MATHEMATICS OF OPERATIONS RESEARCH
Vol. 26, No. 1, February 2001, pp. 119–146
Printed in U.S.A.

ON A NEW HOMOTOPY CONTINUATION TRAJECTORY FOR NONLINEAR COMPLEMENTARITY PROBLEMS

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Most known continuation methods for P_0 complementarity problems require some restrictive assumptions, such as the strictly feasible condition and the properness condition, to guarantee the existence and the boundedness of certain homotopy continuation trajectory. To relax such restrictions, we propose a new homotopy formulation for the complementarity problem based on which a new homotopy continuation trajectory is generated. For P_0 complementarity problems, the most promising feature of this trajectory is the assurance of the existence and the boundedness of the trajectory under a condition that is strictly weaker than the standard ones used widely in the literature of continuation methods. Particularly, the often-assumed strictly feasible condition is not required here. When applied to P_* complementarity problems, the boundedness of the proposed trajectory turns out to be equivalent to the solvability of the problem, and the entire trajectory converges to the (unique) least element solution provided that it exists. Moreover, for monotone complementarity problems, the whole trajectory always converges to a least 2-norm solution provided that the solution set of the problem is nonempty. The results presented in this paper can serve as a theoretical basis for constructing a new path-following algorithm for solving complementarity problems, even for the situations where the solution set is unbounded.

1. Introduction. We denote by R^n the n-dimensional Euclidean space, R^n_+ the nonnegative orthant, and R^n_{++} the positive orthant. For simplicity, we also denote $x \in R^n_+ (R^n_{++})$ by $x \geq 0$ $(x > 0)$. In this paper, all vectors are column vectors, and superscript T denotes the transpose. e is the vector of all ones in R^n. For any $x \in R^n$, $\|x\|$ denotes the 2-norm and $[x]_+$($[x]_-$, respectively) denotes the vector whose ith component is $\max\{0, x_i\}$ ($\min\{0, x_i\}$, respectively). For any mapping $g : R^n \to R^n$ and any subset D of R^n, $g^{-1}(D)$, unless otherwise stated, denotes the set $\{x \in R^n : g(x) \in D\}$.

Given a continuous mapping $f : R^n \to R^n$, the well-known complementarity problem (abbreviated, CP(f)) is to determine a vector x satisfying

$$x \geq 0, \quad f(x) \geq 0, \quad x^T f(x) = 0.$$

Let $\mathrm{SOL}_{cp}(f)$ denote the solution set of the above problem. Define

$$\mathcal{S}_{++}(f) = \{z = (x, y) > 0 : y = f(x)\}.$$

We refer to $\mathcal{S}_{++}(f)$ as the set of strictly feasible points. There are several equivalent equation-based reformulations of CP(f) reported in the literature. One of the well-known forms is based on the following mapping:

$$\mathcal{F}(x, y) = \begin{pmatrix} Xy \\ y - f(x) \end{pmatrix}, \tag{1}$$

Received August 7, 1999; revised April 8, 2000.
AMS 2000 subject classification. Primary: 90C30, 90C33.
OR/MS subject classification. Primary: Programming/nonlinear; secondary: programming/complementarity.
Key words. Complementarity problems, continuation methods, homotopy continuation trajectories, strictly feasible condition, P_0-functions, P_*-functions.

0364-765X/01/2601/0119/$05.00
1526-5471 electronic ISSN,

where $X = \mathrm{diag}(x)$. It is easy to see that $x^* \in \mathrm{SOL}_{cp}(f)$ if and only if (x^*, y^*), where $y^* = f(x^*)$, is a solution of the system

$$\mathscr{F}(x, y) = 0, \quad (x, y) \geq 0.$$

This system provides us with a general theoretical framework for various efficient homotopy continuation methods, including the interior-point methods. See, for example, Lemke (1965), Lemke and Howson (1980), Kojima, Mizuno, and Noma (1989, 1990), Kojima, Megiddo, and Noma (1991), Kojima, Megiddo, Noma, and Yoshise (1991), Ye (1997), and Wright (1997). Moré (1996) also used $\mathscr{F}(x, y)$ to study the trust-region algorithm for CP(f). The fundamental idea of a homotopy continuation method is to solve the problem by tracing a certain continuous trajectory leading to a solution of the problem. The existence and the boundedness of a continuation trajectory play an essential role in constructing the homotopy continuation algorithms for the problem. The following two conditions are standard ones widely used in the literature to ensure the existence and the boundedness of a continuation trajectory. See, e.g., McLinden (1980), Megiddo (1989), Kojima, Mizuno, and Noma (1989, 1990), Kojima, Megiddo, and Noma (1991), Kojima, Megiddo, and Mizuno (1993), Kanzow (1996), Hotta and Yoshise (1999), Burke and Xu (2000), Qi and Sun (2000), and Hotta et al. (1998).

CONDITION 1.1.
(a) f is monotone on R^n, i.e., $(x-y)^T(f(x) - f(y)) \geq 0$ for all $x, y \in R^n$.
(b) There is a strictly feasible point, i.e., $\mathscr{S}_{++}(f) \neq \varnothing$.

CONDITION 1.2.
(a) f is a P_0-function, that is, $\max_{y_i \neq x_i}(x_i - y_i)(f_i(x) - f_i(y)) \geq 0$ for all $x \neq y$ in R^n.
(b) $\mathscr{S}_{++}(f) \neq \varnothing$.
(c) The set

$$\mathscr{F}^{-1}(D) = \{z = (x, y) \in R^{2n}_+ \colon \mathscr{F}(z) \in D\}$$

is bounded for every compact subset D of $R^n_+ \times B_{++}(f)$, where

$$B_{++}(f) = \{u \in R^n : u = y - f(x) \text{ for all } (x, y) > 0\}.$$

It is easy to see that Condition 1.1 implies Condition 1.2 (see, Kojima, Megiddo, and Noma 1991). Let $(a, b) \in R^n_+ \times R^n_+$ and $(a, b) \neq 0$. Kojima, Mizuno, and Noma (1989, 1990) and Kojima, Megiddo, and Noma (1991) considered a family of the system of equations with a parameter $t > 0$:

$$\mathscr{F}(x, y) = t\begin{pmatrix} a \\ b \end{pmatrix}, \ (x, y) \geq 0. \tag{2}$$

Let $(x(t), y(t))$ denote a solution to the above system. If $(x(t), y(t))$ is unique for each $t > 0$ and continuous in the parameter t, then the set

$$\left\{(x(t), y(t)) : \mathscr{F}(x, y) = t\begin{pmatrix} a \\ b \end{pmatrix}, \ (x, y) > 0, \ t > 0\right\} \tag{3}$$

forms a continuous curve in R^{2n}_+ which is called the trajectory of solutions of the system (2). As $t \to 0$, any accumulation point $(\bar{x}, \bar{y})$ of the trajectory, if exists, must be a solution to CP(f). Thus by tracing such a trajectory as $t \to 0$, we can obtain a solution of the problem. In this paper, we say that a trajectory is bounded if any slice (subtrajectory) of the trajectory, i.e., $\{(x(t), y(t)) : 0 < t \leq \hat{t}\}$ where $\hat{t} > 0$ is a positive number, is bounded.

