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1-Bit compressive sensing: Reformulation and RRSP-based sign recovery theory

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Abstract Recently, the 1-bit compressive sensing (1-bit CS) has been studied in the field of sparse signal recovery. Since the amplitude information of sparse signals in 1-bit CS is not available, it is often the support or the sign of a signal that can be exactly recovered with a decoding method. We first show that a necessary assumption (that has been overlooked in the literature) should be made for some existing theories and discussions for 1-bit CS. Without such an assumption, the found solution by some existing decoding algorithms might be inconsistent with 1-bit measurements. This motivates us to pursue a new direction to develop uniform and nonuniform recovery theories for 1-bit CS with a new decoding method which always generates a solution consistent with 1-bit measurements. We focus on an extreme case of 1-bit CS, in which the measurements capture only the sign of the product of a sensing matrix and a signal. We show that the 1-bit CS model can be reformulated equivalently as an ℓ_0-minimization problem with linear constraints. This reformulation naturally leads to a new linear-program-based decoding method, referred to as the 1-bit basis pursuit, which is remarkably different from existing formulations. It turns out that the uniqueness condition for the solution of the 1-bit basis pursuit yields the so-called restricted range space property (RRSP) of the transposed sensing matrix. This concept provides a basis to develop sign recovery conditions for sparse signals through 1-bit measurements. We prove that if the sign of a sparse signal can be exactly recovered from 1-bit measurements with 1-bit basis pursuit, then the sensing matrix must admit a certain RRSP, and that if the sensing matrix admits a slightly enhanced RRSP, then the sign of a k-sparse signal can be exactly recovered with 1-bit basis pursuit.

Keywords 1-bit compressive sensing, restricted range space property, 1-bit basis pursuit, linear program, ℓ_0-minimization, sparse signal recovery

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1 Introduction

Compressive sensing (CS) has attracted plenty of recent attention in the field of signal and image processing. One of the key mathematical issues addressed in CS is how a sparse signal can be reconstructed by a decoding algorithm. An extreme case of CS can be cast as the problem of seeking the sparsest solution of an underdetermined linear system, i.e.,

$$\min\{\|x\|_0 : \Phi x = b\},$$

where $\|x\|_0$ counts the number of nonzero components of x, $\Phi \in \mathbb{R}^{m\times n}$ $(m < n)$ is called a sensing matrix, and $b \in \mathbb{R}^m$ is the vector of nonadaptive measurements. It is known that the reconstruction of

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a sparse signal from a reduced number of acquired measurements is possible when the sensing matrix Φ admits certain properties (see [10–12, 17–19, 37, 41]). Note that measurements must be quantized. Fine quantization provides more information on a signal, making the signal more likely to be exactly recovered. However, fine quantization imposes a huge burden on measurement systems, leading to slower sampling rates and increased costs for hardware systems (see [5, 29, 35, 38]). Also, fine quantization introduces error to measurements. This motivates one to consider sparse signal recovery through lower bits of measurements. An extreme quantization is only one bit per measurement. As demonstrated in [4–6], it is possible, in some situations, to reconstruct a sparse signal within certain factors from 1-bit measurements, e.g., the sign of measurements. This motivates the recent development of CS with 1-bit measurements, called 1-bit compressive sensing (see [4, 6, 23, 26–28, 31]). An ideal model for 1-bit CS is the ℓ_0-minimization with sign constraints

$$\min\{\|x\|_0 : \operatorname{sign}(\Phi x) = y\}, \tag{1.1}$$

where $\Phi \in \mathbb{R}^{m\times n}$ is a sensing matrix and $y \in \mathbb{R}^m$ is the vector of 1-bit measurements. Throughout the paper, we assume that $m < n$. The sign function in (1.1) is applied element-wise. Due to the NP-hardness of (1.1), some relaxations of (1.1) have been investigated in the literature. A common relaxation is replacing $\|x\|_0$ with $\|x\|_1$ and replacing the constraint of (1.1) with the linear system

$$Y\Phi x \geqslant 0, \tag{1.2}$$

where $Y = \operatorname{diag}(y)$. In addition, an extra constraint, such as $\|x\|_2 = 1$ and $\|\Phi x\|_1 = m$, is introduced into this relaxation model in order to exclude some trivial solutions.

Only the acquired 1-bit information is insufficient to exactly reconstruct a sparse signal. For example, if $\operatorname{sign}(\Phi x^*) = y$ where $y \in \{1, -1\}^m$, then any small perturbation $x^* + u$ also satisfies this equation, making the exact recovery of x^* almost impossible by whichever decoding algorithms. While the sign information of measurements might not be enough to exactly reconstruct a signal, it might be adequate to recover the support or the sign of the signal. Thus 1-bit CS still has found applications in signal recovery [4–6, 23, 26], imaging processing [7, 8], and matrix completion [16].

The 1-bit CS was first proposed and investigated by Boufounos and Baraniuk [6]. Since 2008, numerous algorithms have been developed in this direction, including greedy algorithms (see [2, 4, 22–25, 39]) and convex and nonconvex programming algorithms (see [1, 6, 28, 30–32, 34]). To find a polynomial-time solver for the 1-bit CS problems, a linear programming model based on (1.2) has been formulated, and certain stability results for reconstruction have been shown in [31] as well.

In classic CS setting, it is well known that when a sensing matrix admits some properties such as mutual coherence [9, 18], null space property (NSP) [14, 40], restricted isometry property (RIP) [12] or range space property (RSP) of Φ^{T} [41, 42], the signals with low sparsity levels can be exactly recovered by the basis pursuit and other algorithms. This motivates one to investigate whether similar recovery theories can also be established for 1-bit CS problems. In [24], the binary iterative hard thresholding (BIHT) algorithm for 1-bit CS problems is discussed and the so-called binary ε-stable embedding (BϵSE) condition is introduced. The BϵSE can be seen as an extension of the RIP. However, at the current stage, the theoretical analysis for the guaranteed performance of 1-bit CS algorithms is far from complete, in contrast to the classic CS. Recovery conditions in terms of the property of Φ and/or y are still under development.

The fundamental assumption on 1-bit CS is that any solution x generated by an algorithm should be consistent with the acquired 1-bit measurements in the sense that

$$\operatorname{sign}(\Phi x) = y = \operatorname{sign}(\Phi x^*), \tag{1.3}$$

where x^* is the targeted signal. Clearly, it is very difficult to directly solve a problem with such a constraint if it does not have a tractable reformation. From a computational point of view, an ideal relaxation or reformulation of the sign constraint is a linear system. The current algorithms and theories for 1-bit CS (see [5, 6, 31, 34]) have been developed largely based on the system (1.2), which is a linear relaxation of (1.3). In Section 2, we show that the existing relaxation based on (1.2) is not equivalent to the original 1-bit CS model. In fact, a vector satisfying (1.2) together with a trivial-solution excluder, such as $\|x\|_2 = 1$ or $\|\Phi x\|_1 = m$, may not be consistent with the acquired 1-bit measurements y. Some necessary conditions must be imposed on the matrix in order to ensure that the solution of a decoding algorithm based on (1.2) is consistent with y. These necessary conditions have been overlooked in the literature (see the discussion in Section 2 for details).

Many existing discussions for 1-bit CS do not distinguish between zero and positive measurements. Both are mapped to 1 (or -1) by a nonstandard sign function. In Section 2, we point out that it is beneficial to allow y admitting zero components and to treat zero and nonzero measurements separately from both practical and mathematical points of view. Failing to distinguish zero and nonzero magnitude of measurements might yield ambiguity of measurements when sensing vectors are nearly orthogonal to the signal. Such ambiguity might prevent from acquiring a correct sign of measurements due to signal noises or errors in computation.

This motivates us to pursue a new direction to establish a recovery theory for 1-bit CS. Our study is remarkably different from existing ones in several aspects.

(a) The acquired sign measurement y is allowed to admit zero components. When y does not contain zero components, our model immediately reduces to the existing 1-bit CS model.

(b) We introduce a truly equivalent reformulation of the 1-bit CS model (1.1). The model (1.1) is reformulated equivalently as an ℓ_0-minimization problem with linear constraints. Replacing $\|x\|_0$ with $\|x\|_1$ leads naturally to a new linear-program-based decoding method, referred to as the 1-bit basis pursuit. Different from existing formulations, the new reformulation ensures that the solution of the 1-bit basis pursuit is always consistent with the acquired 1-bit measurement y.

(c) The sign recovery theory developed in the paper is from the perspective of the restricted range space properties (RRSP) of transposed sensing matrices. In classic CS, it has been shown in [41, 42] that any k-sparse signal can be exactly recovered with basis pursuit if and only if the transposed sensing matrix admits the so-called range space property (RSP) of order k. This property is equivalent to the well known NSP of order k in the sense that both are the necessary and sufficient conditions for the uniform recovery of k-sparse signals. The new reformulation of the 1-bit CS model proposed in this paper makes it possible to develop an analogous recovery guarantee for the sign of sparse signals with 1-bit basis pursuit. This development naturally yields the concept of the restricted range space property (RRSP) which gives rise to some necessary and sufficient conditions for the nonuniform and uniform recovery of the sign of sparse signals from 1-bit measurements.

The main results of the paper can be summarized as follows:

• (Theorem 3.6, nonuniform) If the 1-bit basis pursuit can exactly recover the sign of k-sparse signals consistent with 1-bit measurements y, then Φ must admit the N-RRSP of order k with respect to y (see Definition 3.5).

• (Theorem 3.9, nonuniform) If Φ admits the S-RRSP of order k with respect to y (see Definition 3.7), then from 1-bit measurements, the 1-bit basis pursuit can exactly recover the sign of k-sparse signals which are the sparsest vectors consistent with y.

• (Theorem 4.2, uniform) If the 1-bit basis pursuit can exactly recover the sign of all k-sparse signals from 1-bit measurements, then Φ must admit the so-called N-RRSP of order k (see Definition 4.1).

• (Theorem 4.4, uniform) If the matrix admits the S-RRSP of order k (see Definition 4.3), then from 1-bit measurements, the 1-bit basis pursuit can exactly recover the sign of all k-sparse signals which are the sparsest vectors consistent with 1-bit measurements.

The above-mentioned definitions and theorems are given in Sections 3 and 4. Central to the proof of these results is Theorem 3.2 which provides a full characterization for the uniqueness of solutions to the 1-bit basis pursuit, and thus yields a fundamental basis to develop recovery conditions.

This paper is organized as follows. We provide motivations for a new reformulation of the 1-bit CS model in Section 2. Based on the reformulation, nonuniform sign recovery conditions with 1-bit basis pursuit are developed in Section 3, and uniform sign recovery conditions are developed in Section 4. The proof of Theorem 3.2 is given in Section 5. Conclusions are presented in Section 6.

We use the following notation in the paper. Let $\mathbb{R}^n_+$ be the set of non-negative vectors in $\mathbb{R}^n$. The vector $x \in \mathbb{R}^n_+$ is also written as $x \geqslant 0$. Given a set S, $|S|$ denotes the cardinality of S. For $x \in \mathbb{R}^n$ and $S \subseteq \{1, \ldots, n\}$, let $x_S \in \mathbb{R}^{|S|}$ denote the subvector of x obtained by deleting those components x_i with $i \notin S$, and let $\mathrm{supp}(x) = \{i : x_i \neq 0\}$ denote the support of x. The ℓ_0-norm $\|x\|_0$ counts the number of nonzero components of x, and the ℓ_1-norm of x is defined as $\|x\|_1 = \sum_{i=1}^n |x_i|$. For a matrix $\Phi \in \mathbb{R}^{m\times n}$, we use Φ^{T} to denote the transpose of Φ, $\mathcal{N}(\Phi) = \{x : \Phi x = 0\}$ the null space of Φ, $\mathcal{R}(\Phi^{\mathrm{T}}) = \{\Phi^{\mathrm{T}} u : u \in \mathbb{R}^m\}$ the range space of Φ^{T}, $\Phi_{J,n}$ the submatrix of Φ formed by deleting the rows of Φ which are not indexed by J, and $\Phi_{m,J}$ the submatrix of Φ formed by deleting the columns of Φ which are not indexed by J. e with a suitable dimension is the vector of ones, i.e., $e = (1, \ldots, 1)^{\mathrm{T}}$.

2 Reformulation of 1-bit compressive sensing

In this section, we point out that for a given matrix, existing 1-bit CS algorithms based on the relaxation (1.2) cannot guarantee the found solution being consistent with the acquired 1-bit measurement y, unless the matrix satisfies some condition. This motivates one to propose a new reformulation of the 1-bit CS problem so that the resulting algorithm can automatically ensure its solution being consistent with 1-bit measurements.

2.1 Consistency conditions for existing 1-bit CS methods

The standard sign function is defined as $\mathrm{sign}(t) = 1$ if $t > 0$, $\mathrm{sign}(t) = -1$ if $t < 0$, and $\mathrm{sign}(t) = 0$ otherwise. In the 1-bit CS literature, many researchers do not distinguish between zero and positive values of measurements and thus define $\mathrm{sign}(t) = 1$ for $t \geqslant 0$ and $\mathrm{sign}(t) = -1$ otherwise. The function $\mathrm{sign}(\cdot)$ defined this way is referred to as a nonstandard sign function in this paper. We now point out that no matter a standard or nonstandard sign function is used, *the equation* $y = \mathrm{sign}(\Phi x)$ *is generally not equivalent to the system* (1.2) *even if a trivial-solution excluder such as* $\|x\|_2 = 1$ *or* $\|\Phi x\|_1 = m$ *is used, unless certain necessary assumptions are made on* Φ. First, since $y = \mathrm{sign}(\Phi x)$ implies $Y\Phi x \geqslant 0$ (this fact was observed in [6]), the following statement is obvious:

Lemma 2.1. *If* $\Phi \in \mathbb{R}^{m\times n}$ *and* $y \in \{1, -1\}^m$ *or* $y \in \{1, 0, -1\}^m$, *then* $\{x : \mathrm{sign}(\Phi x) = y\} \subseteq \{x : Y\Phi x \geqslant 0\}$.

Without a further assumption on Φ, however, the system (1.2) does not imply $\mathrm{sign}(\Phi x) = y$ even if some trivial solutions of (1.2) are excluded by adding a widely used trivial-solution excluder, such as $\|x\|_2 = 1$ or $\|\Phi x\|_1 = m$, to the system. In fact, for any given y with $J_- = \{i : y_i = -1\} \neq \emptyset$, we see that all vectors $0 \neq \widetilde{x} \in \mathcal{N}(\Phi)$ (or more generally, $\widetilde{x} \neq 0$ satisfying $\Phi_{J_-,n}\widetilde{x} = 0$ and $\Phi_{J_+,n}\widetilde{x} \geqslant 0$) satisfy $Y\Phi\widetilde{x} \geqslant 0$, but for these vectors, $\mathrm{sign}(\Phi\widetilde{x}) \neq y$ no matter $\mathrm{sign}(\cdot)$ is standard or nonstandard. The trivial-solution excluder $\|x\|_2 = 1$ (see [6]) cannot exclude vectors satisfying $0 \neq \widetilde{x} \in \mathcal{N}(\Phi)$ from the set $\{x : Y\Phi x \geqslant 0\}$. The excluder $\|\Phi x\|_1 = m$ (see [31, 34]) cannot exclude $\widetilde{x}$ satisfying $\Phi_{J_-,n}\widetilde{x} = 0$ and $0 \neq \Phi_{J_+,n}\widetilde{x} \geqslant 0$ from $\{x : Y\Phi x \geqslant 0\}$. This implies that the solutions of some existing 1-bit CS algorithms such as

$$\min\{\|x\|_1 : Y\Phi x \geqslant 0, \|x\|_2 = 1\}, \tag{2.1}$$

$$\min\{\|x\|_1 : Y\Phi x \geqslant 0, \|\Phi x\|_1 = m\} \tag{2.2}$$

may not be consistent with the acquired 1-bit measurements. For example, let

$$\Phi = \begin{bmatrix} 2 & -1 & 0 & 2 \\ -1 & 1 & 1 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \tag{2.3}$$

Clearly, for any scalar $\alpha > 0$, $\widetilde{x}(\alpha) = (\alpha, \alpha, 0, 0)^{\mathrm{T}} \in \{x : Y\Phi x \geqslant 0\}$, but $\widetilde{x}(\alpha) \notin \{x : y = \mathrm{sign}(\Phi x)\}$ no matter a standard or nonstandard sign function is used, and no matter which of the above-mentioned trivial-solution excluders is used. Clearly, there exists a positive number α^* such that $\widetilde{x}(\alpha^*) = (\alpha^*, \alpha^*, 0, 0)^{\mathrm{T}}$ is an optimal solution to (2.1) or (2.2). But this optimal solution is not consistent with y.

