficients of the spline. Once a dual optimal solution is obtained, conversion to a primal optimal solution is no more than a linear program. The results obtained in this paper provide a platform for further study on the mathematical treatment of shape-preserving properties and the development of specific algorithms for this new class of cubic spline functions.

Extensions of the results of this paper from interpolating splines to approximating splines and from splines to solution of partial differential equations are possible. Smoothing splines that minimize a linear combination of the ℓ_1 norm of the difference between the spline and the data and L_1 norm of the second derivative of the spline have been shown in [17] to have excellent shape-preservation properties. A theoretical investigation of second-derivative-based L_1 smoothing splines has been carried out in [7]. Algorithms that take into account the special structure of the problem have been developed [6,30]. Development of first-derivative-based L_1 smoothing splines is of considerable theoretical and practical interest. L_1-norm-based approaches have also been shown to be advantageous for solution of differential equations [15].

It is worth mentioning that, while no boundary condition has been explicitly considered in our model, we can impose boundary conditions or other conditions on the coefficients of the underlying spline function without substantial changes in the analysis presented in this paper. For instance, if the conditions $\mathcal{Z}'(x_0) = \gamma_0$ and $\mathcal{Z}'(x_n) = \gamma_1$ are imposed on the spline, where γ_0, γ_1 are two constants, then only three more constraints, that is, $q_{n-1} + h_{n-1}u_{n-1} + \frac{h_{n-1}^2}{2}v_{n-1} = q_n$, $q_0 = \gamma_0$, and $q_n = \gamma_1$, have to be added to problem (2). Furthermore, the latter two simple constraints can be incorporated into the objective function by using the indicator function as we have done for the condition of $p_i = z_i$.

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