It should be noted that (3) includes several important particular situations. For example, when $a = 0$ and $b > 0$, the trajectory reduces to the one induced by the well-known Lemke's method for linear complementarity problems (Lemke 1965, Lemke and Howson 1980). When $b = 0$ and $a = e$, the trajectory (3) reduces to the central path based on which many interior-point and non-interior-point algorithms are constructed. See, for example, Kojima, Mizuno, and Yoshise (1989), Kojima, Megiddo, Noma, and Yoshise (1991), Ye (1997), Wright (1997), Burke and Xu (1998, 1999, 2000).

For monotone maps and P_0-functions, Kojima et al. (1990) and Kojima, Megiddo, and Noma (1991) proved the existence and the boundedness of the trajectory (3) under the aforementioned Condition 1.1 and Condition 1.2, respectively. The strictly feasible condition is indispensable to the existence and the boundedness of their continuation trajectories. In fact, we give examples (see §6) to show that the trajectory (3) may not exist if the strictly feasible condition fails to hold. It is well-known that Condition 1.1 implies the existence and the boundedness of the central path (McLinden 1980, Megiddo 1989, Güler 1993). On the other hand, since each point on the central path is strictly feasible, we deduce that for a monotone CP(f) the central path exists if and only if there exists a strictly feasible point. This property actually holds for P_* complementarity problems (Kojima, Megiddo, and Yoshise 1991, Zhao and Isac 2000b). Therefore, it is not possible to remove the strictly feasible condition for those central path-based continuation methods without destroying their convergence.

Recently, some non-interior-point continuation methods have been intensively investigated for solving CP(f) (Chen and Harker 1993; Kanzow 1996; Hotta and Yoshise 1999; Burke and Xu 1998, 1999, 2000; Qi and Sun 2000; Hotta et al. 1998). While these algorithms start from a noninterior point, the strictly feasible condition is still assumed for such a class of methods to ensure the existence of non-interior-point trajectories and the convergence of algorithms. Specifically, Hotta and Yoshise (1999) considered the following homotopy trajectory:

$$\{(u, x, y) \in R^n_{++} \times R^{2n} : U(u, x, y) = t(\bar{u}, \bar{v}, \bar{r})^T, \, t \in (0, 1)\},$$

where $(\bar{u}, \bar{v}, \bar{r})^T \in R^n_{++} \times R^{2n}$ is a given vector, and

$$U(x, y, u) = \begin{pmatrix} u \\ x + y - \sqrt{(x-y)^2 + 4u} \\ y - f(x) \end{pmatrix},$$

where all algebraic operations are performed componentwise. Their analysis for the existence and the boundedness of such a continuation trajectory also requires Condition 1.1 or the one analogous to Condition 1.2 in which (c) of Condition 1.2 is replaced by

(c′) $U^{-1}(D) := \{(u, x, y) \in R^n_+ \times R^{2n} : U(u, x, y) \in D\}$ is bounded for every compact subset D of range $U(R^n_{++} \times R^{2n})$.

Hotta and Yoshise (1999) pointed out that their condition is not weaker than Condition 1.2. Thus, most of the continuation methods in the literature, including both the interior-point algorithms and the non-interior-point algorithms, are limited to solving the class of complementarity problems satisfying Condition 1.2 or its similar versions. Since Condition 1.2 implies that the solution set of CP(f) is nonempty and bounded (see §4 for details), it seems to be restrictive. Some continuation methods use other assumptions such as the P_0 and R_0 (see Burke and Xu 1998, Chen and Chen 1999, Zhao and Li 1998) which, however, still imply the nonemptyness and the boundedness of the solution set. It is worth mentioning that Monteiro and Pang (1996) established several results on the existence of certain interior-point paths for mixed complementarity problems. Restricted to CP(f), their results generalize and, to some extent, unify those already obtained for CP(f). It is easy to see, however, that the conditions used by Monteiro and Pang (1996) also imply the boundedness of the solution set of the problems. In fact, the properness assumptions used in Monteiro and Pang (1996), such as "uniformly norm coercive," imply that the solution set of the problem is bounded.

Since the nonemptyness and the boundedness of the solution set imply that the P_0 complementarity problem must be strictly feasible (Corollary 5 in Ravindran and Gowda 1997 and Corollary 3.14 in Chen et al. 1997), we conclude that most existing continuation methods actually confine themselves to solve only the class of P_0 complementarity problems with bounded solution sets, which are strictly feasible problems.

It is known that for monotone CP(f), the strictly feasible condition is equivalent to the nonemptyness and the boundedness of the solution set of CP(f) (McLinden 1990, Chen et al. 1997, Ravindran and Gowda 1997). This result has been extended to P_* complementarity problems by Zhao and Li (2000) who showed that if f is a P_*-function (see Definition 2.1 in the next section), the solution set of CP(f) is nonempty and bounded if and only if there exists a strictly feasible point. In summary, we conclude that the following three conditions are equivalent for P_* complementarity problems.

- There exists a strictly feasible point, i.e., $\mathcal{S}_{++}(f) \neq \varnothing$.
- Solution set of CP(f) is nonempty and bounded.
- The central path exists and any slice of it is bounded.

Because of the dependence on the strictly feasible condition, most existing continuation methods, when applied to a P_0 complementarity problem (in particular, a P_* problem and a monotone problem), fail to solve it if it is not strictly feasible (in this case, the solution set of CP(f) is unbounded).

From the above discussion, a natural question arises: *How does one construct a homotopy continuation trajectory such that its existence and boundedness do not require the strictly feasible condition in the setting of P_0-complementarity problems?* At a cost of losing the generality of the vector $c \in R^n_+ \times R^n_+$ in (2), a continuation trajectory, which does not need the strictly feasible condition, does exist. In fact, Kojima, Megiddo, and Mizuno (1993) established a result (Theorem 3.1 therein) which states that if $a \in R^n_+$ is fixed then for almost all $b \in R^n_+$, the Trajectory (3) exists. However, their result cannot exclude a zero measure set on which a trajectory fails to exist. A similar result is also obtained by Zhang and Zhang (1997).

In this paper, we propose a new homotopy formulation for the complementarity problem. Utilizing this formulation, we study the existence, boundedness and limiting behavior of a new continuation trajectory which can serve as a theoretical basis for designing a new path-following algorithm for CP(f) even when its solution set is unbounded. Our continuation trajectory possesses several prominent features: (f_1) The Tikhonov regularization technique is used in the formulation of the homotopy mapping; (f_2) In the P_0 situation, the proposed trajectory (a unique and continuous curve) always exists from arbitrary starting point $(a, b) \in R^n_+ \times R^n$ without any assumption other than the continuity of the mapping f, and the boundedness of this trajectory only requires a condition that is strictly weaker than (b) and (c) of Condition 1.2. Particularly, the strictly feasible condition is not assumed in our derivation; (f_3) In the setting of P_* complementarity problems, the boundedness of the trajectory is equivalent to the solvability of the problem, i.e., the trajectory is bounded if and only if the problem has a solution. Moreover, for monotone complementarity problems, the whole trajectory converges to a least 2-norm solution provided that the solution set of CP(f) is nonempty. This is a very desirable property of the homotopy continuation trajectory with which we may design a continuation method (path-following method) to solve a CP even when the solution set is not bounded. For semimonotone functions (in particular, P_0-functions), each accumulation point of the proposed trajectory turns out to be a weak Pareto minimal solution (see Theorem 5.2). The above-mentioned properties essentially distinguish the proposed continuation trajectory from the previous ones in the literature.