The above discussion indicates that when $J_- \neq \emptyset$, $x = 0$ and $x \in \mathcal{N}(\Phi)$ are not contained in the set $\{x : \mathrm{sign}(\Phi x) = y\}$. In this case, we see from Lemma 2.1 that

$$\{x : \mathrm{sign}(\Phi x) = y\} \subseteq \{x : Y\Phi x \geqslant 0, x \neq 0\}, \tag{2.4}$$

$$\{x : \mathrm{sign}(\Phi x) = y\} \subseteq \{x : Y\Phi x \geqslant 0, \Phi x \neq 0\}. \tag{2.5}$$

We now find a condition to ensure the opposite direction of the above containing relations.

Lemma 2.2. *Let* $\mathrm{sign}(\cdot)$ *be the nonstandard sign function. Let* $\Phi \in \mathbb{R}^{m\times n}$ *and* $y \in \{1, -1\}^m$ *with* $J_- = \{i : y_i = -1\} \neq \emptyset$ *being given. Then*

$$\{x : Y\Phi x \geqslant 0, x \neq 0\} \subseteq \{x : \mathrm{sign}(\Phi x) = y\} \tag{2.6}$$

if and only if

$$\left[\bigcup_{i\in J_-} \mathcal{N}(\Phi_{i,n})\right] \cap \{d : \Phi_{J_+,n} d \geqslant 0, \Phi_{J_-,n} d \leqslant 0\} = \{0\}, \tag{2.7}$$

where $J_+ = \{i : y_i = 1\}$.

Proof. Let x be an arbitrary vector in the set $\{x : Y\Phi x \geqslant 0, x \neq 0\}$. Note that $y \in \{1, -1\}^m$. So $Y\Phi x \geqslant 0$ together with $x \neq 0$ is equivalent to

$$\Phi_{J_+,n} x \geqslant 0, \quad \Phi_{J_-,n} x \leqslant 0, \quad x \neq 0. \tag{2.8}$$

Under the condition (2.7), we see that for any x satisfying (2.8), it must hold that $x \notin \bigcup_{i\in J_-} \mathcal{N}(\Phi_{i,n})$ which implies that $\Phi_{i,n} x \neq 0$ for all $i \in J_-$. Thus under (2.7), the system (2.8) becomes

$$\Phi_{J_+,n} x \geqslant 0, \quad \Phi_{J_-,n} x < 0, \quad x \neq 0$$

which, by the definition of the nonstandard sign function, implies that $\mathrm{sign}(\Phi x) = y$. Thus (2.6) holds.

We now assume that (2.7) does not hold. Then there exists a vector $d^* \neq 0$ satisfying that

$$d^* \in \left[\bigcup_{i\in J_-} \mathcal{N}(\Phi_{i,n})\right] \cap \{d : \Phi_{J_+,n} d \geqslant 0, \Phi_{J_-,n} d \leqslant 0\}. \tag{2.9}$$

The fact

$$d^* \in \{d : \Phi_{J_+,n} d \geqslant 0, \Phi_{J_-,n} d \leqslant 0\}$$

implies that

$$d^* \in \{x : Y\Phi x \geqslant 0, x \neq 0\},$$

and

$$0 \neq d^* \in \bigcup_{i\in J_-} \mathcal{N}(\Phi_{i,n})$$

implies that there is $i \in J_-$ such that $\Phi_{i,n} d^* = 0$. By the definition of nonstandard sign function, this implies that $\mathrm{sign}(\Phi_{i,n} d^*) = 1 \neq y_i$ (since $y_i = -1$ for $i \in J_-$). So $d^* \notin \{x : \mathrm{sign}(\Phi x) = y\}$, and thus (2.6) does not hold.

The above proof shows that (2.6) and (2.7) are equivalent. □

Replacing $x \neq 0$ with $\Phi x \neq 0$ and using the same argument as above yields the next statement.

Lemma 2.3. *Under the same conditions of Lemma* 2.2*, the following statement holds*:

$$\{x : Y\Phi x \geqslant 0, \Phi x \neq 0\} \subseteq \{x : \mathrm{sign}(\Phi x) = y\}$$

if and only if

$$\left[\bigcup_{i\in J_-} \mathcal{N}(\Phi_{i,n})\right] \cap \{d : \Phi_{J_+,n} d \geqslant 0, \Phi_{J_-,n} d \leqslant 0, \Phi d \neq 0\} = \emptyset, \tag{2.10}$$

where $\emptyset$ *denotes the empty set.*

Therefore, we have the following result.

Theorem 2.4. *Let* sign$(\cdot)$ *be the nonstandard sign function, and let* $\Phi \in \mathbb{R}^{m\times n}$ *and* $y \in \{1,-1\}^m$ *be given.*

(i) *If* $J_- = \emptyset$, *then* $\{x : \mathrm{sign}(\Phi x) = y\} = \{x : Y\Phi x \geqslant 0\}$.

(ii) *If* $J_- \neq \emptyset$, *then* $\{x : \mathrm{sign}(\Phi x) = y\} = \{x : Y\Phi x \geqslant 0, x \neq 0\}$ *if and only if* (2.7) *holds.*

(iii) *If* $J_- \neq \emptyset$, *then* $\{x : \mathrm{sign}(\Phi x) = y\} = \{x : Y\Phi x \geqslant 0, \Phi x \neq 0\}$ *if and only if* (2.10) *holds.*

The result (i) above is obvious. Results (ii) and (iii) follow by combining (2.4), (2.5), Lemmas 2.2 and 2.3. It is easy to verify that the example (2.3) does not satisfy (2.7) and (2.10).

We now consider the standard sign function. In this case, for $y = 0$, the set $\{x : Y\Phi x \geqslant 0\} = \mathbb{R}^n$ and

$$\{x : 0 = \mathrm{sign}(\Phi x)\} = \{x : \Phi x = 0\} = \mathcal{N}(\Phi) \neq \mathbb{R}^n$$

provided that $\Phi \neq 0$; for $y \neq 0$, we see that $\mathcal{N}(\Phi) \subseteq \{x : Y\Phi x \geqslant 0\}$ but any vector in $\mathcal{N}(\Phi)$ fails to satisfy the equation $\mathrm{sign}(\Phi x) = y$. Thus we have the following observation:

Lemma 2.5. *For standard sign functions and any nonzero* $\Phi \in \mathbb{R}^{m\times n}$, *we have*

$$\{x : Y\Phi x \geqslant 0\} \neq \{x : \mathrm{sign}(\Phi x) = y\}.$$

In general, the set $\{x : Y\Phi x \geqslant 0\}$ can be significantly larger than $\{x : \mathrm{sign}(\Phi x) = y\}$. In what follows, we only focus on the nontrivial case $y \neq 0$. For a given $0 \neq y \in \{1,-1,0\}^m$, when $J_0 = \{i : y_i = 0\} \neq \emptyset$, the vectors in $\mathcal{N}(\Phi)$ and the vectors x satisfying $\Phi_{J_0,n}x \neq 0$ do not satisfy the constraint $\mathrm{sign}(\Phi x) = y$. These vectors must be excluded from $\{x : Y\Phi x \geqslant 0\}$ in order to get a tighter relaxation for the sign equation. In other words, only vectors satisfying $\Phi x \neq 0$ and $\Phi_{J_0,n}x = 0$, i.e., $x \in \mathcal{N}(\Phi_{J_0,n})\backslash\mathcal{N}(\Phi)$, should be considered. (Note that $\mathcal{N}(\Phi) \subseteq \mathcal{N}(\Phi_{J_0,n})$ due to the fact $\Phi_{J_0,n}$ being a submatrix of Φ.) Thus we have the following result.

Theorem 2.6. *Let* $\Phi \in \mathbb{R}^{m\times n}$ *and* $0 \neq y \in \{1,0,-1\}^m$ *be given. For the standard sign function, the following statements hold*:

(i) $\{x : y = \mathrm{sign}(\Phi x)\} \subseteq \{x : Y\Phi x \geqslant 0, \Phi_{J_0,n}x = 0, \Phi x \neq 0\}$.

(ii) $\{x : Y\Phi x \geqslant 0, \Phi_{J_0,n}x = 0, \Phi x \neq 0\} \subseteq \{x : \mathrm{sign}(\Phi x) = y\}$ *if and only if*

$$\left[\bigcup_{i\in J_+\cup J_-} \mathcal{N}(\Phi_{i,n})\right] \cap \{d : \Phi_{J_+,n}d \geqslant 0, \Phi_{J_-,n}d \leqslant 0, \Phi_{J_0,n}d = 0, \Phi d \neq 0\} = \emptyset. \tag{2.11}$$

Proof. The statement (i) follows from Lemma 2.1 and the discussion before Theorem 2.6. We now prove the statement (ii). First we assume that (2.11) holds, and let $\hat{x}$ be an arbitrary vector in the set

$$\{x : Y\Phi x \geqslant 0, \Phi_{J_0,n}x = 0, \Phi x \neq 0\}.$$

Then

$$\Phi_{J_+,n}\hat{x} \geqslant 0, \quad \Phi_{J_-,n}\hat{x} \leqslant 0, \quad \Phi_{J_0,n}\hat{x} = 0, \quad \Phi\hat{x} \neq 0. \tag{2.12}$$

As $y \neq 0$, the set $J_+ \cup J_- \neq \emptyset$. It follows from (2.11) and (2.12) that $\hat{x} \notin \bigcup_{i\in J_+\cup J_-} \mathcal{N}(\Phi_{i,n})$, which implies that the inequalities $\Phi_{J_+,n}\hat{x} \geqslant 0$ and $\Phi_{J_-}\hat{x} \leqslant 0$ in (2.12) must hold strictly, i.e.,

$$\Phi_{J_+,n}\hat{x} > 0, \quad \Phi_{J_-,n}\hat{x} < 0, \quad \Phi_{J_0,n}\hat{x} = 0, \quad \Phi\hat{x} \neq 0,$$

and hence $\mathrm{sign}(\Phi\hat{x}) = y$. So

$$\{x : Y\Phi x \geqslant 0, \Phi_{J_0,n}x = 0, \Phi x \neq 0\} \subseteq \{x : \mathrm{sign}(\Phi x) = y\}. \tag{2.13}$$

We now further prove that if (2.11) does not hold, then (2.13) does not hold. Indeed, assume that (2.11) is not satisfied. Then there exists a vector $\hat{d}$ satisfying

$$\Phi_{J_+,n}\hat{d} \geqslant 0, \quad \Phi_{J_-,n}\hat{d} \leqslant 0, \quad \Phi_{J_0,n}\hat{d} = 0, \quad \Phi\hat{d} \neq 0$$

and

$$\hat{d} \in \bigcup_{i \in J_+ \cup J_-} \mathcal{N}(\Phi_{i,n}).$$

This implies that

$$\hat{d} \in \{x : Y\Phi x \geqslant 0, \Phi_{J_0,n} x = 0, \Phi x \neq 0\}$$

and there exists $i \in J_+ \cup J_-$ such that $\Phi_{i,n}\hat{d} = 0$. Thus $\mathrm{sign}(\Phi_{i,n}\hat{d}) = 0 \neq y_i$ where $y_i = 1$ or -1 (since $i \in J_+ \cup J_-$). So (2.13) does not hold. □

Therefore, under the conditions of Theorem 2.6, the set $\{x : \mathrm{sign}(\Phi x) = y\}$ coincides with

$$\{x : Y\Phi x \geqslant 0, \Phi_{J_0,n} x = 0, \Phi x \neq 0\}$$

if and only if (2.11) holds. Recall that the 1-bit CS problem (see [4, 6, 31]) can be cast as the ℓ_0-minimization problem (1.1), which admits the relaxation

$$\min\{\|x\|_0 : Y\Phi x \geqslant 0, \|x\|_2 = 1\}, \tag{2.14}$$

$$\min\{\|x\|_0 : Y\Phi x \geqslant 0, \|\Phi x\|_1 = m\}, \tag{2.15}$$

where m is not essential and can be replaced with any positive constant. Replacing $\|x\|_0$ by $\|x\|_1$ immediately leads to (2.1) and (2.2) which are linear programming models.

To guarantee that problems (2.14) and (2.15) are equivalent to (1.1) and that problems (2.1) and (2.2) are equivalent to the problem

$$\min\{\|x\|_1 : \mathrm{sign}(\Phi x) = y\}, \tag{2.16}$$

as shown in Theorems 2.4 and 2.6, the conditions (2.7), (2.10) or (2.11), depending on the definition of the sign function, must be imposed on the matrix. These conditions have been overlooked in the literature. If (2.7), (2.10) or (2.11) is not satisfied, the feasible sets of (2.14), (2.15), (2.1) and (2.2) are larger than those of (1.1) and (2.16), and thus their solutions might not satisfy the sign equation $\mathrm{sign}(\Phi x) = y$. In other words, the constructed signal through the algorithms for solving (2.14), (2.15), (2.1) and (2.2) might be inconsistent with the acquired 1-bit measurements.

2.2 Allowing zero in sign measurement y

The 1-bit CS model with a nonstandard sign function does not cause any inconvenience or difficulty when the magnitude of all components of $|\Phi x^*|$ is relatively large, in which case $\mathrm{sign}(\Phi x^*)$ is stable in the sense that any small perturbation of Φx^* does not affect its sign. However, when $|\Phi x^*|$ admits a very small components (this case does happen in some situations, as we point out later), the nonstandard sign function might introduce certain ambiguity into the 1-bit CS model since $\Phi x^* > 0$, $\Phi x^* = 0$ and $0 \neq \Phi x^* \geqslant 0$ yield the same measurement $y = (1, 1, \ldots, 1)^{\mathrm{T}}$. Once y is acquired, the information concerning which of the above cases yields y in 1-bit CS models is lost. In this situation, through sign information only, it might be difficult to reconstruct the information of the targeted signal no matter what 1-bit CS algorithms are used.

When the magnitude of $|\Phi_{i,n} x^*|$ is very small, errors or noises do affect the reliability of the measurement y. The reliability of y is vital since the unknown signal is expected to be partially or fully reconstructed from y. Suppose that x^* is the signal to recovery. We consider a sensing matrix $\Phi \in \mathbb{R}^{m \times n}$ whose rows are uniformly drawn from the surface of the n-dimensional unit ball $\{u \in \mathbb{R}^n : \|u\|_2 = 1\}$. Note that for any small positive number $\epsilon > 0$, with positive probability, a drawn vector lies in the region of the unit surface

$$\{u \in \mathbb{R}^n : \|u\|_2 = 1, |u^{\mathrm{T}} x^*| \leqslant \epsilon\}.$$

The sensing row vector $\Phi_{i,n}$ drawn in this region yields a very small product $\Phi_{i,n} x^* \approx 0$, at which $\mathrm{sign}(\Phi_{i,n} x^*)$ becomes sensitive or uncertain in the sense that any small error in measuring $\Phi_{i,n} x^*$ can totally flip its sign, leading to an opposite of the correct sign measurement. In this situation, not only the acquired information y_i might be unreliable to be used for the recovery of the sign of a signal, but also the measured value $y_i = 1$ or -1 does not reflect the fact $\Phi_{i,n}x^* \approx 0$, which indicates that x^* is nearly orthogonal to the known sensing vector $\Phi_{i,n}$. The information $\Phi_{i,n}x^* \approx 0$ is particularly useful to help locate the position of the unknown vector x^*. Using only 1 or -1 as the sign of $\Phi_{i,n}x^*$, however, the information $\Phi_{i,n}x^* \approx 0$ is completely lost in the 1-bit CS model. Allowing $y_i = 0$ in this case can correctly reflect the relation of $\Phi_{i,n}$ and x^* when they are nearly orthogonal. Taking into account the small magnitude of $|\Phi_{i,n}x^*|$ and allowing y to admit zero components provides a practical means to avoid the aforementioned ambiguity of sign measurements resulting from the nonstandard sign function. By using the standard sign function to distinguish the three different cases $\Phi x^* > 0$, $\Phi x^* = 0$, and $0 \neq \Phi x^* \geqslant 0$, the resulting sign measurement y would carry more information of the signal, which might increase the chance for the sign recovery of the signal.

Thus we consider the 1-bit CS model with the standard sign function in this paper. In fact, the standard sign function was already used by Plan and Vershynin [31], but their discussions are based on the linear relaxation of (1.2).

2.3 Reformulation of the 1-bit CS model

From the above discussions, the system (1.2) is generally a loose relaxation of the sign constraint of (1.1). The 1-bit CS algorithms based on this relaxation might generate a solution inconsistent with 1-bit measurements if a sensing matrix does not satisfy the conditions specified in Theorems 2.4 and 2.6. We now introduce a new reformulation of the 1-bit CS model, which can ensure that the solution of our 1-bit CS algorithm is always consistent with the acquired 1-bit measurements.

In the remainder of the paper, we focus on the 1-bit CS problem with standard sign function. For a given $y \in \{-1, 1, 0\}^m$, we use J_+, J_- and J_0 to denote the indices of positive, negative, and zero components of y, respectively, i.e.,

$$J_+ = \{i : y_i = 1\}, \quad J_- = \{i : y_i = -1\}, \quad J_0 = \{i : y_i = 0\}. \tag{2.17}$$

Since these indices are determined by y, we also write them as $J_+(y), J_-(y)$ and $J_0(y)$ when necessary. By using (2.17), the constraint $\mathrm{sign}(\Phi x) = y$ can be written as

$$\mathrm{sign}(\Phi_{J_+,n}x) = e_{J_+}, \quad \mathrm{sign}(\Phi_{J_-,n}x) = -e_{J_-}, \quad \Phi_{J_0,n}x = 0. \tag{2.18}$$

Thus the model (1.1) with $y \in \{-1, 1, 0\}^m$ can be stated as

$$\begin{aligned} \min \quad & \|x\|_0 \\ \text{s.t.} \quad & \mathrm{sign}(\Phi_{J_+,n}x) = e_{J_+}, \quad \mathrm{sign}(\Phi_{J_-,n}x) = -e_{J_-}, \\ & \Phi_{J_0,n}x = 0. \end{aligned} \tag{2.19}$$

Consider the system in $u \in \mathbb{R}^n$,

$$\Phi_{J_+,n}u \geqslant e_{J_+}, \quad \Phi_{J_-,n}u \leqslant -e_{J_-}, \quad \Phi_{J_0,n}u = 0. \tag{2.20}$$

Clearly, if x satisfies (2.18), then there exists a positive number $\alpha > 0$ such that $u = \alpha x$ satisfies the system (2.20); conversely, if u satisfies the system (2.20), then $x = u$ satisfies the system (2.18). Note that $\|x\|_0 = \|\alpha x\|_0$ for any $\alpha \neq 0$. Thus (2.19) can be reformulated as the ℓ_0-minimization problem

$$\begin{aligned} \min \quad & \|x\|_0 \\ \text{s.t.} \quad & \Phi_{J_+,n}x \geqslant e_{J_+}, \quad \Phi_{J_-,n}x \leqslant -e_{J_-}, \quad \Phi_{J_0,n}x = 0. \end{aligned} \tag{2.21}$$

From the relation of (2.18) and (2.20), we immediately have the following observation.

Proposition 2.7. *If x^* is an optimal solution to the* 1*-bit CS model* (2.19)*, then there exists a positive number $\alpha > 0$ such that αx^* is an optimal solution to the ℓ_0-problem* (2.21)*; conversely, if x^* is an optimal solution to the ℓ_0-problem* (2.21)*, then x^* must be an optimal solution to* (2.19)*.*

As a result, to study the 1-bit CS model (2.19), it is sufficient to investigate the model (2.21). This makes it possible to use the CS methodology to study the 1-bit CS problem (2.19). Motivated by (2.21), we consider the ℓ_1-minimization

$$\begin{aligned} \min \quad & \|x\|_1 \\ \text{s.t.} \quad & \Phi_{J_+,n}x \geqslant e_{J_+}, \quad \Phi_{J_-,n}x \leqslant -e_{J_-}, \quad \Phi_{J_0,n}x = 0, \end{aligned} \tag{2.22}$$

which can be seen as a natural decoding method for the 1-bit CS problems. In this paper, the problem (2.22) is referred to as the 1-bit basis pursuit. It is worth stressing that the optimal solution of (2.22) is always consistent with y as indicated by Proposition 2.7. More importantly, the later analysis indicates that our reformulation makes it possible to develop a sign recovery theory for sparse signals from 1-bit measurements.

For the convenience of analysis, we define the sets $\mathcal{A}(\cdot)$, $\widetilde{\mathcal{A}}_+(\cdot)$ and $\widetilde{\mathcal{A}}_-(\cdot)$ which are used frequently in this paper. Let $x^* \in \mathbb{R}^n$ satisfy the constraints of (2.22). At x^*, let

$$\mathcal{A}(x^*) = \{i : (\Phi x^*)_i = 1\} \cup \{i : (\Phi x^*)_i = -1\}, \tag{2.23}$$

$$\tilde{\mathcal{A}}_+(x^*) = J_+ \setminus \mathcal{A}(x^*), \quad \tilde{\mathcal{A}}_-(x^*) = J_- \setminus \mathcal{A}(x^*). \tag{2.24}$$

Clearly, $\mathcal{A}(x^*)$ is the index set of active constraints among the inequality constraints of (2.22), $\tilde{\mathcal{A}}_+(x^*)$ is the index set of inactive constraints in the first group of inequalities of (2.22) (i.e., $\Phi_{J_+,n}x^* \geqslant e_{J_+}$), and $\tilde{\mathcal{A}}_-(x^*)$ is the index set of inactive constraints in the second group of inequalities of (2.22) (i.e., $\Phi_{J_-,n}x^* \leqslant -e_{J_-}$). Thus we see that

$$\begin{aligned} &(\Phi x^*)_i = 1 \quad \text{for} \quad i \in \mathcal{A}(x^*) \cap J_+, \quad (\Phi x^*)_i > 1 \quad \text{for} \quad i \in \tilde{\mathcal{A}}_+(x^*), \\ &(\Phi x^*)_i = -1 \quad \text{for} \quad i \in \mathcal{A}(x^*) \cap J_-, \quad (\Phi x^*)_i < -1 \quad \text{for} \quad i \in \tilde{\mathcal{A}}_-(x^*). \end{aligned}$$

We also need symbols $\pi(\cdot)$ and $\varrho(\cdot)$ defined as follows. Denote the elements in J_+ by $i_k \in \{1, \dots, m\}$, $k = 1, \dots, p$, i.e., $J_+ = \{i_1, i_2, \dots, i_p\}$ where $p = |J_+|$. Without loss of generality, we let the elements be sorted in ascending order $i_1 < i_2 < \cdots < i_p$. Then we define the bijective mapping $\pi : J_+ \to \{1, \dots, p\}$ as

$$\pi(i_k) = k \quad \text{for all} \quad k = 1, \dots, p. \tag{2.25}$$

Similarly, let $J_- = \{j_1, j_2, \dots, j_q\}$, where $q = |J_-|$, $j_k \in \{1, \dots, m\}$ for $k = 1, \dots, q$ and $j_1 < j_2 < \cdots < j_q$. We define the bijective mapping $\varrho : J_- \to \{1, \dots, q\}$ as

$$\varrho(j_k) = k \quad \text{for all} \quad k = 1, \dots, q. \tag{2.26}$$

By introducing variables $\alpha \in \mathbb{R}_+^{|J_+|}$ and $\beta \in \mathbb{R}_+^{|J_-|}$, the problem (2.22) can be written as

$$\begin{aligned} \min \quad & \|x\|_1 \\ \text{s.t.} \quad & \Phi_{J_+,n}x - \alpha = e_{J_+}, \\ & \Phi_{J_-,n}x + \beta = -e_{J_-}, \\ & \Phi_{J_0,n}x = 0, \\ & \alpha \geqslant 0, \quad \beta \geqslant 0. \end{aligned} \tag{2.27}$$

Note that for any optimal solution (x^*, α^*, β^*) of (2.27), we have $\alpha^* = \Phi_{J_+,n}x^* - e_{J_+}$ and $\beta^* = -e_{J_-} - \Phi_{J_-,n}x^*$. Using (2.23)–(2.26), we immediately have the following observation.

Lemma 2.8. (i) *For any optimal solution* (x^*, α^*, β^*) *to the problem* (2.27), *we have*

$$\begin{cases} \alpha^*_{\pi(i)} = 0 & \textit{for} \ \ i \in \mathcal{A}(x^*) \cap J_+, \\ \alpha^*_{\pi(i)} = (\Phi x^*)_i - 1 > 0 & \textit{for} \ \ i \in \tilde{\mathcal{A}}_+(x^*), \\ \beta^*_{\varrho(i)} = 0 & \textit{for} \ \ i \in \mathcal{A}(x^*) \cap J_-, \\ \beta^*_{\varrho(i)} = -1 - (\Phi x^*)_i > 0 & \textit{for} \ \ i \in \tilde{\mathcal{A}}_-(x^*). \end{cases} \tag{2.28}$$

(ii) x^* *is the unique optimal solution to the* 1-*bit basis pursuit* (2.22) *if and only if* (x^*, α^*, β^*) *is the unique optimal solution to the problem* (2.27), *where* (α^*, β^*) *is determined by* (2.28).

2.4 Recovery criteria

When $y = \mathrm{sign}(\Phi x^*) \in \{1, -1\}^m$, any small perturbation $x^* + u$ is also consistent with y. When $y \in \{1, -1, 0\}^m$, any small perturbation $x^* + u$ with $u \in \mathcal{N}(\Phi_{J_0,n})$ is also consistent with y. Thus a 1-bit CS problem generally has infinitely many solutions and the sparsest solution of a sign equation is also not unique in general. Since the amplitude of signals is not available, the recovery criteria in 1-bit CS scenarios can be sign recovery, support recovery or others, depending on signal environments. The exact sign recovery of a signal means that the found solution $\widetilde{x}$ by an algorithm satisfies

$$\mathrm{sign}(\widetilde{x}) = \mathrm{sign}(x^*).$$

The support recovery, i.e., the found solution $\widetilde{x}$ satisfying $\mathrm{supp}(\widetilde{x}) = \mathrm{supp}(x^*)$ is a relaxed version of the sign recovery. It is worth mentioning that the following criterion

$$\left\| \frac{x}{\|x\|_2} - \frac{x^*}{\|x^*\|_2} \right\| \leqslant \varepsilon$$

has been widely used in the 1-bit CS literature, where $\varepsilon > 0$ is a certain small number.

In the remainder of the paper, we work toward developing some necessary and sufficient conditions for the exact recovery of the sign of sparse signals from 1-bit measurements.

3 Nonuniform sign recovery

We assume that the measurement $y = \mathrm{sign}(\Phi x^*)$ is available. From this information, we use the 1-bit basis pursuit (2.22) to recover the sign of x^*. We ask when the optimal solution of (2.22) admits the same sign of x^*. The recovery of the sign of an individual sparse signal is referred to as the nonuniform sign recovery. In this section, we develop certain necessary and sufficient conditions for the nonuniform sign recovery from the perspective of the range space property of a transposed sensing matrix.

Assume that $y \in \{1, -1, 0\}^m$ is given and (J_+, J_-, J_0) is specified as (2.17). We first introduce the concept of the RRSP.

Definition 3.1 (RRSP of Φ^{T} at x^*)**.** Let $x^* \in \mathbb{R}^n$ satisfy $y = \mathrm{sign}(\Phi x^*)$. We say that Φ^{T} satisfies the restricted range space property (RRSP) at x^* if there exist vectors $\eta \in \mathcal{R}(\Phi^{\mathrm{T}})$ and $w \in \mathcal{F}(x^*)$ such that $\eta = \Phi^{\mathrm{T}} w$ and

$$\eta_i = 1 \quad \text{for} \quad x_i^* > 0, \quad \eta_i = -1 \quad \text{for} \quad x_i^* < 0, \quad |\eta_i| < 1 \quad \text{for} \quad x_i^* = 0,$$

where $\mathcal{F}(x^*)$ is the set defined as

$$\begin{aligned} \mathcal{F}(x^*) = \{ & w \in \mathbb{R}^m : w_i > 0 \text{ for } i \in \mathcal{A}(x^*) \cap J_+,\ w_i < 0 \text{ for } i \in \mathcal{A}(x^*) \cap J_-, \\ & w_i = 0 \text{ for } i \in \tilde{\mathcal{A}}_+(x^*) \cup \tilde{\mathcal{A}}_-(x^*)\}. \end{aligned} \tag{3.1}$$

The RRSP of Φ^{T} at x^* is a natural condition for the uniqueness of optimal solutions to the 1-bit basis pursuit (2.22), as shown by the following theorem.

Theorem 3.2 (Necessary and sufficient conditions)**.** *x^* is the unique optimal solution to the 1-bit basis pursuit (2.22) if and only if the RRSP of Φ^{T} at x^* holds and the matrix*

$$H(x^*) = \begin{bmatrix} \Phi_{\mathcal{A}(x^*) \bigcap J_+, S_+} & \Phi_{\mathcal{A}(x^*) \bigcap J_+, S_-} \\ \Phi_{\mathcal{A}(x^*) \bigcap J_-, S_+} & \Phi_{\mathcal{A}(x^*) \bigcap J_-, S_-} \\ \Phi_{J_0, S_+} & \Phi_{J_0, S_-} \end{bmatrix} \tag{3.2}$$

has a full-column rank, where $S_+ = \{i : x_i^* > 0\}$ and $S_- = \{i : x_i^* < 0\}$.

The proof of Theorem 3.2 requiring some fundamental facts for linear programs is given in Section 5.