Since Tikhonov regularization trajectory (see Isac 1991, Venkateswaran 1993, Facchinei 1998, Facchinei and Kanzow 1999, Ravindran and Gowda 1997, Gowda and Tawhid 1999, Sznajder and Gowda 1998, Tseng 1998, and Facchinei and Pang 1998) can be viewed as an extreme variant of the proposed trajectory, some new properties of the regularization trajectory (see Theorem 5.4) are also revealed as by-products from the discussion of this paper.

The paper is organized as follows: In §2, we list some definitions and basic results that will be utilized later. In §3, we give a new homotopy formulation for CP(f) and prove a useful equivalent version of it. A useful alternative theorem is also shown. In §4, we study the existence and the boundedness of a new continuation trajectory. The limiting behavior of the trajectory is studied in §5. Two examples are given in §6 to show that the central path and other interior-point trajectories studied by Kojima, Megiddo, and Noma (1991) fail to exist if the strictly feasible condition is not satisfied. Based on the proposed trajectory, a framework of a new path-following algorithm for CP(f) is also given in §6. Conclusions are given in the last section.

2. Preliminaries. For reference purposes, we introduce some basic results and definitions. Let Ω be a bounded open set in R^n. The symbols $\overline{\Omega}$ and $\partial\Omega$ denote the closure and boundary of Ω, respectively. Let v be a continuous function from $\overline{\Omega}$ into R^n. For any vector $y \in R^n$ such that $y \notin v(\partial\Omega)$, the degree of v at y with respect to Ω is denoted by $\deg(v, \Omega, y)$. The following result can be found in Lloyd (1978).

LEMMA 2.1 (LLOYD 1978).
(a) *If v is injective on R^n, then, for any $y \in \Omega$, $\deg(v, \Omega, y) = \pm 1$.*
(b) *If $\deg(v, \Omega, y) \neq 0$, then the equation $v(x) = y$ has a solution in Ω.*
(c) *Let g be a continuous function from $\overline{\Omega} \to R^n$. Let*

$$H(x, t) = tg(x) + (1-t)v(x), \quad 0 \le t \le 1.$$

If $y \notin \{H(x, t) : x \in \partial\Omega, t \in [0, 1]\}$, then $\deg(v, \Omega, y) = \deg(g, \Omega, y)$.

The next result is an upper-semicontinuity theorem concerning weakly univalent maps established recently by Ravindran and Gowda (1997). Since each P_0-function must be weakly univalent (Ravindran and Gowda 1997, Gowda and Sznajder 1999), the following result is very helpful for the analysis in this paper.

LEMMA 2.2 (RAVINDRAN AND GOWDA 1997). *Let $g : R^n \to R^n$ be weakly univalent, that is, g is continuous, and there exist one-to-one continuous functions $g_k : R^n \to R^n$ such that $g_k \to g$ uniformly on every bounded subset of R^n. Suppose that $q^* \in R^n$ such that $g^{-1}(q^*)$ is nonempty and compact. Then for any given $\varepsilon > 0$, there exists a scalar $\delta > 0$ such that for any weakly univalent function h and for any q with*

$$\sup_{\overline{\Omega}} \|h(x) - g(x)\| < \delta, \|q - q^*\| < \delta,$$

we have

$$\varnothing \neq h^{-1}(q) \subseteq g^{-1}(q^*) + \varepsilon B,$$

where B denotes the open unit ball in R^n and $\Omega = g^{-1}(q^*) + \varepsilon B$. In particular, $h^{-1}(q)$ and $g^{-1}(q)$ are nonempty and uniformly bounded for q in a neighborhood of q^*.

We now introduce two classes of functions.

DEFINITION 2.1.
(a) A function f is said to be a semimonotone function if for any $x \neq y$ in R^n such that $x - y \geq 0$, there exists some i such that $x_i > y_i$ and $f_i(x) \geq f_i(y)$.

(b) (Kojima, Megiddo, Noma, and Yoshise 1991, Zhao and Isac 2000a) A function $f: R^n \to R^n$ is said to be a P_*-function if there exists a constant $\tau \geq 0$ such that

$$(1+\tau) \sum_{i \in I_+(x,y)} (x_i - y_i)(f_i(x) - f_i(y)) + \sum_{i \in I_-(x,y)} (x_i - y_i)(f_i(x) - f_i(y)) \geq 0,$$

where

$$I_+(x, y) = \{i : (x_i - y_i)(f_i(x) - f_i(y)) > 0\}, I_-(x, y) = \{1, \ldots, n\} \setminus I_+(x, y).$$

We note the following easily verifiable relations:

Monotone functions $\subseteq$ P_*-functions $\subseteq$ P_0-functions $\subseteq$ Semimonotone functions.

For a linear map $f = Mx + q$ where M is $n \times n$ matrix and $q \in R^n$, it is evident that f is a semimonotone function if and only if M is a semimonotone matrix (Cottle et al. 1992), and f is a P_*-function if and only if M is a P_*-matrix defined in Kojima, Megiddo, Noma, and Yoshise (1991). Väliaho (1996) showed that the class of P_*-matrices coincides with that of sufficient matrices introduced in Cotlle et al. (1989). A new equivalent definition for P_*-function is given in Zhao and Isac (2000a).

3. A new homotopy formulation and an alternative theorem. Given a scalar $\mu > 0$, we define $\mathscr{F}_\mu : R^{2n} \to R^{2n}$ by

$$\mathscr{F}_\mu(x, y) := \begin{pmatrix} Xy \\ y - (f(x) + \mu x) \end{pmatrix},$$

where $X = \mathrm{diag}(x)$. This map can be viewed as a perturbed version of $\mathscr{F}(x, y)$ defined by (1) with the parameter $\mu > 0$. Clearly, $(x, y) \in R^{2n}_+$ such that $\mathscr{F}_\mu(x, y) = 0$ if and only if

$$x \geq 0, \quad f(x) + \mu x \geq 0, \quad x^T(f(x) + \mu x) = 0. \tag{4}$$

Under suitable conditions, the solutions of the above system approach to a solution of CP(f) as $\mu \to 0$. This is the basic idea of the well-known Tikhonov regularization methods for CP(f). See, for example, Isac (1991), Venkateswaran (1993), Facchinei (1998), Facchinei and Kanzow (1999), Ravindran and Gowda (1997), Gowda and Tawhid (1999), Sznajder and Gowda (1998), Qi (2000), Tseng (1998), and Facchinei and Pang (1998).

Let $(a, b) \in R^n_+ \times R^n$ and let $G : R^{2n} \to R^{2n}$ be given by

$$G(x, y) := \begin{pmatrix} Xy - a \\ y - b \end{pmatrix}.$$

Define the convex homotopy $\mathscr{H} : R^n \times R^n \times [0, 1] \times [0, \infty) \to R^{2n}$ as follows:

$$\begin{aligned} \mathscr{H}(x, y, \theta, \mu) &= \theta G(x, y) + (1 - \theta)\mathscr{F}_\mu(x, y) \\ &= \begin{pmatrix} Xy - \theta a \\ y - (1 - \theta)(f(x) + \mu x) - \theta b \end{pmatrix}. \end{aligned} \tag{5}$$

We note that the above homotopy formulation has two important extreme cases. If $\theta = 0$ and $\mu > 0$, then the Map (5) reduces to $\mathscr{F}_\mu(x, y)$ which is used in Tikhonov regularization methods. If $\mu = 0$ and $\theta > 0$, (5) reduces to

$$H(x, y, \theta) = \mathscr{H}(x, y, \theta, 0) = \begin{pmatrix} Xy - \theta a \\ y - (1 - \theta) f(x) - \theta b \end{pmatrix}, \tag{6}$$

which is investigated by Kojima, Megiddo, and Mizuno (1993), who proved that the above map is equivalent to (2) under a suitable one-to-one transformation such as $t = \theta/(1-\theta)$. However, the version of (6) is mathematically easier to be handled than (2). Let $(x^0, y^0) \in R^n_{++}$. Setting $b = y^0$ and $a = X^0 y^0$, it is easy to verify that (6) coincides with the homotopy formulation studied by Zhang and Zhang (1997). Therefore, Zhang and Zhang's homotopy function is actually a special case of that of Kojima, Megiddo, and Mizuno (1993).

When $\theta > 0$ and $\mu > 0$, the Homotopy (5) is essentially different from previous ones in the literature. We may treat the two parameters μ and θ independently in general cases. But for simplicity and convenience for designing algorithms, we use only one parameter by setting $\mu = \phi(\theta)$, where ϕ satisfies the following properties:

- $\phi : R^1_+ \to R^1_+$ *is a continuous and strictly increasing function satisfying* $\phi(0) = 0$, $\phi(\theta) > 0$ *if* $\theta > 0$, *and* $\phi(\theta) \to 0$ *if and only if* $\theta \to 0$.

An example of ϕ satisfying the above properties is $\phi(\theta) = \theta^p$, where $p \in (0, \infty)$ is a fixed positive number. Then (5) can be written as

$$\overline{\mathscr{H}}(x, y, \theta) := \mathscr{H}(x, y, \theta, \phi(\theta)) = \begin{pmatrix} Xy - \theta a \\ y - (1-\theta)(f(x) + \phi(\theta)x) - \theta b \end{pmatrix}. \tag{7}$$

Basing on the above homotopy, we consider the following system:

$$\overline{\mathscr{H}}(x, y, \theta) = 0, \quad (x, y) > 0, \tag{8}$$

where the parameter $\theta \in (0, 1)$. For given $\theta \in (0, 1)$, we denote $(x(\theta), y(\theta))$ by a solution of the above system, and consider the following set

$$T = \{(x(\theta), y(\theta)) > 0 : \ \overline{\mathscr{H}}(x(\theta), y(\theta), \theta) = 0, \ \theta \in (0, 1)\}.$$

If $(x(\theta), y(\theta))$ is unique for each $\theta \in (0, 1)$ and continuous in θ, the above set is called the homotopy continuation trajectory generated by the System (8). We say that the trajectory is bounded if for any scalar $\delta \in (0, 1)$, the set

$$\{(x(\theta), y(\theta)) > 0 : \ \overline{\mathscr{H}}(x(\theta), y(\theta), \theta) = 0, \ \theta \in (0, \delta]\}$$

is bounded. For any subsequence denoted by $\{(x(\theta_k), y(\theta_k))\} \subseteq T$ such that $(x(\theta_k), y(\theta_k)) \to (x^*, y^*)$ as $\theta_k \to 0$, the cluster point (x^*, y^*), where $y^* = f(x^*)$, must be a solution to CP(f) (since f is continuous). Thus by tracing such a trajectory, if it exists, we may find a solution to the problem.

In this paper, we study the existence, boundedness, and limiting behavior of the above trajectory. The following result, which gives an equivalent description of the System (8), is useful in the analysis throughout the remainder of the paper.

LEMMA 3.1. *Given a pair* $(a, b) \in R^n_{++} \times R^n$. *For each* $\theta > 0$, *the vector* (x, y), *where* $y = (1-\theta)(f(x) + \phi(\theta)x) + \theta b$, *is a solution to the System* (8) *if and only if* x *is a solution to the following equation*

$$\mathscr{E}(x, \theta) = 0 \tag{9}$$

where $\mathscr{E}(x, \theta) = (\mathscr{E}_1(x, \theta), \ldots, \mathscr{E}_n(x, \theta))^T$, *and for each* i,

$$\begin{aligned} \mathscr{E}_i(x, \theta) = {} & x_i + (1-\theta)(f_i(x) + \phi(\theta)x_i) + \theta b_i \\ & - \sqrt{x_i^2 + [(1-\theta)(f_i(x) + \phi(\theta)x_i) + \theta b_i]^2 + 2\theta a_i}. \end{aligned} \tag{10}$$

PROOF. Since $a \in R^n_{++}$ and $\theta > 0$, it is easy to verify that x is a solution to (9) if and only if it satisfies the following system:

$$x_i > 0, \tag{11}$$

$$(1-\theta)(f_i(x) + \phi(\theta)x_i) + \theta b_i > 0, \tag{12}$$

$$x_i[(1-\theta)(f_i(x) + \phi(\theta)x_i) + \theta b_i] = \theta a_i, \tag{13}$$

where $i = 1, \ldots, n$. Denote $y = (1-\theta)(f(x) + \phi(\theta)x) + \theta b$. Then it is easy to see that (x, y) is a solution to the System (8) if and only if x is a solution to the System (11)–(13). (This fact will be used again in later sections). $\square$

In what follows, we establish a basic result used to show the existence of the proposed continuation trajectory. The following concept is useful.

DEFINITION 3.1. Given $(a, b) \in R^n_{++} \times R^n$. Let $\theta \in (0, 1)$ be a given scalar. A sequence $\{x^k\} \subseteq R^n_{++}$ with $\|x^k\| \to \infty$ is said to be a $(\theta, \phi, \overline{\mathscr{H}})$-exceptional sequence for f if for each x^k there exists a scalar $\beta_k \in (0, 1)$ such that

$$f_i(x^k) = \left[\frac{1}{2(1-\theta)}\left(\beta_k - \frac{1}{\beta_k}\right) - \phi(\theta)\right]x_i^k + \frac{\theta\beta_k a_i}{(1-\theta)x_i^k} - \frac{\theta b_i}{1-\theta} \tag{14}$$

for all $i = 1, \ldots, n$.

The above concept tailored to our needs here can be viewed as a modified form of those introduced to investigate the solvability of complementarity problems (Isac et al. 1997, Zhao and Isac 2000a) and variational inequalities (Zhao 1999, Zhao et al. 1999). For given θ, the concept of $(\theta, \phi, \overline{\mathscr{H}})$-exceptional sequence is closely related to the existence of a solution of the System (8) as the following result shows.