The uniqueness of solutions to a decoding method like (2.22) is an important property required in signal reconstruction. As indicated in [19, 20, 33, 41], the uniqueness conditions often lead to certain criteria for the nonuniform and uniform recovery of sparse signals. Later, we will see that Theorem 3.2, together with the matrix properties N-RRSP and S-RRSP of order k that will be introduced in this and next sections, provides a fundamental basis to develop a sign recovery theory for sparse signals from 1-bit measurements. Let us begin with the following lemma.

Lemma 3.3. *Let x^* be a sparsest solution of the ℓ_0-problem* (2.21) *and let S_+ and S_- be defined as in Theorem* 3.2*. Then*

$$\widetilde{H}(x^*) = \begin{bmatrix} \Phi_{\mathcal{A}(x^*)\cap J_+, S_+} & \Phi_{\mathcal{A}(x^*)\cap J_+, S_-} \\ \Phi_{\mathcal{A}(x^*)\cap J_-, S_+} & \Phi_{\mathcal{A}(x^*)\cap J_-, S_-} \\ \Phi_{J_0, S_+} & \Phi_{J_0, S_-} \\ \Phi_{\tilde{\mathcal{A}}_+(x^*), S_+} & \Phi_{\tilde{\mathcal{A}}_+(x^*), S_-} \\ \Phi_{\tilde{\mathcal{A}}_-(x^*), S_+} & \Phi_{\tilde{\mathcal{A}}_-(x^*), S_-} \end{bmatrix} \tag{3.3}$$

has a full-column rank. Furthermore, at any sparsest solution x^* of (2.21)*, which admits the maximum cardinality $|\mathcal{A}(x^*)| = \max\{|\mathcal{A}(x)| : x \in F^*\}$, where F^* is the set of optimal solutions of* (2.21)*, $H(x^*)$ given by* (3.2) *has a full-column rank.*

Proof. Note that x^* is a sparsest solution to the system

$$\Phi_{J_+,n}x^* \geqslant e_{J_+}, \quad \Phi_{J_-,n}x^* \leqslant -e_{J_-}, \quad \Phi_{J_0,n}x^* = 0. \tag{3.4}$$

Including α^* and β^*, given by (2.28), into (3.4) leads to

$$\Phi_{J_+,n}x^* - \alpha^* = e_{J_+}, \quad \Phi_{J_-,n}x^* + \beta^* = -e_{J_-}, \quad \Phi_{J_0,n}x^* = 0. \tag{3.5}$$

Eliminating the zero components of x^* from (3.5) leads to

$$\begin{cases} \Phi_{J_+,S_+}x^*_{S_+} + \Phi_{J_+,S_-}x^*_{S_-} - \alpha^* = e_{J_+}, \\ \Phi_{J_-,S_+}x^*_{S_+} + \Phi_{J_-,S_-}x^*_{S_-} + \beta^* = -e_{J_-}, \\ \Phi_{J_0,S_+}x^*_{S_+} + \Phi_{J_0,S_-}x^*_{S_-} = 0. \end{cases} \tag{3.6}$$

Note that x^* is the sparsest solution of (2.21). It is not very difficult to see that the coefficient matrix

$$\widehat{H} = \begin{bmatrix} \Phi_{J_+,S_+} & \Phi_{J_+,S_-} \\ \Phi_{J_-,S_+} & \Phi_{J_-,S_-} \\ \Phi_{J_0,S_+} & \Phi_{J_0,S_-} \end{bmatrix}$$

has a full-column rank, since otherwise at least one column of $\widehat{H}$ can be linearly represented by its other columns and hence the system (3.6), which is equivalent to (3.4), has a solution sparser than x^*. From (2.23) and (2.24), we see that

$$J_+ = (\mathcal{A}(x^*) \cap J_+) \cup \widetilde{\mathcal{A}}_+(x^*), \quad J_- = (\mathcal{A}(x^*) \cap J_-) \cup \widetilde{\mathcal{A}}_-(x^*). \tag{3.7}$$

Performing row permutations on $\widehat{H}$, if necessary, yields $\widetilde{H}(x^*)$ given as (3.3).

Since row permutations do not affect the column rank of $\widehat{H}$, $\widetilde{H}(x^*)$ must have a full-column rank.

We now show that $H(x^*)$ has a full-column rank if $\mathcal{A}(x^*)$ admits the maximum cardinality in the sense that $|\mathcal{A}(x^*)| = \max\{|\mathcal{A}(x)| : x \in F^*\}$, where F^* is the set of optimal solutions of (2.21). We prove this by contradiction. Assume that the columns of $H(x^*)$ are linearly dependent. Then there is a nonzero vector $d = (u, v) \in \mathbb{R}^{|S_+|} \times \mathbb{R}^{|S_-|}$ such that

$$H(x^*)d = H(x^*)\begin{bmatrix} u \\ v \end{bmatrix} = 0.$$

Since $d \neq 0$ and $\widetilde{H}(x^*)$, given by (3.3), has a full-column rank, we see that

$$\begin{bmatrix} \Phi_{\tilde{\mathcal{A}}_+(x^*),S_+} & \Phi_{\tilde{\mathcal{A}}_+(x^*),S_-} \\ \Phi_{\tilde{\mathcal{A}}_-(x^*),S_+} & \Phi_{\tilde{\mathcal{A}}_-(x^*),S_-} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \neq 0. \tag{3.8}$$

Let $x(\lambda)$ be the vector with components $x(\lambda)_{S_+} = x^*_{S_+} + \lambda u$, $x(\lambda)_{S_-} = x^*_{S_-} + \lambda v$ and $x(\lambda)_i = 0$ for all $i \notin S_+ \cup S_-$, where $\lambda \in R$. Clearly, we have $\mathrm{supp}(x(\lambda)) \subseteq \mathrm{supp}(x^*)$ for any $\lambda \in R$. By (2.28) and (3.7), the system (3.6) is equivalent to

$$\begin{cases} \Phi_{\mathcal{A}(x^*)\cap J_+,S_+} x^*_{S_+} + \Phi_{\mathcal{A}(x^*)\cap J_+,S_-} x^*_{S_-} = e_{\mathcal{A}(x^*)\cap J_+}, \\ \Phi_{\mathcal{A}(x^*)\cap J_-,S_+} x^*_{S_+} + \Phi_{\mathcal{A}(x^*)\cap J_-,S_-} x^*_{S_-} = -e_{\mathcal{A}(x^*)\cap J_-}, \\ \Phi_{J_0,S_+} x^*_{S_+} + \Phi_{J_0,S_-} x^*_{S_-} = 0, \\ \Phi_{\tilde{\mathcal{A}}_+(x^*),S_+} x^*_{S_+} + \Phi_{\tilde{\mathcal{A}}_+(x^*),S_-} x^*_{S_-} > e_{\tilde{\mathcal{A}}_+(x^*)}, \\ \Phi_{\tilde{\mathcal{A}}_-(x^*),S_+} x^*_{S_+} + \Phi_{\tilde{\mathcal{A}}_-(x^*),S_-} x^*_{S_-} < -e_{\tilde{\mathcal{A}}_-(x^*)}, \end{cases} \tag{3.9}$$

From the above system and the definition of $x(\lambda)$, we see that for any sufficiently small $|\lambda| \neq 0$, the vector $(x(\lambda)_{S_+}, x(\lambda)_{S_-})$ satisfies the system

$$H(x^*) \begin{bmatrix} x(\lambda)_{S_+} \\ x(\lambda)_{S_-} \end{bmatrix} = \begin{bmatrix} e_{\mathcal{A}(x^*)\cap J_+} \\ -e_{\mathcal{A}(x^*)\cap J_-} \\ 0 \end{bmatrix}, \tag{3.10}$$

$$[\Phi_{\widetilde{\mathcal{A}}_+(x^*),S_+}, \Phi_{\widetilde{\mathcal{A}}_+(x^*),S_-}] \begin{bmatrix} x(\lambda)_{S_+} \\ x(\lambda)_{S_-} \end{bmatrix} > e_{\widetilde{\mathcal{A}}_+(x^*)}, \tag{3.11}$$

$$[\Phi_{\widetilde{\mathcal{A}}_-(x^*),S_+}, \Phi_{\widetilde{\mathcal{A}}_-(x^*),S_-}] \begin{bmatrix} x(\lambda)_{S_+} \\ x(\lambda)_{S_-} \end{bmatrix} < -e_{\widetilde{\mathcal{A}}_-(x^*)}. \tag{3.12}$$

Equality (3.10) actually holds for any $\lambda \in \mathbb{R}^n$. Starting from $\lambda = 0$, we continuously increase the value of $|\lambda|$. In this process, if one of the components of the vector $(x(\lambda)_{S_+}, x(\lambda)_{S_-})$ satisfying (3.10)–(3.12) becomes zero, then a sparser solution than x^* is found, leading to a contradiction. Thus without loss of generality, we assume that $\mathrm{supp}(x(\lambda)) = \mathrm{supp}(x^*)$ is maintained when $|\lambda|$ is continuously increased. It follows from (3.8) that there exists $\lambda^* \neq 0$ such that $(x(\lambda^*)_{S_+}, x(\lambda^*)_{S_-})$ satisfies (3.10)–(3.12) and at this vector, one of the inactive constraints in (3.11) and (3.12) becomes active. Therefore $|\mathcal{A}(x(\lambda^*))| > |\mathcal{A}(x^*)|$. This contradicts the fact $|\mathcal{A}(x^*)|$ has the maximal cardinality amongst the sparsest solutions. Thus we conclude that $H(x^*)$ must have a full-column rank. $\square$

From Lemma 3.3, we see that the full-rank property of (3.2) can be guaranteed if x^* is a sparsest solution consistent with 1-bit measurements and $|\mathcal{A}(x^*)|$ is maximal. Thus by Theorem 3.2, the central condition for x^* to be the unique optimal solution to (2.22) is the RRSP described in Definition 3.1. From the above discussions, we obtain the following connection between 1-bit CS and 1-bit basis pursuit.

Theorem 3.4. (i) *Suppose that x^* is an optimal solution to the ℓ_0-problem* (2.21) *with maximal $|\mathcal{A}(x^*)|$. Then x^* is the unique optimal solution to* (2.22) *if and only if the RRSP of Φ^{T} at x^* holds.* (ii) *Suppose that x^* is an optimal solution to* (2.19) *or* (2.21). *Then the sign of x^* coincides with the sign of the unique solution of* (2.22) *if and only if there exists a weight $z \in \mathbb{R}^n$ satisfying $z_i > 0$ for $i \in \mathrm{supp}(x^*)$ and $z_i = 0$ for $i \notin \mathrm{supp}(x^*)$ such that Zx^*, where $Z = \mathrm{diag}(z)$, is feasible to* (2.22) *and $H(Zx^*)$ has a full-column rank and the RRSP of Φ^{T} at Zx^* holds.*

Proof. Result (i) follows directly from Lemma 3.3 and Theorem 3.2. We now prove result (ii). If the sign of x^* coincides with the sign of the unique optimal solution $\widetilde{x}$ of (2.22), then $\widetilde{x}$ can be written as $\widetilde{x} = Zx^*$ for a certain weight satisfying that $z_i > 0$ for $i \in \mathrm{supp}(x^*)$ and $z_i = 0$ for $i \notin \mathrm{supp}(x^*)$. It follows from Theorem 3.2 that $H(Zx^*)$ has a full-column rank and the RRSP of Φ^{T} at Zx^* holds. Conversely, if there exists a weight $z \in \mathbb{R}^n$ satisfying $z_i > 0$ for $i \in \mathrm{supp}(x^*)$ and $z_i = 0$ for $i \notin \mathrm{supp}(x^*)$ such that $\widetilde{x} = Zx^*$, where $Z = \mathrm{diag}(z)$, is feasible to (2.22) and $H(Zx^*)$ has a full-column rank and the RRSP of Φ^{T} at Zx^* holds, then by Theorem 3.2 again $\widetilde{x} = Zx^*$ is the unique optimal solution to (2.22). Clearly, by the definition of Z, we have $\mathrm{sign}(\widetilde{x}) = \mathrm{sign}(Zx^*) = \mathrm{sign}(x^*)$. □

The above result provides some insight into the nonuniform recovery of the sign of an individual sparse signal via the 1-bit measurements and 1-bit basis pursuit. This result indicates that central to the sign recovery of x^* is the RRSP of Φ^{T} at x^*. However, this property is defined at x^*, which is unknown in advance. Thus we need to further strengthen this concept in order to develop certain recovery conditions independent of the specific signal x^*. To this purpose, we introduce the notion of N- *and* S-RRSP *of order k with respect to* 1*-bit measurements,* which turns out to be a necessary condition and a sufficient condition, respectively, for the nonuniform sign recovery.

For given measurement $y \in \{1, -1, 0\}^m$, let $P(y)$ denote the set of all possible partitions of the support of signals consistent with y:

$$P(y) = \{(S_+(x), S_-(x)) : y = \mathrm{sign}(\Phi x)\},$$

where $S_+(x) = \{i : x_i > 0\}$ and $S_-(x) = \{i : x_i < 0\}$.

Definition 3.5 (N-RRSP of order k with respect to y)**.** The matrix Φ^{T} is said to satisfy the necessary restricted range space property (N-RRSP) of order k with respect to y if there exist a pair $(S_+, S_-) \in P(y)$ with $|S_+ \cup S_-| \leqslant k$ and a pair (T_1, T_2) with $T_1 \subseteq J_+$, $T_2 \subseteq J_-$, $T_1 \cup T_2 \neq J_+ \cup J_-$ and

$$\begin{bmatrix} \Phi_{J_+\setminus T_1,S} \\ \Phi_{J_-\setminus T_2,S} \\ \Phi_{J_0,S} \end{bmatrix},$$

where $S = S_+ \cup S_-$, having a full-column rank such that there is a vector $\eta \in \mathcal{R}(\Phi^{\mathrm{T}})$ satisfying the following properties:

(i) $\eta_i = 1$ for $i \in S_+$, $\eta_i = -1$ for $i \in S_-$, $|\eta_i| < 1$ otherwise;

(ii) $\eta = \Phi^{\mathrm{T}} w$ for some $w \in \mathcal{F}(T_1, T_2)$, where

$$\mathcal{F}(T_1, T_2) = \{w \in \mathbb{R}^m : w_{J_+\setminus T_1} > 0, w_{J_-\setminus T_2} < 0, w_{T_1\cup T_2} = 0\}. \quad (3.13)$$

The above matrix property turns out to be a necessary condition for the nonuniform recovery of the sign of a k-sparse signal, as shown by the next theorem.

Theorem 3.6. *Let x^* be an unknown k-sparse signal (i.e., $\|x^*\|_0 \leqslant k$) and assume that the measurements $y = \mathrm{sign}(\Phi x^*)$ are known. If the* 1*-bit basis pursuit* (2.22) *admits a unique optimal solution $\widetilde{x}$ satisfying $\mathrm{sign}(\widetilde{x}) = \mathrm{sign}(x^*)$ (i.e., the sign of x^* can be exactly recovered by* (2.22)*), then Φ^{T} has the N-RRSP of order k with respect to y.*

Proof. Suppose that the measurement $y = \mathrm{sign}(\Phi x^*)$ is given, where x^* is an unknown k-sparse signal. By the definition of $P(y)$, we see that $(S_+(x^*), S_-(x^*)) \in P(y)$. Denote by $S = S_+(x^*) \cup S_-(x^*)$. Suppose that (2.22) has a unique optimal solution $\widetilde{x}$ satisfying $\mathrm{sign}(\widetilde{x}) = \mathrm{sign}(x^*)$, which implies that $(S_+(\widetilde{x}), S_-(\widetilde{x})) = (S_+(x^*), S_-(x^*))$. By Theorem 3.2, the uniqueness of $\widetilde{x}$ implies that the RRSP of Φ^{T} at $\widetilde{x}$ holds and $H(\widetilde{x})$ has a full-column rank. Let

$$T_1 = \widetilde{\mathcal{A}}_+(\widetilde{x}) = J_+ \setminus \mathcal{A}(\widetilde{x}), \quad T_2 = \widetilde{\mathcal{A}}_-(\widetilde{x}) = J_- \setminus \mathcal{A}(\widetilde{x}). \quad (3.14)$$

Note that at any optimal solution of (2.22), at least one of the inequality constraints of (2.22) must be active. Thus $\mathcal{A}(\widetilde{x}) \neq \emptyset$, which implies that $T_1 \cup T_2 \neq J_+ \cup J_-$. We also note that $J_+ \setminus T_1 = J_+ \cap \mathcal{A}(\widetilde{x})$ and $J_-\setminus T_2 = J_- \cap \mathcal{A}(\widetilde{x})$. Hence the matrix

$$\begin{bmatrix} \Phi_{J_+\setminus T_1,S} \\ \Phi_{J_-\setminus T_2,S} \\ \Phi_{J_0,S} \end{bmatrix},$$

coinciding with $H(\widetilde{x})$, has a full-column rank. The RRSP of Φ^{T} at $\widetilde{x}$ implies that Properties (i) and (ii) of Definition 3.5 are satisfied with $(S_+, S_-) = (S_+(\widetilde{x}), S_-(\widetilde{x})) = (S_+(x^*), S_-(x^*))$ and (T_1, T_2) being given as (3.14). This implies that the N-RRSP of order k with respect to y must hold. □

A slight enhancement of the N-RRSP property by varying the choices of (S_+, S_-) and (T_1, T_2) leads to the next property which turns out to be a sufficient condition for the exact recovery of the sign of a k-sparse signal.

Definition 3.7 (S-RRSP of order k with respect to y)**.** The matrix Φ^{T} is said to satisfy the sufficient restricted range space property (S-RRSP) of order k with respect to y if for any $(S_+, S_-) \in P(y)$ with $|S_+ \cup S_-| \leqslant k$, there exists a pair (T_1, T_2) such that $T_1 \subseteq J_+$, $T_2 \subseteq J_-$, $T_1 \cup T_2 \neq J_+ \cup J_-$ and

$$\begin{bmatrix} \Phi_{J_+ \backslash T_1, S} \\ \Phi_{J_- \backslash T_2, S} \\ \Phi_{J_0, S} \end{bmatrix},$$

where $S = S_+ \cup S_-$, has a full-column rank, and for any such a pair (T_1, T_2), there is a vector $\eta \in \mathcal{R}(\Phi^{\mathrm{T}})$ satisfying the following properties:

(i) $\eta_i = 1$ for $i \in S_+$, $\eta_i = -1$ for $i \in S_-$, $|\eta_i| < 1$ otherwise;

(ii) $\eta = \Phi^{\mathrm{T}} w$ for some $w \in \mathcal{F}(T_1, T_2)$ defined by (3.13).

Note that when

$$\begin{bmatrix} \Phi_{J_+ \backslash T_1, S} \\ \Phi_{J_- \backslash T_2, S} \\ \Phi_{J_0, S} \end{bmatrix}$$

has a full-column rank, so does $\Phi_{m,S}$. Thus we have the next lemma.

Lemma 3.8. *If Φ^{T} satisfies the S-RRSP of order k with respect to y, then for any $(S_+, S_-) \in P(y)$ with $|S_+ \cup S_-| \leqslant k$, $\Phi_{m,S}$ must have a full-column rank, where $S = S_+ \cup S_-$.*

For a given y, the equation $y = \mathrm{sign}(\Phi x)$ might possess infinitely many solutions. We now prove that if x^* is a sparsest solution to this equation, then its sign can be exactly recovered by (2.22) if Φ^{T} has the S-RRSP of order k with respect to y.

Theorem 3.9. *Let measurement $y \in \{-1, 1, 0\}^m$ be given and assume that Φ^{T} has the S-RRSP of order k with respect to y. Then the* 1*-bit basis pursuit* (2.22) *admits a unique optimal solution x' satisfying* $\mathrm{supp}(x') \subseteq \mathrm{supp}(x^*)$ *for any k-sparse signal x^* consistent with the measurement y, i.e., $y = \mathrm{sign}(\Phi x^*)$. Furthermore, if x^* is a sparsest signal consistent with y, then* $\mathrm{sign}(x') = \mathrm{sign}(x^*)$*, and thus the sign of x^* can be exactly recovered by* (2.22)*.*

Proof. Let x^* be a k-sparse signal consistent with y, i.e., $\mathrm{sign}(\Phi x^*) = y$. Denote by $S_+ = \{i : x_i^* > 0\}$, $S_- = \{i : x_i^* < 0\}$ and $S = \mathrm{supp}(x^*) = S_+ \cup S_-$. Clearly, $(S_+, S_-) \in P(y)$ and $|S_+ \cup S_-| \leqslant k$. Consistency implies that $(\Phi x^*)_i > 0$ for all $i \in J_+$, $(\Phi x^*)_i < 0$ for all $i \in J_-$ and $(\Phi x^*)_i = 0$ for all $i \in J_0$. This implies that there is a scalar $\alpha > 0$ such that $\alpha(\Phi x^*)_i \geqslant 1$ for all $i \in J_+$ and $\alpha(\Phi x^*)_i \leqslant -1$ for all $i \in J_-$. Thus αx^* is feasible to (2.22), i.e.,

$$\Phi_{J_+, n}(\alpha x^*) \geqslant e_{J_+}, \tag{3.15}$$

$$\Phi_{J_-, n}(\alpha x^*) \leqslant -e_{J_-}, \tag{3.16}$$

$$\Phi_{J_0, n}(\alpha x^*) = 0. \tag{3.17}$$

We see that $\alpha \geqslant \frac{1}{(\Phi x^*)_i}$ for $i \in J_+$ and $\alpha \geqslant \frac{1}{-(\Phi x^*)_i}$ for $i \in J_-$. Let α^* be the smallest α satisfying these inequalities, i.e.,

$$\alpha^* = \max\left\{\max_{i \in J_+} \frac{1}{(\Phi x^*)_i}, \max_{i \in J_-} \frac{1}{-(\Phi x^*)_i}\right\} = \max_{i \in J_+ \cup J_-} \frac{1}{|(\Phi x^*)_i|}.$$

By the choice of α^*, at $\alpha^* x^*$ one of the inequalities in (3.15) and (3.16) becomes an equality. Let T_0' and T_0'' be the sets of indices for active constraints in (3.15) and (3.16), i.e.,

$$T_0' = \{i \in J_+ : \Phi(\alpha^* x^*)_i = 1\}, \quad T_0'' = \{i \in J_- : \Phi(\alpha^* x^*)_i = -1\}.$$

If the null space

$$\mathcal{N}\left(\begin{bmatrix} \Phi_{T_0',S} \\ \Phi_{T_0'',S} \\ \Phi_{J_0,S} \end{bmatrix}\right) \neq \{0\},$$

then let $d \neq 0$ be a vector in this null space. It follows from Lemma 3.8 that $\Phi_{m,S}$ has a full-column rank. This implies that

$$\begin{bmatrix} \Phi_{J_+ \setminus T_0',S} \\ \Phi_{J_- \setminus T_0'',S} \end{bmatrix} d \neq 0. \tag{3.18}$$

Consider the vector $x(\lambda)$ with components $x(\lambda)_S = \alpha^* x_S^* + \lambda d$ and $x(\lambda)_i = 0$ for $i \notin S$, where $\lambda \in R$. By the choice of d, we see that $\mathrm{supp}(x(\lambda)) \subseteq \mathrm{supp}(x^*)$ for any $\lambda \in R$. For all sufficiently small $|\lambda|$, the vector $x(\lambda)$ is feasible to the problem (2.22) and the active constraints at $\alpha^* x^*$ in (3.15) and (3.16) are still active at $x(\lambda)$ and the inactive constraints at $\alpha^* x^*$ are still inactive at $x(\lambda)$. Due to (3.18), if letting $|\lambda|$ continuously vary from zero to a positive number, there exists $\lambda^* \neq 0$ such that $x(\lambda^*)$ is still feasible to (2.22) and one of the above-mentioned inactive constraints becomes active at $x(\lambda^*)$. Let $x' = x(\lambda^*)$ and

$$T' = \{i \in J_+ : (\Phi x')_i = 1\}, \quad T'' = \{i \in J_- : (\Phi x')_i = -1\}.$$

By the construction of x', we see that $T_0' \subseteq T'$ and $T_0'' \subseteq T''$. So we obtain an augmented set of active constraints at x'.

Now replace the role of $\alpha^* x^*$ by x' and repeat the above process. If

$$\mathcal{N}\left(\begin{bmatrix} \Phi_{T',S} \\ \Phi_{T'',S} \\ \Phi_{J_0,S} \end{bmatrix}\right) \neq \{0\},$$

pick a vector $d' \neq 0$ from this null space. Since $\Phi_{m,S}$ has a full-column rank, we must have that

$$\begin{bmatrix} \Phi_{J_+ \setminus T',S} \\ \Phi_{J_- \setminus T'',S} \end{bmatrix} d' \neq 0.$$

So we can continue to update the components of x' by setting $x_S' \leftarrow x_S' + \lambda' d'$ and keeping $x_i' = 0$ for $i \notin S$, where λ' is chosen such that $x_S' + \lambda' d'$ is still feasible to (2.22) and one of the inactive constraints at the current point x' becomes active at $x_S' + \lambda' d'$. Thus the index sets T' and T'' for active constraints are further augmented.

Since $\Phi_{m,S}$ has a full-column rank, after repeating the above process a finite number of times, we stop at a point, denoted still by x', at which

$$\mathcal{N}\left(\begin{bmatrix} \Phi_{T',S} \\ \Phi_{T'',S} \\ \Phi_{J_0,S} \end{bmatrix}\right) = \{0\},$$

i.e.,

$$\begin{bmatrix} \Phi_{T',S} \\ \Phi_{T'',S} \\ \Phi_{J_0,S} \end{bmatrix}$$

has a full-column rank. Note that $\mathrm{supp}(x') \subseteq \mathrm{supp}(x^*)$ is always maintained in the above process. Define the sets

$$T_1 = \widetilde{\mathcal{A}}_+(x'), \quad T_2 = \widetilde{\mathcal{A}}_-(x'). \tag{3.19}$$

Thus $T_1 \subseteq J_+$ and $T_2 \subseteq J_-$. By the construction of x', we see that $\mathcal{A}(x') \neq \emptyset$. Thus (T_1, T_2) given by (3.19) satisfies that $T_1 \cup T_2 \neq J_+ \cup J_-$.

We now further prove that x' must be the unique optimal solution to the 1-bit basis pursuit (2.22). By Theorem 3.2, it is sufficient to prove that Φ^{T} has the RRSP at x' and the matrix

$$H(x') = \begin{bmatrix} \Phi_{\mathcal{A}(x')\cap J_+, S'_+} & \Phi_{\mathcal{A}(x')\cap J_+, S'_-} \\ \Phi_{\mathcal{A}(x')\cap J_-, S'_+} & \Phi_{\mathcal{A}(x')\cap J_-, S'_-} \\ \Phi_{J_0, S'_+} & \Phi_{J_0, S'_-} \end{bmatrix}$$

has a full-column rank, where $S'_+ = \{i : x'_i > 0\}$ and $S'_- = \{i : x'_i < 0\}$.

Indeed, let S'_+, S'_-, T_1 and T_2 be defined as above. Since x' is consistent with y and satisfies that $\mathrm{supp}(x') \subseteq \mathrm{supp}(x^*)$, we see that $(S'_+, S'_-) \in P(y)$ satisfying $S' = S'_+ \cup S'_- \subseteq S$. Since

$$\begin{bmatrix} \Phi_{T',S} \\ \Phi_{T'',S} \\ \Phi_{J_0,S} \end{bmatrix}$$

has a full-column rank,

$$\begin{bmatrix} \Phi_{T',S'} \\ \Phi_{T'',S'} \\ \Phi_{J_0,S'} \end{bmatrix}$$

must have a full-column rank. Note that

$$T' = J_+ \setminus T_1 = \mathcal{A}(x') \cap J_+, \quad T'' = J_- \setminus T_2 = \mathcal{A}(x') \cap J_-. \tag{3.20}$$

Thus

$$H(x') = \begin{bmatrix} \Phi_{J_+\setminus T_1, S'} \\ \Phi_{J_-\setminus T_2, S'} \\ \Phi_{J_0, S'} \end{bmatrix}$$

has a full-column rank.

Since Φ^{T} has the S-RRSP of order k with respect to y, there exists a vector $\eta \in \mathcal{R}(\Phi^{\mathrm{T}})$ and $w \in \mathcal{F}(T_1, T_2)$ satisfying that $\eta = \Phi^{\mathrm{T}} w$ and $\eta_i = 1$ for $i \in S'_+$, $\eta_i = -1$ for $i \in S'_-$, and $|\eta_i| < 1$ otherwise. The set $\mathcal{F}(T_1, T_2)$ is defined as (3.13). From (3.19), we see that the condition $w_{T_1 \cup T_2} = 0$ in (3.13) coincides with the condition $w_i = 0$ for $i \in \tilde{\mathcal{A}}_+(x') \cup \tilde{\mathcal{A}}_-(x')$. This, together with (3.20), implies that $\mathcal{F}(T_1, T_2)$ coincides with $\mathcal{F}(x')$ defined as (3.1). Thus the RRSP of Φ^{T} at x' holds (see Definition 3.1). This, together with the full-column-rank property of $H(x')$, implies that x' is the unique optimal solution to (2.22).

Furthermore, suppose that x^* is a k-sparse signal and x^* is a sparsest signal consistent with y. Since x' is also consistent with y, it follows from $\mathrm{supp}(x') \subseteq \mathrm{supp}(x^*)$ that $\mathrm{supp}(x') = \mathrm{supp}(x^*)$. So x' is also a sparsest vector consistent with y. From the aforementioned construction process of x', it is not difficult to see that the updating scheme $x'_S \leftarrow x'_S + \lambda' d'$ does not change the sign of nonzero components of the vectors. In fact, when we vary the parameter λ in $x'_S + \lambda d'$ to determine the critical value λ' which yields new active constraints, this value λ' still ensures that the new vector $x'_S + \lambda' d'$ is feasible to (2.22). If there is a nonzero component of $x'_S + \lambda' d'$, say the i-th component, holding a different sign from the corresponding nonzero component of x'_S, then by continuity and by convexity of the feasible set of (2.22), there is a suitable λ lying between zero and λ' such that the i-th component of $x'_S + \lambda d'$ is equal to zero. Thus $x'_S + \lambda d'$ is sparser than x^*. Since $x'_S + \lambda d'$ is also feasible to (2.22), it is consistent with y. This is a contradiction as x^* is a sparsest signal consistent with y. Thus we have $\mathrm{sign}(x') = \mathrm{sign}(x^*)$. $\square$

4 Uniform sign recovery

Theorems 3.6 and 3.9 provide some conditions for the nonuniform recovery of the sign of an individual k-sparse signal. In this section, we develop some necessary and sufficient conditions for the uniform recovery of the sign of all k-sparse signals through a sensing matrix Φ. Let us first define

$$Y^k = \{y : y = \mathrm{sign}(\Phi x), x \in \mathbb{R}^n, \|x\|_0 \leqslant k\}.$$

For any two disjoint subsets $S_1, S_2 \subseteq \{1, \ldots, n\}$ satisfying $|S_1 \cup S_2| \leqslant k$, there exists a k-sparse signal x such that $S_1 = S_+(x)$ and $S_2 = S_-(x)$. Thus any such disjoint subsets (S_1, S_2) must be in the set $P(y)$ for some $y \in Y^k$. We now introduce the notion of the N-RRSP of order k which turns out to be a necessary condition for uniform sign recovery.