THEOREM 3.1. *Let $(a, b) \in R^n_{++} \times R^n$ and $f : R^n \to R^n$ be a continuous function. Then for each $\theta \in (0, 1)$, there exists either a solution to the System* (8) *or a $(\theta, \phi, \overline{\mathscr{H}})$-exceptional sequence for f.*

PROOF. Let θ be an arbitrary number in $(0, 1)$. Assume that there exists no solution to the System (8). We now show that there exists a $(\theta, \phi, \overline{\mathscr{H}})$-exceptional sequence for f. Consider the homotopy between the identity mapping and $\mathscr{E}(x, \theta)$ defined by (10), i.e.,

$$\begin{aligned} H(x, l) &= lx + (1-l)\mathscr{E}(x, \theta) \\ &= x + (1-l)[(1-\theta)(f(x) + \phi(\theta)x) + \theta b] \\ &\quad -(1-l)\sqrt{x^2 + [(1-\theta)(f(x) + \phi(\theta)x) + \theta b]^2 + 2\theta a}, \end{aligned}$$

where $0 \le \lambda \le 1$ and all algebraic operations are performed componentwise. Define the set

$$\mathscr{S}_\theta = \{x \in R^n : H(x, l) = 0 \text{ for some } l \in [0, 1]\}.$$

We now show that the set is unbounded. Assume the contrary, that is, that the set is bounded. In this case, there is a bounded open set $\Omega \subset R^n$ such that $\{0\} \cup \mathscr{S}_\theta \subset \Omega$. Thus $0 \notin \partial\Omega$ and $\mathscr{S}_\theta \cap \partial\Omega = \varnothing$. By the definition of $\mathscr{S}_\theta$, for any $\bar{x} \in \partial\Omega$, we have $H(\bar{x}, l) \neq 0$ for any $l \in [0, 1]$. By (c) of Lemma 2.1, we deduce that $\deg(I, \Omega, 0) = \deg(\mathscr{E}, \Omega, 0)$, where I denotes the identity mapping which is one-to-one in R^n. Thus by (a) and (b) of Lemma 2.1, we deduce that $\mathscr{E}(x, \theta) = 0$ has at least a solution, and hence by Lemma 3.1 the System (8) has a solution. This contradicts the assumption at the beginning of the proof. Thus, the set $\mathscr{S}_\theta$ is indeed unbounded, and thus there exists a sequence $\{x^k\} \subseteq \mathscr{S}_\theta$ with $\|x^k\| \to \infty$. Without loss of generality, we may assume $\|x^k\| > 0$ for all k. We now show that $\{x^k\}$ is a

$(\theta, \phi, \overline{\mathscr{H}})$-exceptional sequence for f. For each x^k, it follows from $x^k \in \mathscr{S}_\theta$ that there exists a number $l^k \in [0, 1]$ such that $H(x^k, l^k) = 0$, i.e.,

$$
\begin{aligned}
&x^k + (1-l^k)[(1-\theta)(f(x^k) + \phi(\theta)x^k) + \theta b] \\
&\quad = (1-l^k)\sqrt{(x^k)^2 + [(1-\theta)(f(x^k) + \phi(\theta)x^k) + \theta b]^2 + 2\theta a}. \qquad (15)
\end{aligned}
$$

Clearly, $l^k \neq 1$ since $\|x^k\| > 0$. On the other hand, if $l^k = 0$, it follows from (15) that x^k is a solution to the equation $\mathscr{E}(x, \theta) = 0$. By Lemma 3.1, the pair (x^k, y^k), where $y^k = (1-\theta)(f(x^k) + \phi(\theta)x^k) + \theta b$, is a solution to the System (8). This contradicts again the assumption at the beginning of the proof. Thus, in the rest of the proof, we only need to consider the situation of $l^k \in (0, 1)$. We show that $\{x^k\}$ is a $(\theta, \phi, \overline{\mathscr{H}})$-exceptional sequence for f. We first show that $\{x^k\} \subset R^n_{++}$. Indeed, noting that $\sqrt{v^2} - v \in R^n_+$ for all $v \in R^n$ and $2\theta a \in R^n_{++}$, we have from (15) that

$$
\begin{aligned}
x^k &= (1-l^k)\sqrt{(x^k)^2 + [(1-\theta)(f(x^k) + \phi(\theta)x^k) + \theta b]^2 + 2\theta a} \\
&\quad -(1-l^k)\left[(1-\theta)(f(x^k) + \phi(\theta)x^k) + \theta b\right] \\
&> (1-l^k)[\sqrt{[(1-\theta)(f(x^k) + \phi(\theta)x^k) + \theta b]^2} \\
&\quad -((1-\theta)(f(x^k) + \phi(\theta)x^k) + \theta b)] \\
&\geq 0.
\end{aligned}
$$

Thus $\{x^k\} \subset R^n_{++}$. It is sufficient to verify (14). Squaring both sides of (15), we have

$$
(1-(1-l^k)^2)(x^k)^2 + 2(1-l^k)X^k[(1-\theta)(f(x^k) + \phi(\theta)x^k) + \theta b] = 2(1-l^k)^2\theta a,
$$

where $X^k = \operatorname{diag}(x^k)$. Multiplying both sides of the above by $[2(1-\lambda^k)X^k]^{-1}$, we have

$$
(1-\theta)(f(x^k) + \phi(\theta)x^k) + \theta b = \frac{1}{2}\left[\frac{1}{1-l^k} - (1-l^k)\right]x^k + (1-l^k)\theta(X^k)^{-1}a.
$$

Let $\beta_k = 1 - l^k$. The above equation can be written as

$$
f(x^k) = \left[\frac{1}{2(1-\theta)}\left(\frac{1}{\beta_k} - \beta_k\right) - \phi(\theta)\right]x^k + \frac{\beta_k\theta(X^k)^{-1}a}{1-\theta} - \frac{\theta b}{1-\theta}.
$$

By definition, $\{x^k\}$ is a $(\theta, \phi, \overline{\mathscr{H}})$-exceptional sequence for f. $\square$

4. Existence and boundedness of the trajectory. In this section, we show the existence and the boundedness of the new trajectory under a very weak condition. The strictly feasible condition is not required in our results. For any vectors $v, v' \in R^n$ with $v' \leq v$, we denote by $[v', v]$ the rectangular set $[v_1', v_1] \times \cdots \times [v_n', v_n]$. The following is our assumption.

CONDITION 4.1. For any $(a, b) \in R^n_{++} \times R^n$, there exists a scalar $0 < \gamma < 1$ such that the set

$$
\bigcup_{\mu \in (0, \gamma]} \mathscr{F}_\mu^{-1}(D_\mu)
$$

is bounded, where $\mathscr{F}_\mu^{-1}(D_\mu) = \{(x, y) \in R^{2n}_+ : \mathscr{F}_\mu(x, y) \in D_\mu\}$ and

$$
D_\mu = \left\{(u, v) \in R^{2n} : u \in \left[0, \frac{\mu}{1-\mu}a\right], v \in \left[\frac{\mu}{1-\mu}[b]_-, \frac{\mu}{1-\mu}[b]_+\right]\right\}.
$$

Clearly, D_μ is a rectangular set in $R^n_+ \times R^n$. We now show a general result.