Definition 4.1 (N-RRSP of order k)**.** The matrix Φ^{T} is said to satisfy the necessary restricted range space property (N-RRSP) of order k if for any disjoint subsets S_+, S_- of $\{1, \ldots, n\}$ with $|S| \leqslant k$, where $S = S_+ \cup S_-$, there exist $y \in Y^k$ and (T_1, T_2) such that $(S_+, S_-) \in P(y)$, $T_1 \subseteq J_+(y)$, $T_2 \subseteq J_-(y)$, $T_1 \cup T_2 \neq J_+(y) \cup J_-(y)$ and

$$\begin{bmatrix} \Phi_{J_+(y)\backslash T_1, S} \\ \Phi_{J_-(y)\backslash T_2, S} \\ \Phi_{J_0, S} \end{bmatrix}$$

has a full-column rank, and there is a vector $\eta \in \mathcal{R}(\Phi^{\mathrm{T}})$ satisfying the following properties:

(i) $\eta_i = 1$ for $i \in S_+$, $\eta_i = -1$ for $i \in S_-$, $|\eta_i| < 1$ otherwise;

(ii) $\eta = \Phi^{\mathrm{T}} w$ for some $w \in \mathcal{F}(T_1, T_2)$ defined by (3.13).

The N-RRSP of order k is a necessary condition for the uniform recovery of the sign of all k-sparse signals via 1-bit measurements and basis pursuit.

Theorem 4.2. *Let $\Phi \in \mathbb{R}^{m\times n}$ be a given matrix and assume that for any k-sparse signal x^*, the sign measurement* $\mathrm{sign}(\Phi x^*)$ *can be acquired. If the sign of any k-sparse signal x^* can be exactly recovered by the* 1*-bit basis pursuit* (2.22) *with $J_+ = \{i : \mathrm{sign}(\Phi x^*)_i = 1\}$, $J_- = \{i : \mathrm{sign}(\Phi x^*)_i = -1\}$ and $J_0 = \{i : \mathrm{sign}(\Phi x^*)_i = 0\}$ in the sense that* (2.22) *admits a unique optimal solution $\widetilde{x}$ satisfying* $\mathrm{sign}(\widetilde{x}) = \mathrm{sign}(x^*)$, *then Φ^{T} must admit the N-RRSP of order k.*

Proof. Let x^* be an arbitrary k-sparse signal with $S_+ = \{i : x_i^* > 0\}$, $S_- = \{i : x_i^* < 0\}$ and $S = S_+ \cup S_-$. Clearly, $|S| \leqslant k$. Let $y = \mathrm{sign}(\Phi x^*)$ be the acquired measurement. Assume that $\widetilde{x}$ is the unique optimal solution to (2.22) and $\mathrm{sign}(\widetilde{x}) = \mathrm{sign}(x^*)$. Then we see that $y \in Y^k$, $(S_+, S_-) \in P(y)$, and

$$(S_+(\widetilde{x}), S_-(\widetilde{x})) = (S_+, S_-). \tag{4.1}$$

It follows from Theorem 3.2 that the uniqueness of $\widetilde{x}$ implies that the matrix $H(\widetilde{x})$ admits a full-column rank and there exists a vector $\eta \in \mathcal{R}(\Phi^{\mathrm{T}})$ such that

(a) $\eta_i = 1$ for $i \in S_+(\widetilde{x})$, $\eta_i = -1$ for $i \in S_-(\widetilde{x})$, and $|\eta_i| < 1$ otherwise;

(b) $\eta = \Phi^{\mathrm{T}} w$ for some $w \in \mathcal{F}(\widetilde{x})$ given as

$$\begin{aligned} \mathcal{F}(\widetilde{x}) = \{ & w \in \mathbb{R}^m : w_i > 0 \text{ for } i \in \mathcal{A}(\widetilde{x}) \cap J_+(y), w_i < 0 \text{ for } i \in \mathcal{A}(\widetilde{x}) \cap J_-(y), \\ & w_i = 0 \text{ for } i \in \tilde{\mathcal{A}}_+(\widetilde{x}) \cup \tilde{\mathcal{A}}_-(\widetilde{x})\}. \end{aligned}$$

Let $T_1 = \widetilde{\mathcal{A}}_+(\widetilde{x}) \subseteq J_+(y)$ and $T_2 = \widetilde{\mathcal{A}}_-(\widetilde{x}) \subseteq J_-(y)$. Since $\widetilde{x}$ is an optimal solution to (2.22), we must have that $\mathcal{A}(\widetilde{x}) \neq \emptyset$, which implies that $T_1 \cup T_2 \neq J_+(y) \cup J_-(y)$. Clearly,

$$\mathcal{A}(\widetilde{x}) \cap J_+(y) = J_+(y)\backslash T_1, \quad \mathcal{A}(\widetilde{x}) \cap J_-(y) = J_-(y)\backslash T_2. \tag{4.2}$$

Therefore, the full-column-rank property of $H(\widetilde{x})$ implies that

$$\begin{bmatrix} \Phi_{J_+(y)\backslash T_1, S} \\ \Phi_{J_-(y)\backslash T_2, S} \\ \Phi_{J_0, S} \end{bmatrix}$$

has a full-column rank. By (4.1) and (4.2), the above properties (a) and (b) coincide with the properties (i) and (ii) described in Definition 4.1. By considering all possible k-sparse signals x^*, which yield all possible disjoint subsets S_+, S_- of $\{1, \ldots, n\}$ satisfying $|S_+ \cup S_-| \leqslant k$. Thus Φ^{T} admits the N-RRSP of order k. $\square$

It should be pointed out that for random matrices Φ, with probability 1 the optimal solution to the linear program (2.22) is unique. In fact, the non-uniqueness of optimal solutions happens only if the optimal face of the feasible set (which is a polyhedron) is parallel to the objective hyperplane, and the probability for this event is zero. This means that the uniqueness assumption for the optimal solution of (2.22) is very mild and it holds almost for sure. Thus when the sensing matrix Φ is randomly generated according to a probability distribution, with probability 1 the RRSP of Φ^{T} at its optimal solution $\widetilde{x}$ holds and the associated matrix $H(\widetilde{x})$ has a full-column rank. The N-RRSP of order k is defined based on such a mild assumption. Theorem 4.2 has indicated that the N-RRSP of order k is a necessary requirement for the uniform recovery of the sign of all k-sparse signals from 1-bit measurements with the linear program (2.22). Using linear programs as decoding methods will necessarily and inevitably yield a certain range space property like the RRSP (since this property results directly from the fundamental optimality condition of linear programs). From the study in this paper, we conclude that if the sign of k-sparse signals can be exactly recovered from 1-bit measurements with a linear programming decoding method, then Φ^{T} must satisfy the N-RRSP of order k or its variants. At the moment, it is not clear whether this necessary condition is also sufficient for the exact sign recovery in 1-bit CS setting.

In classic CS, a sensing matrix is required to admit a general positioning property in order to achieve the uniform recovery of k-sparse signals. This property is reflected in all concepts such as RIP, NSP and RSP. Similarly, in order to achieve the uniform recovery of the sign of k-sparse signals in 1-bit CS setting, the matrix should admit a certain general positioning property as well. Since N-RRSP is a necessary property for uniform sign recovery, a sufficient sign recovery condition can be developed by slightly enhancing this necessary property, i.e., by considering all possible sign measurement $y \in Y^k$ together with the pairs (T_1, T_2) described in Definition 4.1. This naturally leads to the next definition.

Definition 4.3 (S-RRSP of order k). The matrix Φ^{T} is said to satisfy the sufficient restricted range space property (S-RRSP) of order k if for any disjoint subsets (S_+, S_-) of $\{1, \ldots, n\}$ with $|S| \leqslant k$, where $S = S_+ \cup S_-$, and for any $y \in Y^k$ such that $(S_+, S_-) \in P(y)$, there exist T_1 and T_2 such that $T_1 \subseteq J_+(y)$, $T_2 \subseteq J_-(y)$, $T_1 \cup T_2 \neq J_+(y) \cup J_-(y)$ and

$$\begin{bmatrix} \Phi_{J_+(y)\backslash T_1, S} \\ \Phi_{J_-(y)\backslash T_2, S} \\ \Phi_{J_0, S} \end{bmatrix}$$

has a full-column rank, and for any such a pair (T_1, T_2), there is a vector $\eta \in \mathcal{R}(\Phi^{\mathrm{T}})$ satisfying the following properties:

(i) $\eta_i = 1$ for $i \in S_+$, $\eta_i = -1$ for $i \in S_-$, $|\eta_i| < 1$ otherwise;

(ii) $\eta = \Phi^{\mathrm{T}} w$ for some $w \in \mathcal{F}(T_1, T_2)$ defined by (3.13).

The above concept taking into account all possible vectors y is stronger than Definition 3.7. If a matrix has the S-RRSP of order k, it must have the S-RRSP of order k with respect to any individual vector $y \in Y^k$. The S-RRSP of order k makes it possible to recover the sign of all k-sparse signals from 1-bit measurements with (2.22), as shown in the next theorem.

Theorem 4.4. *Suppose that Φ^{T} has the S-RRSP of order k and that for any k-sparse signal x^*, the sign measurement $\mathrm{sign}(\Phi x^*)$ can be acquired. Then the 1-bit basis pursuit (2.22) with $J_+ = \{i : \mathrm{sign}(\Phi x^*)_i = 1\}$, $J_- = \{i : \mathrm{sign}(\Phi x^*)_i = -1\}$ and $J_0 = \{i : \mathrm{sign}(\Phi x^*)_i = 0\}$ has a unique optimal solution $\widetilde{x}$ satisfying that $\mathrm{supp}(\widetilde{x}) \subseteq \mathrm{supp}(x^*)$. Furthermore, for any k-sparse signal x^* which is a sparsest signal satisfying*

$$\mathrm{sign}(\Phi x) = \mathrm{sign}(\Phi x^*), \tag{4.3}$$

the sign of x^* *can be exactly recovered by* (2.22), *i.e., the unique optimal solution* $\widetilde{x}$ *of* (2.22) *satisfies that* $\mathrm{sign}(\widetilde{x}) = \mathrm{sign}(x^*)$.

Proof. Let x^* be an arbitrary k-sparse signal, and let measurement $y = \mathrm{sign}(\Phi x^*)$ be taken, which determines a partition (J_+, J_-, J_0) of $\{1, \dots, m\}$ as (2.17). Since Φ^{T} has the S-RRSP of order k, this implies that Φ^{T} has the S-RRSP of order k with respect to this vector y. By Theorem 3.9, the problem (2.22) has a unique optimal solution, denoted by $\widetilde{x}$, which satisfies that $\mathrm{supp}(\widetilde{x}) \subseteq \mathrm{supp}(x^*)$. Furthermore, if x^* is a sparsest signal satisfying the system (4.3), then by Theorem 3.9 again, we must have that $\mathrm{sign}(\widetilde{x}) = \mathrm{sign}(x^*)$, and hence the sign of x^* can be exactly recovered by (2.22). □

The above theorem indicates that under the S-RRSP of order k if x^* is a sparsest solution to (4.3), then the sign of x^* can be exactly recovered by (2.22). If x^* is not a sparsest solution to (4.3), then at least part of the support of x^* can be exactly recovered by (2.22) in the sense that $\mathrm{supp}(\widetilde{x}) \subseteq \mathrm{supp}(x^*)$, where $\widetilde{x}$ is the optimal solution to (2.22).

The study in this paper indicates that the models (2.21) and (2.22) make it possible to establish a sign recovery theory for k-sparse signals from 1-bit measurements. It is worth noting that these models can also make it possible to extend reweighted ℓ_1-algorithms (see [13, 34, 43, 44]) to 1-bit CS problems.

The RIP and NSP recovery conditions are widely assumed in classic CS scenarios. Recent study has shown that it is NP-hard to compute the RIP and NSP constants of a given matrix (see [3, 36]). The RSP recovery condition introduced in [41] is equivalent to the NSP since both are the necessary and sufficient conditions for the uniform recovery of all k-sparse signals. The NSP characterizes the uniform recovery from the perspective of the null space of a sensing matrix, while the RSP characterizes the uniform recovery from its orthogonal space, i.e., the range space of a transposed sensing matrix. So it is also difficult to certify the RSP of a given matrix. Clearly, the N-RRSP and S-RRSP are more complex than the standard RSP, and thus they are hard to certify as well. Note that the existence of a matrix with the RSP follows directly from the fact that any matrix with RIP of order $2k$ or NSP of order $2k$ must admit the RSP of order k (see [41]). In 1-bit CS setting, however, the analogous theory is still under development. The existence analysis of an S-RRSP matrix has not yet been properly addressed at the current stage.

5 Proof of Theorem 3.2

We now prove Theorem 3.2 which provides a complete characterization for the uniqueness of solutions to the 1-bit basis pursuit (2.22). We start by developing necessary conditions.

5.1 Necessary condition (I): Range space property

By introducing $u, v, t \in \mathbb{R}^n_+$, where t satisfies that $|x_i| \leqslant t_i$ for $i = 1, \dots, n$, then (2.27) can be written as the linear program

$$\begin{aligned} \min\quad & e^{\mathrm{T}} t \\ \text{s.t.}\quad & x + u = t, \quad -x + v = t, \quad \Phi_{J_+,n} x - \alpha = e_{J_+}, \\ & \Phi_{J_-,n} x + \beta = -e_{J_-}, \quad \Phi_{J_0,n} x = 0, \\ & (t, u, v, \alpha, \beta) \geqslant 0. \end{aligned} \tag{5.1}$$

Clearly, we have the following statement.

Lemma 5.1. (i) *For any optimal solution* $(x^*, t^*, u^*, v^*, \alpha^*, \beta^*)$ *of* (5.1), *we have that* $t^* = |x^*|$, $u^* = |x^*| - x^*$, $v^* = |x^*| + x^*$ *and* (α^*, β^*) *is given by* (2.28). (ii) x^* *is the unique optimal solution to* (2.22) *if and only if* $(x, t, u, v, \alpha, \beta) = (x^*, |x^*|, |x^*| - x^*, |x^*| + x^*, \alpha^*, \beta^*)$ *is the unique optimal solution to* (5.1), *where* (α^*, β^*) *is given by* (2.28).

Any linear program can be written in the form $\min\{c^{\mathrm{T}}z : Az = b, z \geqslant 0\}$, to which the Lagrangian dual problem is given by $\max\{b^{\mathrm{T}}y : A^{\mathrm{T}}y \leqslant c\}$ (see [15]). So it is very easy to verify that the dual problem of (5.1) is given as

$$
\begin{aligned}
\text{(DLP)} \quad \max \quad & e_{J_+}^{\mathrm{T}} h_3 - e_{J_-}^{\mathrm{T}} h_4 \\
\text{s.t.} \quad & h_1 - h_2 + (\Phi_{J_+,n})^{\mathrm{T}} h_3 + (\Phi_{J_-,n})^{\mathrm{T}} h_4 + (\Phi_{J_0,n})^{\mathrm{T}} h_5 = 0, \\
& -h_1 - h_2 \leqslant e, && (5.2) \\
& h_1 \leqslant 0, && (5.3) \\
& h_2 \leqslant 0, && (5.4) \\
& -h_3 \leqslant 0, && (5.5) \\
& h_4 \leqslant 0. && (5.6)
\end{aligned}
$$

(DLP) is always feasible in the sense that there exists a point, for example, $(h_1, \ldots, h_5) = (0, \ldots, 0)$, satisfying all constraints. Furthermore, let $s^{(1)}, \ldots, s^{(5)}$ be the non-negative slack variables associated with the constraints (5.2) through (5.6), respectively. Then (DLP) can be also written as

$$
\begin{aligned}
\max \quad & e_{J_+}^{\mathrm{T}} h_3 - e_{J_-}^{\mathrm{T}} h_4 \\
\text{s.t.} \quad & h_1 - h_2 + (\Phi_{J_+,n})^{\mathrm{T}} h_3 + (\Phi_{J_-,n})^{\mathrm{T}} h_4 + (\Phi_{J_0,n})^{\mathrm{T}} h_5 = 0, && (5.7) \\
& s^{(1)} - h_1 - h_2 = e, && (5.8) \\
& s^{(2)} + h_1 = 0, && (5.9) \\
& s^{(3)} + h_2 = 0, && (5.10) \\
& s^{(4)} - h_3 = 0, && (5.11) \\
& s^{(5)} + h_4 = 0, && (5.12) \\
& s^{(1)}, \ldots, s^{(5)} \geqslant 0.
\end{aligned}
$$

We now prove that if x^* is the unique optimal solution to (2.22), the range space $\mathcal{R}(\Phi^{\mathrm{T}})$ must satisfy some properties.

Lemma 5.2. *If x^* is the unique optimal solution to* (2.22)*, then there exist vectors $h_1, h_2 \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$ satisfying*

$$
\begin{cases}
h_2 - h_1 = \Phi^{\mathrm{T}} w, \\
(h_1)_i = -1, \quad (h_2)_i = 0 \quad \textit{for} \;\; x_i^* > 0, \\
(h_1)_i = 0, \quad (h_2)_i = -1 \quad \textit{for} \;\; x_i^* < 0, \\
(h_1)_i, (h_2)_i < 0, \quad (h_1 + h_2)_i > -1 \quad \textit{for} \;\; x_i^* = 0, \\
w_i > 0 \quad \textit{for} \;\; i \in \mathcal{A}(x^*) \cap J_+, \\
w_i < 0 \quad \textit{for} \;\; i \in \mathcal{A}(x^*) \cap J_-, \\
w_i = 0 \quad \textit{for} \;\; i \in \tilde{\mathcal{A}}_+(x^*) \cup \tilde{\mathcal{A}}_-(x^*).
\end{cases}
\tag{5.13}
$$

Proof. Assume that x^* is the unique optimal solution to (2.22). By Lemma 5.1,

$$
(x, t, u, v, \alpha, \beta) = (x^*, |x^*|, |x^*| - x^*, |x^*| + x^*, \alpha^*, \beta^*) \tag{5.14}
$$

is the unique optimal solution to (5.1), where (α^*, β^*) is given by (2.28). By the strict complementarity theory of linear programs (see [21]), there exists a solution $(h_1, \ldots, h_5)$ of (DLP) such that the associated vectors $s^{(1)}, \ldots, s^{(5)}$ determined by (5.8)–(5.12) and the vectors (t, u, v, α, β) given by (5.14) are strictly complementary, i.e., these vectors satisfy the conditions

$$
t^{\mathrm{T}} s^{(1)} = u^{\mathrm{T}} s^{(2)} = v^{\mathrm{T}} s^{(3)} = \alpha^{\mathrm{T}} s^{(4)} = \beta^{\mathrm{T}} s^{(5)} = 0 \tag{5.15}
$$

and

$$\begin{cases} t + s^{(1)} > 0, \quad u + s^{(2)} > 0, \quad v + s^{(3)} > 0, \\ \alpha + s^{(4)} > 0, \quad \beta + s^{(5)} > 0. \end{cases} \tag{5.16}$$

For the above-mentioned solution $(h_1, \dots, h_5)$ of (DLP), let $w \in \mathbb{R}^m$ be the vector defined by $w_{J_+} = h_3, w_{J_-} = h_4$, and $w_{J_0} = h_5$. Then it follows from (5.7) that

$$h_2 - h_1 = (\Phi_{J_+,n})^{\mathrm{T}} h_3 + (\Phi_{J_-,n})^{\mathrm{T}} h_4 + (\Phi_{J_0,n})^{\mathrm{T}} h_5 = \Phi^{\mathrm{T}} w. \tag{5.17}$$

From (5.14), we see that the solution of (5.1) satisfies the following properties:

$$\begin{aligned} &t_i = x_i^* > 0, \quad u_i = 0, \quad v_i = 2x_i^* > 0 \quad \text{for} \;\; x_i^* > 0, \\ &t_i = |x_i^*| > 0, \quad u_i = 2|x_i^*| > 0, \quad v_i = 0 \quad \text{for} \;\; x_i^* < 0, \\ &t_i = 0, \quad u_i = 0, \quad v_i = 0 \quad \text{for} \;\; x_i^* = 0. \end{aligned}$$

Thus, from (5.15) and (5.16), it follows that

$$\begin{aligned} &s_i^{(1)} = 0, \quad s_i^{(2)} > 0, \quad s_i^{(3)} = 0 \quad \text{for} \;\; x_i^* > 0, \\ &s_i^{(1)} = 0, \quad s_i^{(2)} = 0, \quad s_i^{(3)} > 0 \quad \text{for} \;\; x_i^* < 0, \\ &s_i^{(1)} > 0, \quad s_i^{(2)} > 0, \quad s_i^{(3)} > 0 \quad \text{for} \;\; x_i^* = 0. \end{aligned}$$

From (5.8), (5.9) and (5.10), the above relations imply that

$$\begin{aligned} &(h_1 + h_2)_i = -1, \quad (h_1)_i < 0, \quad (h_2)_i = 0 \quad \text{for} \;\; x_i^* > 0, \\ &(h_1 + h_2)_i = -1, \quad (h_1)_i = 0, \quad (h_2)_i < 0 \quad \text{for} \;\; x_i^* < 0, \\ &(h_1 + h_2)_i > -1, \quad (h_1)_i < 0, \quad (h_2)_i < 0 \quad \text{for} \;\; x_i^* = 0. \end{aligned}$$

From (5.11) and (5.12), we see that $s^{(4)} = h_3 \geqslant 0$ and $s^{(5)} = -h_4 \geqslant 0$. Let $\pi(\cdot)$ and $\varrho(\cdot)$ be defined as (2.25) and (2.26), respectively. It follows from (2.28), (5.15) and (5.16) that

$$\begin{aligned} &(h_3)_{\pi(i)} = s_{\pi(i)}^{(4)} > 0 \quad \text{for} \;\; i \in \mathcal{A}(x^*) \cap J_+, \\ &(h_3)_{\pi(i)} = s_{\pi(i)}^{(4)} = 0 \quad \text{for} \;\; i \in \tilde{\mathcal{A}}_+(x^*), \\ &(-h_4)_{\varrho(i)} = s_{\varrho(i)}^{(5)} > 0 \quad \text{for} \;\; i \in \mathcal{A}(x^*) \cap J_-, \\ &(-h_4)_{\varrho(i)} = s_{\varrho(i)}^{(5)} = 0 \quad \text{for} \;\; i \in \tilde{\mathcal{A}}_-(x^*). \end{aligned}$$

By the definition of w (i.e., $w_{J_+} = h_3$, $w_{J_-} = h_4$ and $w_{J_0} = h_5$), the above conditions imply that

$$\begin{aligned} &w_i = (h_3)_{\pi(i)} > 0 \quad \text{for} \;\; i \in \mathcal{A}(x^*) \cap J_+, \\ &w_i = (h_3)_{\pi(i)} = 0 \quad \text{for} \;\; i \in \tilde{\mathcal{A}}_+(x^*), \\ &w_i = (h_4)_{\varrho(i)} < 0 \quad \text{for} \;\; i \in \mathcal{A}(x^*) \cap J_-, \\ &w_i = (h_4)_{\varrho(i)} = 0 \quad \text{for} \;\; i \in \tilde{\mathcal{A}}_-(x^*). \end{aligned}$$

Thus, h_1, h_2 and w satisfy (5.17) and the following properties:

$$\begin{aligned} &(h_1)_i = -1, \quad (h_2)_i = 0 \quad \text{for} \;\; x_i^* > 0, \\ &(h_1)_i = 0, \quad (h_2)_i = -1 \quad \text{for} \;\; x_i^* < 0, \\ &(h_1)_i, (h_2)_i < 0, \quad (h_1 + h_2)_i > -1 \quad \text{for} \;\; x_i^* = 0, \\ &w_i > 0 \quad \text{for} \;\; i \in \mathcal{A}(x^*) \cap J_+, \\ &w_i = 0 \quad \text{for} \;\; i \in \tilde{\mathcal{A}}_+(x^*), \\ &w_i < 0 \quad \text{for} \;\; i \in \mathcal{A}(x^*) \cap J_-, \\ &w_i = 0 \quad \text{for} \;\; i \in \tilde{\mathcal{A}}_-(x^*). \end{aligned}$$

Therefore, (5.13) is a necessary condition for x^* to be the unique optimal solution to (2.22). $\square$

It should be pointed out that the uniqueness of x^* implies that x^* is the strictly complementary solution. This leads to the condition (5.13) in which all inequalities hold strictly. If x^* is not the unique optimal solution of (2.22), then x^* is not necessarily a strictly complementary solution, and thus (5.13) does not necessarily hold. We now present an equivalent statement for (5.13) as follows.

Lemma 5.3. *Let $x^* \in \mathbb{R}^n$ be a given vector satisfying the constraints of* (2.22)*. There exist vectors h_1, h_2 and w satisfying* (5.13) *if and only if there exists a vector $\eta \in \mathcal{R}(\Phi^{\mathrm{T}})$ satisfying the following two conditions*:

(i) $\eta_i = 1$ *for* $x_i^* > 0$, $\eta_i = -1$ *for* $x_i^* < 0$, *and* $|\eta_i| < 1$ *for* $x_i^* = 0$;

(ii) $\eta = \Phi^{\mathrm{T}} w$ *for some* $w \in \mathcal{F}(x^*)$ *defined as* (3.1).

It is straightforward to verify this lemma. Its proof is omitted here. By Definition 3.1, combining Lemmas 5.2 and 5.3 yields the following result.

Corollary 5.4. *If x^* is the unique optimal solution to* (2.22)*, then the RRSP of Φ^{T} at x^* holds.*

The RRSP at x^* is not sufficient to ensure the uniqueness of x^*. We need to develop another necessary condition (called the full-column-rank property).

5.2 Necessary condition (II): Full column rank

Assume that x^* is the unique optimal solution to (2.22). Denote still by $S_+ = \{i : x_i^* > 0\}$ and $S_- = \{i : x_i^* < 0\}$. We have the following lemma.

Lemma 5.5. *If x^* is the unique optimal solution to* (2.22)*, then $H(x^*)$, defined by* (3.2)*, has a full-column rank.*

Proof. Assume the contrary that $H(x^*)$ has linearly dependent columns. Then there exists a vector

$$d = \begin{bmatrix} u \\ v \end{bmatrix} \neq 0,$$

where $u \in \mathbb{R}^{|S_+|}$ and $v \in \mathbb{R}^{|S_-|}$, such that $H(x^*)d = 0$. Since x^* is the unique optimal solution to (2.22), there exist non-negative α^* and β^*, determined by (2.28), such that (x^*, α^*, β^*) is the unique optimal solution to (2.27) with the least objective value $\|x^*\|_1$. Note that (x^*, α^*, β^*) satisfies

$$\Phi_{J_+,n} x^* - \alpha^* = e_{J_+}, \quad \Phi_{J_-,n} x^* + \beta^* = -e_{J_-}, \quad \Phi_{J_0,n} x^* = 0.$$

Similar to the proof of Lemma 3.3, eliminating the zero components of x^*, α^* and β^* from the above system yields the same system as (3.9). Similarly, we define $x(\lambda) \in \mathbb{R}^n$ as $x(\lambda)_{S_+} = x^*_{S_+} + \lambda u$, and $x(\lambda)_{S_-} = x^*_{S_-} + \lambda v$, and $x(\lambda)_i = 0$ for $i \notin S_+ \cup S_-$. We see that for all sufficiently small $|\lambda|$, $(x(\lambda)_{S_+}, x(\lambda)_{S_-})$ satisfies the conditions (3.10)–(3.12). In other words, there exists a small number $\delta > 0$ such that for any $\lambda \neq 0$ with $|\lambda| \in (0, \delta)$, the vector $x(\lambda)$ is feasible to (2.22). In particular, choose $\lambda^* \neq 0$ such that $|\lambda^*| \in (0, \delta), x^*_{S_+} + \lambda^* u > 0, x^*_{S_-} + \lambda^* v > 0$ and

$$\lambda^* (e_{S_+}^{\mathrm{T}} u - e_{S_-}^{\mathrm{T}} v) \leqslant 0. \tag{5.18}$$

Then we see that $x(\lambda^*) \neq x^*$ since $\lambda^* \neq 0$ and $(u, v) \neq 0$. Moreover, we have

$$\begin{aligned} \|x(\lambda^*)\|_1 &= e_{S_+}^{\mathrm{T}} (x^*_{S_+} + \lambda^* u) - e_{S_-}^{\mathrm{T}} (x^*_{S_-} + \lambda^* v) \\ &= e_{S_+}^{\mathrm{T}} x^*_{S_+} - e_{S_-}^{\mathrm{T}} x^*_{S_-} + \lambda^* e_{S_+}^{\mathrm{T}} u - \lambda^* e_{S_-}^{\mathrm{T}} v \\ &= \|x^*\|_1 + \lambda^* (e_{S_+}^{\mathrm{T}} u - e_{S_-}^{\mathrm{T}} v) \\ &\leqslant \|x^*\|_1, \end{aligned}$$

where the inequality follows from (5.18). As $\|x^*\|_1$ is the least objective value of (2.22), it implies that $x(\lambda^*)$ is also an optimal solution to this problem, contradicting the uniqueness of x^*. Hence, $H(x^*)$ must have a full-column rank. □

Combination of Corollary 5.4 and Lemma 5.5 yields the desired necessary conditions.

Theorem 5.6. *If x^* is the unique optimal solution to (2.22), then $H(x^*)$, given by (3.2), has a full-column rank and the RRSP of Φ^{T} at x^* holds.*

5.3 Sufficient conditions

We now prove that the converse of Theorem 5.6 is also valid, i.e., the RRSP of Φ^{T} at x^* combined with the full-column-rank property of $H(x^*)$ is a sufficient condition for the uniqueness of x^*. We start with a property of (DLP).

Lemma 5.7. *Suppose that x^* satisfies the constraints of (2.22). If the vector $(h_1, h_2, w) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ satisfies that*

$$\begin{cases} (h_1)_i = -1, \quad (h_2)_i = 0 \quad \text{for } x_i^* > 0, \\ (h_1)_i = 0, \quad (h_2)_i = -1 \quad \text{for } x_i^* < 0, \\ (h_1)_i < 0, \quad (h_2)_i < 0, \quad (h_1 + h_2)_i > -1 \quad \text{for } x_i^* = 0, \\ h_2 - h_1 = \Phi^{\mathrm{T}} w, \\ w_{J_+} \geqslant 0, \\ w_{J_-} \leqslant 0, \\ w_i = 0 \quad \text{for } i \in \tilde{\mathcal{A}}_+(x^*) \cup \tilde{\mathcal{A}}_-(x^*), \end{cases} \tag{5.19}$$

then the vector $(h_1, h_2, h_3, h_4, h_5)$, with $h_3 = w_{J_+}, h_4 = w_{J_-}$ and $h_5 = w_{J_0}$, is an optimal solution to (DLP) and x^* is an optimal solution to (2.22).