THEOREM 4.1. *Let (a, b) be an arbitrary point in $R^n_{++} \times R^n$ and $f : R^n \to R^n$ be a continuous semimonotone function. Then for each $\theta \in (0, 1)$, the System* (8) *has a solution. Denote the solution set by*

$$C_\theta = \{(x, y) > 0 : \overline{\mathscr{H}}(x, y, \theta) = 0\}. \tag{16}$$

Moreover, if Condition 4.1 *holds, then, for any $\delta \in (0, 1)$, the set*

$$\{(x(\theta), y(\theta)) \in C_\theta : \theta \in (0, \delta]\} \tag{17}$$

is bounded, where C_θ is given by (16).

To prove the above result, we use the following lemma. Its proof, similar to the one of Lemma 1 in Gowda and Tawhid (1999), is very easy and omitted here.

LEMMA 4.1. *Let $f : R^n \to R^n$ be a continuous semimonotone function. Then for any sequence $\{u^k\} \subset R^n_+$ satisfying $\|u^k\| \to \infty$, there exists some i_0 and a subsequence $\{u^{k_j}\}$ such that $u^{k_j}_{i_0} \to \infty$ and $\{f_{i_0}(u^{k_j})\}$ is bounded from below.*

PROOF OF THEOREM 4.1. To show the former part of the result, by Theorem 3.1, it is sufficient to show that the function f has no $(\theta, \phi, \overline{\mathscr{H}})$-exceptional sequence for each $\theta \in (0, 1)$. Assume the contrary, that is, there exists a scalar $\theta \in (0, 1)$ such that $\{x^k\}$ is a $(\theta, \phi, \overline{\mathscr{H}})$-exceptional sequence for f. By definition, $\|x^k\| \to \infty$, $\{x^k\} \subseteq R^n_{++}$ and (14) is satisfied. Since f is semimonotone, by Lemma 4.1, there exist some subsequence $\{x^{k_j}\}$ and index m such that $x^{k_j}_m \to \infty$ and $\{f_m(x^{k_j})\}$ is bounded from below. However, by (14) we have

$$\begin{aligned} f_m(x^{k_j}) &= \left[\frac{1}{2(1-\theta)}\left(\beta_{k_j} - \frac{1}{\beta_{k_j}}\right) - \phi(\theta)\right] x^{k_j}_m + \frac{\theta \beta_{k_j} a_i}{(1-\theta) x^{k_j}_m} - \frac{\theta b_i}{1-\theta} \\ &\le -\phi(\theta) x^{k_j}_m + \frac{\theta a_i}{(1-\theta) x^{k_j}_m} - \frac{\theta b_i}{1-\theta} \to -\infty. \end{aligned} \tag{18}$$

This is a contradiction since the left-hand side of the above is bounded from below. Thus f has no $(\theta, \phi, \overline{\mathscr{H}})$-exceptional sequence for any $\theta \in (0, 1)$, and hence it follows from Theorem 3.1 that the System (8) has a solution for each $\theta \in (0, 1)$.

We now show the boundedness of the Set (17). Let $\delta \in (0, 1)$ be a fixed scalar. Assume that there exists a sequence $\{\theta_k\} \subseteq (0, \delta]$ such that $\{(x(\theta_k), y(\theta_k))\}$ is an unbounded sequence contained in the Set (17). We derive a contradiction. Indeed, in this case, $\{x(\theta_k)\}$ must be unbounded. By Lemma 4.1, there exist an index p and a subsequence, denoted also by $\{x(\theta_k)\}$, such that $x_p(\theta_k) \to \infty$ and $f_p(x(\theta_k))$ is bounded from below. Since $(x(\theta_k), y(\theta_k)) \in C_{\theta_k}$, we see from the proof of Lemma 3.1 that $x(\theta_k)$ must satisfy (11)–(13). By (13), we have

$$f_p(x(\theta_k)) = -\phi(\theta_k) x_p(\theta_k) + \frac{\theta_k a_p}{(1-\theta_k) x_p(\theta_k)} - \frac{\theta_k b_p}{1-\theta_k}.$$

Since $x_p(\theta_k) \to \infty$ and the left-hand side of the above is bounded from below, we deduce from the above that $\phi(\theta_k) \to 0$. Since

$$y(\theta_k) = (1-\theta_k)(f(x(\theta_k)) + \phi(\theta_k) x(\theta_k)) + \theta_k b > 0,$$

we have

$$\bar{y}_k := f(x(\theta_k)) + \phi(\theta_k) x(\theta_k) + \frac{\theta_k b}{1-\theta_k} = \frac{1}{1-\theta_k} y(\theta_k) > 0.$$

By using (13) again, we have

$$x_i(\theta_k)(\bar{y}_k)_i = \frac{x_i(\theta_k)y_i(\theta_k)}{1-\theta_k} = \frac{\theta_k a_i}{1-\theta_k} \quad \text{for all } i = 1, \ldots, n.$$

From the above two relations, we have

$$\mathcal{F}_{\mu_k}(x(\theta_k), \bar{y}_k) = \begin{pmatrix} X(\theta_k)\bar{y}_k \\ \bar{y}_k - (f(x(\theta_k)) + \mu_k x(\theta_k)) \end{pmatrix} = \frac{\theta_k}{1-\theta_k}\begin{pmatrix} a \\ b \end{pmatrix},$$

where $\mu_k = \phi(\theta_k)$ and $X(\theta_k) = \mathrm{diag}(x(\theta_k))$. Let $0 < \gamma < 1$ be given as in Condition 4.1. Since $\phi(\theta_k) \to 0$ implies $\theta_k \to 0$, there exists a k_0 such that $\theta_k < \gamma$ for all $k \geq k_0$ and

$$\mathcal{F}_{\mu_k}(x(\theta_k, \mu_k), \bar{y}_k) = \frac{\theta_k}{1-\theta_k}\begin{pmatrix} a \\ b \end{pmatrix} \in D_{\mu_k}, \text{ for all } k > k_0$$

where

$$D_{\mu_k} = \left\{(x, y) \in R^{2n} : x \in \left[0, \frac{\mu_k}{1-\mu_k} a\right], y \in \left[\frac{\mu_k}{1-\mu_k}[b]_-, \frac{\mu_k}{1-\mu_k}[b]_+\right]\right\}.$$

Therefore, $(x(\theta_k), \bar{y}_k) \in \mathcal{F}_{\mu_k}^{-1}(D_{\mu_k})$ for all $k \geq k_0$, thus

$$\{(x(\theta_k), \bar{y}_k)\}_{k \geq k_0} \subseteq \bigcup_{\mu \in (0, \gamma]} \mathcal{F}_{\mu}^{-1}(D_{\mu}),$$

which is bounded according to Condition 4.1. This contradicts the unboundedness of $\{x(\theta_k)\}$. The proof is complete. □

The following condition is slightly stronger than Condition 4.1.

CONDITION 4.2. There exists a constant $\gamma > 0$ such that the following set,

$$\bigcup_{\mu \in (0, \gamma]} \mathcal{F}_{\mu}^{-1}(D_{\gamma}),$$

is bounded, where $D_\gamma := [0, \gamma e] \times [-\gamma e, \gamma e]$, and

$$\mathcal{F}_{\mu}^{-1}(D_\gamma) = \{(x, y) \in R^{2n}_+ : \mathcal{F}_\mu(x, y) \in D_\gamma\}.$$

COROLLARY 4.1. *Let (a, b) be an arbitrary point in $R^n_{++} \times R^n$ and f be a continuous semimonotone function from R^n into itself. If Condition* 4.2 *is satisfied*, *then for any constant* $\delta \in (0, 1)$ *the set*

$$\{(x(\theta), y(\theta)) \in C_\theta : \ \theta \in (0, \delta]\}$$

is bounded, *where C_θ is given by* (16).

Conditions 4.1 and 4.2 are motivated by the following observation.

PROPOSITION 4.1. *Let f be a continuous semimonotone function from R^n into itself. Then for each fixed scalar $\mu > 0$, the set $\mathcal{F}_\mu^{-1}(D) = \{(x, y) \in R^{2n}_+ : \mathcal{F}_\mu(x, y) \in D\}$ is bounded for any compact set D in $R^n_+ \times R^n$.*

PROOF. Let $\mu > 0$ be a fixed number. We show this result by contradiction. Assume that there exists a compact set D in $R^n_+ \times R^n$ such that $\mathcal{F}_\mu^{-1}(D)$ is unbounded. Then there is a sequence $\{(x^k, y^k)\}$ contained in $\mathcal{F}_\mu^{-1}(D)$ such that $\|(x^k, y^k)\| \to \infty$. Notice that

$$\mathcal{F}_\mu(x^k, y^k) = \begin{pmatrix} X^k y^k \\ y^k - (f(x^k) + \mu x^k) \end{pmatrix} \in D.$$

Since D is compact set in $R^n_+ \times R^n$, there is a bounded sequences $\{(d^k, w^k)\} \subseteq D$ such that

$$X^k y^k = d^k, \quad y^k = f(x^k) + \mu x^k + w^k. \tag{19}$$

If x^k is bounded, then from the above, so is y^k. Thus, by our assumption, $\{x^k\}$ must be unbounded. We may assume that $\|x^k\| \to \infty$. Passing through a subsequence, we may assume that there exists an index set I such that $x_i^k \to \infty$ for $i \in I$ and $\{x_i^k\}$ is bounded for $i \notin I$. From the first relation of (19), we deduce that $y_i^k \to 0$ for all $i \in I$ since $x_i^k \to \infty$ and d_i^k is bounded. Thus, from the second expression of (19), we deduce that $f_i(x^k) = y_i^k - \mu x_i^k - w^k \to -\infty$ for all $i \in I$. This contradicts Lemma 4.1, which asserts that there is an index $p \in I$ such that $x_p^k \to \infty$ and $f_p(x^k)$ is bounded from below. $\square$

Condition 4.1 actually states that the union of all the bounded sets $\mathcal{F}_\mu^{-1}(D_\mu)$ is also bounded. It is equivalent to saying that $\mathcal{F}_\mu^{-1}(D_\mu)$ is uniformly bounded when the positive parameter μ is sufficiently small. Later, we will prove that for P_0 complementarity problems, Condition 4.1 is strictly weaker than Condition 1.2. In fact, Condition 4.1 may hold even if the problem has no strictly feasible point (in this case, the solution set of CP(f) is unbounded).

While the above results show that for any given $(a, b) \in R^n_{++} \times R^n$ a solution $(x(\theta), y(\theta))$ to the System (8) exists for each $\theta \in (0, 1)$, it is not clear how to achieve the uniqueness and the continuity of $(x(\theta), y(\theta))$ for semimonotone complementarity problems. Nevertheless, at a risk of losing the arbitrariness of the starting point (a, b) in $R^n_{++} \times R^n$, it is possible to obtain the uniqueness and the continuity by using the parameterized Sard Theorem, see, e.g., Allgower and Georg (1990), Kojima et al. (1993), and Zhang and Zhang (1997). But such a result cannot exclude a zero measure set (in Lebesgue sense) from which the proposed trajectory fails to exist. For P_0-functions, however, it is not difficult to achieve the uniqueness and the continuity of the proposed trajectory as the next result shows.

THEOREM 4.2. *Let f be a continuous P_0-function from R^n into itself.*

(a) *For each $\theta \in (0, 1)$, the System (8) has a unique solution denoted by $(x(\theta), y(\theta))$.*

(b) *$(x(\theta), y(\theta))$ is continuous in θ on $(0, 1)$.*

(c) *If Condition 4.1 (in particular, Condition 4.2) is satisfied, the set $\{(x(\theta), y(\theta)) : \theta \in (0, \delta]\}$ is bounded for any $\delta \in (0, 1)$.*

(d) *If f is continuously differentiable, then $(x(\theta), y(\theta))$ is also continuously differentiable in θ.*

PROOF. Since f is a P_0-function, the Tikhonov regularization function, i.e., $f(x) + \phi(\theta)x$, where $\theta \in (0, 1)$, is a P-function in x. Therefore, $g(x) := (1-\theta)(f(x) + \phi(\theta)x) + \theta b$ is also a P-function in x, where $\theta \in (0, 1)$. By the same proof of Ravindran and Gowda (1997), we can show that

$$\mathcal{E}(x, \theta) = x + g(x) - \sqrt{x^2 + g(x)^2 + c}$$

is a P-function in x for any given $c \in R^n_{++}$. Since P-function is injective, the equation $\mathcal{E}(x, \theta) = 0$ has at most one solution for each fixed $\theta \in (0, 1)$, and hence, by Lemma 3.1, the System (8) has at most one solution. Using this fact and noting that each P-function is semimonotone, we deduce from (a) of Theorem 4.1 that the System (8) has a unique solution. Part (a) follows. By using this fact and noting that each P_0-function (in particular, P-function) is a weakly univalent function (Gowda and Tawhid 1999, Gowda and Sznajder 1999, Ravindran and Gowda 1997), from Lemma 2.2, we conclude that the solution of (8) is continuous in θ on (0,1). This proves Part (b). Part (c) immediately follows from Theorem 4.1.

We now prove Part (d). Since $f(x)$ is a P_0-function, the Jacobian matrix $f'(x)$ is a P_0-matrix (Moré and Rheinholdt 1973). Hence, the matrix $(1-\theta)(f'(x)+\phi(\theta)I)$ is a P-matrix for each fixed number $\theta \in (0,1)$. Therefore, the matrix

$$\begin{bmatrix} (1-\theta)(f'(x)+\phi(\theta)I) & -I \\ Y & X \end{bmatrix}$$

is a nonsingular matrix (Kojima, Megiddo, and Noma 1991, Kojima, Megiddo, Noma, and Yoshise 1991) for any $(x, y) > 0$. Noting that the above matrix coincides with the Jacobian matrix of the map $\overline{\mathscr{H}}(x, y, \theta)$ with respect to (x, y). Thus, by the implicit function theorem, there exists a δ-neighborhood of θ such that there exists a unique and continuously differentiable curve $(x(t), y(t))$ satisfying

$$\overline{\mathscr{H}}(x(t), y(t), t) = 0 \text{ for all } t \text{ with } \|t-\theta\| < \delta,$$
$$(x(t), y(t))|_{t=\theta} = (x(\theta), y(\theta)).$$

In particular, $(x(\theta), y(\theta))$ is continuously differentiable at θ. The proof is complete. □

It is known that the structure property of the solution set of CP(f) has a close relationship to the existence of some interior-point trajectories. For instance, Chen et al. (1997) and Gowda and Tawhid (1999) have proved the existence of a short central path when the solution set of a P_0 complementarity problem is nonempty and bounded. In fact, a long central path may not exist in this case. Chen and Ye (1998) gave an example of a 2×2 P_0 linear complementarity problem with a bounded solution set to show that there is no long central path. In contrast, the homotopy continuation trajectory proposed in this paper is always a long trajectory provided that f is a continuous P_0-function as shown in Theorem 4.1. The next result shows that any slice of the trajectory (subtrajectory) is always bounded when $\mathrm{SOL}_{cp}(f)$ is nonempty and bounded. The following lemma is employed to prove the result.