This lemma follows directly from the optimality theory of linear programs by verifying that the dual optimal value at $(h_1, h_2, h_3, h_4, h_5)$ is equal to $\|x^*\|_1$. The proof is omitted.

We now prove the desired sufficient condition for the uniqueness of optimal solutions of (2.22).

Theorem 5.8. *Let x^* satisfy the constraints of the problem (2.22). If the RRSP of Φ^{T} at x^* holds and $H(x^*)$, defined by (3.2), has a full-column rank, then x^* is the unique optimal solution to (2.22).*

Proof. By the assumption of the theorem, the RRSP of Φ^{T} at x^* holds. Then by Lemma 5.3, there exists a vector $(h_1, h_2, w) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ satisfying (5.13), which implies that (5.19) holds. As x^* is feasible to (2.22), by Lemma 5.7, $(h_1, h_2, h_3, h_4, h_5)$ with $h_3 = w_{J_+}, h_4 = w_{J_-}$ and $h_5 = w_{J_0}$ is an optimal solution to (DLP). At this solution, let the slack vectors $s^{(1)}, \ldots, s^{(5)}$ be given as (5.8)–(5.12). Also, from Lemma 5.7, x^* is an optimal solution to (2.22). Thus by Lemma 5.1, $(x, t, u, v, \alpha, \beta) = (x^*, |x^*|, |x^*| - x^*, |x^*| + x^*, \alpha^*, \beta^*)$, where (α^*, β^*) is given by (2.28), is an optimal solution to (5.1). We now further show that x^* is the unique solution to (2.22).

The vector (x^*, α^*, β^*) satisfies the system $\Phi_{J_+,n} x^* - \alpha^* = e_{J_+}, \Phi_{J_-,n} x^* + \beta^* = -e_{J_-}$ and $\Phi_{J_0,n} x^* = 0$. As shown in the proof of Lemma 5.5, removing the zero components of (x^*, α^*, β^*) from the above system yields

$$H(x^*) \begin{bmatrix} x_{S_+}^* \\ x_{S_-}^* \end{bmatrix} = \begin{bmatrix} e_{\mathcal{A}(x^*) \cap J_+} \\ -e_{\mathcal{A}(x^*) \cap J_-} \\ 0 \end{bmatrix}. \tag{5.20}$$

Let $(\widetilde{x}, \widetilde{t}, \widetilde{u}, \widetilde{v}, \widetilde{\alpha}, \widetilde{\beta})$ be an arbitrary optimal solution to (5.1). By Lemma 5.1, it must hold that $\widetilde{t} = |\widetilde{x}|, \widetilde{u} = |\widetilde{x}| - \widetilde{x}$ and $\widetilde{v} = |\widetilde{x}| + \widetilde{x}$. By the complementary slackness property of linear programs (see [15, 21]), the non-negative vectors $(\widetilde{t}, \widetilde{u}, \widetilde{v}, \widetilde{\alpha}, \widetilde{\beta})$ and $(s^{(1)}, \ldots, s^{(5)})$ are complementary, i.e.,

$$\widetilde{t}^{\mathrm{T}} s^{(1)} = \widetilde{u}^{\mathrm{T}} s^{(2)} = \widetilde{v}^{\mathrm{T}} s^{(3)} = \widetilde{\alpha}^{\mathrm{T}} s^{(4)} = \widetilde{\beta}^{\mathrm{T}} s^{(5)} = 0. \tag{5.21}$$

As (h_1, h_2, w) satisfies (5.13), the vector (h_1, h_2) satisfies that $(h_1)_i = -1 < 0$ for $x_i^* > 0$, $(h_2)_i = -1 < 0$ for $x_i^* < 0$ and that $(h_1 + h_2)_i > -1$, $(h_1)_i < 0$ and $(h_2)_i < 0$ for $x_i^* = 0$. By the choice of (h_1, h_2) and $(s^{(1)}, \ldots, s^{(5)})$, we see that the following components of slack variables are positive:

$$s_i^{(1)} = 1 + (h_1 + h_2)_i > 0 \quad \text{for } x_i^* = 0,$$

$$s^{(4)}_{\pi(i)} = (h_3)_{\pi(i)} = w_i > 0 \quad \text{for} \quad i \in \mathcal{A}(x^*) \cap J_+,$$
$$s^{(5)}_{\varrho(i)} = -(h_4)_{\varrho(i)} = -w_i > 0 \quad \text{for} \quad i \in \mathcal{A}(x^*) \cap J_-.$$

These conditions, together with (5.21), imply that

$$\begin{cases} \widetilde{t}_i = 0 \quad \text{for} \quad x^*_i = 0, \\ \widetilde{\alpha}_{\pi(i)} = 0 \quad \text{for} \quad i \in \mathcal{A}(x^*) \cap J_+, \\ \widetilde{\beta}_{\varrho(i)} = 0 \quad \text{for} \quad i \in \mathcal{A}(x^*) \cap J_-. \end{cases} \tag{5.22}$$

We still use the symbol $S_+ = \{i : x^*_i > 0\}$ and $S_- = \{i : x^*_i < 0\}$. Since $\widetilde{t} = |\widetilde{x}|$, the first relation in (5.22) implies that $\widetilde{x}_i = 0$ for all $i \notin S_+ \cup S_-$. Note that

$$\Phi_{J_+,n}\widetilde{x} - \widetilde{\alpha} = e_{J_+}, \quad \Phi_{J_-,n}\widetilde{x} + \widetilde{\beta} = -e_{J_-}, \quad \Phi_{J_0,n}\widetilde{x} = 0.$$

Since $\widetilde{x}_i = 0$ for all $i \notin S_+ \cup S_-$, by (3.7) and (5.22), it implies from the above system that

$$H(x^*) \begin{bmatrix} \widetilde{x}_{S_+} \\ \widetilde{x}_{S_-} \end{bmatrix} = \begin{bmatrix} e_{\mathcal{A}(x^*)\cap J_+} \\ -e_{\mathcal{A}(x^*)\cap J_-} \\ 0 \end{bmatrix}. \tag{5.23}$$

By the assumption of the theorem, the matrix $H(x^*)$ has a full-column rank. Thus it follows from (5.20) and (5.23) that $\widetilde{x}_{S_+} = x^*_{S_+}$ and $\widetilde{x}_{S_-} = x^*_{S_-}$ which, together with the fact $\widetilde{x}_i = 0$ for all $i \notin S_+ \cup S_-$, implies that $\widetilde{x} = x^*$. By assumption, $(\widetilde{x}, \widetilde{t}, \widetilde{u}, \widetilde{v}, \widetilde{\alpha}, \widetilde{\beta})$ is an arbitrary optimal solution to (5.1). Thus $(x, t, u, v, \alpha, \beta) = (x^*, |x^*|, |x^*| - x^*, |x^*| + x^*, \alpha^*, \beta^*)$ is the unique optimal solution to (5.1), and hence (by Lemma 5.1) x^* is the unique optimal solution to (2.22). □

Combination of Theorems 5.6 and 5.8 yields Theorem 3.2.

6 Conclusions

Different from the classic compressive sensing, 1-bit measurements are robust to any small perturbation of a signal. The purpose of this paper is to show that the exact recovery of the sign of a sparse signal from 1-bit measurements is possible. We have proposed a new reformulation for the 1-bit CS problem. This reformulation makes it possible to extend the analytical tools in classic CS to 1-bit CS in order to achieve an analogous theory and decoding algorithms for 1-bit CS problems. Based on the fundamental Theorem 3.2, we have introduced the so-called restricted range space property (RRSP) of a sensing matrix. This property has been used to establish a connection between sensing matrices and the sign recovery of sparse signals from 1-bit measurements. For nonuniform sign recovery, we have shown that if the transposed sensing matrix admits the so-called S-RRSP of order k with respect to 1-bit measurements, acquired from an individual k-sparse signal, then the sign of the signal can be exactly recovered by the proposed 1-bit basis pursuit. For uniform sign recovery, we have shown that the sign of any k-sparse signal, which is the sparsest signal consistent with the acquired 1-bit measurements, can be exactly recovered with 1-bit basis pursuit when the transposed sensing matrix admits the so-called S-RRSP of order k.

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References

1 Ai A, Lapanowski A, Plan Y, et al. One-bit compressed sensing with non-Gaussian measurements. Linear Algebra Appl, 2014, 441: 222–239

2 Bahmani S, Boufounos P T, Raj B. Robust 1-bit compressive sensing via gradient support pursuit. ArXiv:1304.6627, 2013

3 Bandeira A, Dobriban E, Mixon D, et al. Certifying the restricted isometry property is hard. IEEE Trans Inform Theory, 2013, 59: 3448–3450

4 Boufounos P. Greedy sparse signal reconstruction from sign measurements. In: Proceedings of the 43rd Asilomar Conference on Signals, Systems and Computers. Piscataway: IEEE, 2009, 1305–1309

5 Boufounos P. Reconstruction of sparse signals from distorted randomized measurements. In: 2010 IEEE International Conference on Acoustics, Speech and Signal Processing. New York: IEEE, 2010, 3998–4001

6 Boufounos P, Baraniuk R. 1-Bit compressive sensing. In: Conference on Information Sciences and Systems. New York: IEEE, 2008, 16–21

7 Bourquard A, Aguet F, Unser M. Optical imaging using binary sensors. Opt Express, 2010, 18: 4876–4888

8 Bourquard A, Unser M. Binary compressed imaging. IEEE Trans Image Process, 2013, 22: 1042–1055

9 Bruckstein A, Elad M, Donoho D. From sparse solutions of systems of equations to sparse modeling of signals and images. SIAM Rev, 2009, 51: 34–81

10 Candès E, Romberg J, Tao T. Stable signal recovery from incomplete and inaccurate measurements. Comm Pure Appl Math, 2006, 59: 1207–1223

11 Candès E, Romberg J, Tao T. Robust uncertainty principles: Exact signal reconstruction from highly incomplete frequency information. IEEE Trans Inform Theory, 2006, 52: 489–509

12 Candès E, Tao T. Decoding by linear programming. IEEE Trans Inform Theory, 2005, 51: 4203–4215

13 Candès E, Wakin M, Boyd S. Enhancing sparsity by reweighted ℓ_1 minimization. J Fourier Anal Appl, 2008, 14: 877–905

14 Cohen A, Dahmen W, Devore R. Compressive sensing and best k-term approximation. J Amer Math Soc, 2009, 22: 211–231

15 Dantzig G. Linear Programming and Extensions. Princeton: Princeton University Press, 1963

16 Davenport M, Plan Y, van den Berg E, et al. 1-Bit matrix completion. Inform Inference, 2014, 3: 189–223

17 Donoho D. Compressed sensing. IEEE Trans Inform Theory, 2006, 52: 1289–1306

18 Donoho D, Elad M. Optimally sparse representation in general (nonorthogonal) dictionaries via ℓ^1 minimization. Proc Natl Acad Sci USA, 2003, 100: 2197–2202

19 Foucart S, Rauhut H. A Mathematical Introduction to Compressive Sensing. New York: Springer, 2013

20 Fuchs J. On sparse representations in arbitrary redundant bases. IEEE Trans Inform Theory, 2004, 50: 1341–1344

21 Goldman A J, Tucker A W. Theory of linear programming. In: Linear Inequalities and Related Systems. Princeton: Princeton University Press, 1956, 53–97

22 Gopi S, Netrapalli P, Jain P, et al. One-bit compressed sensing: Provable support and vector recovery. J Mach Learn Res, 2013, 3: 154–162

23 Gupta A, Nowak A, Recht B. Sample complexity for 1-bit compressed sensing and sparse classification. In: 2010 IEEE International Symposium on Information Theory. New York: IEEE, 2010, 1553–1557

24 Jacques L, Laska J, Boufounos P, et al. Robust 1-bit compressive sensing via binary stable embeddings of sparse vectors. IEEE Trans Inform Theory, 2013, 59: 2082–2102

25 Kamilov U, Bourquard A, Amini A, et al. One-bit measurements with adaptive thresholds. IEEE Signal Process Lett, 2012, 19: 607–610

26 Laska J. Regime change: Sampling rate vs bit-depth in compressive sensing. PhD thesis. Houston: Rice University, 2011

27 Laska J, Baraniuk R. Regime change: Bit-depth versus measurement-rate in compressive sensing. IEEE Trans Signal Process, 2012, 60: 3496–3505

28 Laska J, Wen Z, Yin W, et al. Trust, but verify: Fast and accurate signal recovery from 1-bit compressive measurements. IEEE Trans Signal Process, 2011, 59: 5289–5301

29 Le B, Rondeau T, Reed J, et al. Analog-to-digital converters. IEEE Signal Process Mag, 2005, 22: 69–77

30 Movahed A, Panahi A, Durisi G. A robust RFPI-based 1-bit compressive sensing reconstruction algorithm. In: Information Theory Workshop. New York: IEEE, 2012, 567–571

31 Plan Y, Vershynin R. One-bit compressed sensing by linear programming. Comm Pure Appl Math, 2013, 66: 1275–1297

32 Plan Y, Vershynin R. Robust 1-bit compressed sensing and sparse logistic regression: A convex programming approach. IEEE Trans Inform Theory, 2013, 59: 482–494

33 Plumbley M. On polar polytopes and the recovery of sparse representations. IEEE Trans Inform Theory, 2007, 53: 3188–3195

34 Shen L, Suter B. Blind one-bit compressive sampling. ArXiv:1302.1419, 2013

35 Sun J, Goyal V. Quantization for compressed sensing reconstruction. https://hal.archives-ouvertes.fr/hal-00452256/, 2009

36 Tillmann A, Pfetsch M. The computational complexity of the restricted isometry property, the nullspace property,

and related concepts in compressed sensing. IEEE Trans Inform Theory, 2014, 60: 1248–1259

37 Tropp J. Greed is good: Algorithmic results for sparse approximation. IEEE Trans Inform Theory, 2004, 50: 2231–2242

38 Walden R. Analog-to-digital converter survey and analysis. IEEE J Sel Areas Commun, 199, 17: 539–550

39 Yan M, Yang Y, Osher S. Robust 1-bit compressive sensing using adaptive outlier pursuit. IEEE Trans Signal Process, 2012, 60: 3868–3875

40 Zhang Y. Theory of compressive sensing via ℓ_1-minimization: A non-RIP analysis and extensions. J Oper Res Soc China, 2013, 1: 79–105

41 Zhao Y B. RSP-based analysis for sparsest and least ℓ_1-norm solutions to underdetermined linear systems. IEEE Trans Signal Process, 2013, 61: 5777–5788

42 Zhao Y B. Equivalence and strong equivalence between the sparsest and least ℓ_1-norm nonnegative solutions of linear systems and their applications. J Oper Res Soc China, 2014, 2: 171–193

43 Zhao Y B, Kočvara M. A new computational method for the sparsest solution to systems of linear equations. SIAM J Optim, 2015, 25: 1110–1134

44 Zhao Y B, Li D. Reweighted ℓ_1-minimization for sparse solutions to underdetermined linear systems. SIAM J Optim, 2012, 22: 1065–1088