LEMMA 4.2. *Suppose that S is an arbitrary compact set in R^n. Let $\mathscr{E}(x,\theta)$ be given by* (10).

(a) *Given $(a, b) \in R^n_{++} \times R^n$. For any $\delta > 0$, there must exist a scalar $\delta' \in (0, 1)$ such that*

$$\sup_{x\in S} \|\mathscr{E}(x,\theta) - \mathscr{E}(x,0)\| < \delta \quad \textit{for all } \theta \in (0, \delta'].$$

(b) *Let $h^* : R^n \times R^1_{++} \times R^n_+ \times R^n \to R^n$ be defined by*

$$h^*(x, \mu, w, v) = x + (f(x)+\mu x + v) - \sqrt{x^2 + (f(x)+\mu x+v)^2 + 2w}.$$

For any $\delta > 0$, there exists a scalar $\gamma \in (0,1)$ such that

$$\sup_{x\in S} \|h^*(x,\mu,w,v) - \mathscr{E}(x,0)\| < \delta$$

for all $\mu \in (0,\gamma]$ and $(w, v) \in [0, \gamma e] \times [-\gamma e, \gamma e]$.

PROOF. Denote by

$$T_i = \sqrt{x_i^2 + f_i^2(x)} + \sqrt{x_i^2 + [(1-\theta)(f_i(x)+\phi(\theta)x_i)+\theta b_i]^2 + 2\theta a_i}$$

and

$$W_i = [(1-\theta)(f_i(x)+\phi(\theta)x_i)+\theta b_i]^2 + 2\theta a_i - f_i^2(x)$$

where $i = 1, \dots, n$. Notice that the inequality $\sqrt{t_1^2+t_2^2} + \sqrt{t_1^2+t_2^2+t_3} \ge \sqrt{|t_3|}$ holds for any scalar t_1, t_2, t_3 satisfying $t_1^2+t_2^2+t_3 \ge 0$. We have

$$T_i = \sqrt{x_i^2 + f_i^2(x)} + \sqrt{x_i^2 + f_i^2(x) + W_i} \ge \sqrt{|W_i|}.$$

Since S is a compact set in R^n and $\phi(\theta) \to 0$ as $\theta \to 0$, there exist two positive constants $C_1, C_2 > 0$ such that for any sufficiently small θ the following holds:

$$|W_i| \le \phi(\theta)C_1 + \theta C_2.$$

Thus, for any sufficiently small θ, we have for each i that

$$\begin{aligned}
&|\mathscr{E}_i(x,\theta) - \mathscr{E}_i(x,0)| \\
&\quad = \Big| x_i + (1-\theta)(f_i(x) + \phi(\theta)x_i) + \theta b_i - (x_i + f_i(x)) + \sqrt{x_i^2 + f_i^2(x)} \\
&\qquad - \sqrt{x_i^2 + [(1-\theta)(f_i(x) + \phi(\theta)x_i) + \theta b_i]^2 + 2\theta a_i} \Big| \\
&\quad \le |(1-\theta)\phi(\theta)x_i + \theta(b_i - f_i(x))| \\
&\qquad + |f_i^2(x) - [(1-\theta)(f_i(x) + \phi(\theta)x_i) + \theta b_i)^2 - 2\theta a_i|/T_i \\
&\quad \le |(1-\theta)\phi(\theta)x_i + \theta(b_i - f_i(x))| + \sqrt{|W_i|} \\
&\quad \le |(1-\theta)\phi(\theta)x_i + \theta(b_i - f_i(x))| + \sqrt{\phi(\theta)}\sqrt{C_1} + \sqrt{\theta}\sqrt{C_2}.
\end{aligned}$$

By the compactness of S, the result (a) holds. Result (b) can be proved in a similar way. □

THEOREM 4.3. *Let f be a continuous P_0-function from R^n into itself. Denote by $\{(x(\theta), y(\theta)) :\ \theta \in (0,1)\}$ the trajectory generated by the (unique) solution of the System (8) as θ varies. If the* CP(f) *has a nonempty and bounded solution set, then for any scalar $\delta \in (0,1)$ the slice $\mathscr{T} := \{(x(\theta), y(\theta)) :\ \theta \in (0,\delta]\}$ is bounded.*

PROOF. The existence of the (unique) trajectory follows from (a) and (b) of Theorem 4.2. It suffices to prove that any short part of the trajectory is bounded under the assumption of the theorem. Assume that $\mathrm{SOL}_{cp}(f)$ is nonempty and bounded. Notice that

$$E(x) := \mathscr{E}(x,0) = x + f(x) - \sqrt{x^2 + f(x)^2}$$

is the well-known Fischer-Burmeister function. Thus $E(x) = 0$ if and only if $x \in \mathrm{SOL}_{cp}(f)$, i.e., $\mathrm{SOL}_{cp}(f) = E^{-1}(0)$. Hence, $E^{-1}(0)$ is nonempty and bounded by assumption. Since f is a P_0-function, $E(x)$ is also a P_0-function (see Ravindran and Gowda 1997, Gowda and Tawhid 1999), and thus is weakly univalent (Gowda and Tawhid 1999, Gowda and Sznajder 1999, Ravindran and Gowda 1997). It follows from Lemma 2.2 that for any fixed $\varepsilon > 0$ there exists a $\delta > 0$ such that for any weakly univalent function $h : R^n \to R^n$ with

$$\sup_{x \in \overline{\Omega}} \|h(x) - E(x)\| < \delta, \tag{20}$$

we have

$$\varnothing \ne h^{-1}(0) \subseteq E^{-1}(0) + \varepsilon B, \tag{21}$$

where $\overline{\Omega} = \mathrm{cl}(E^{-1}(0) + \varepsilon B)$ is a compact set. By (a) of Lemma 4.2, there exists a number $0 < \delta' < 1$ such that for any $\theta \in (0, \delta']$ the map $h(x) := \mathscr{E}(x,\theta)$, a P_0-function in x, satisfies the Relation (20). Thus, it follows from (21) that the set

$$\{x \in R^n :\ \mathscr{E}(x,\theta) = 0,\ \theta \in (0,\delta']\}$$

is contained in the bounded set $E^{-1}(0)+\varepsilon B$. By Lemma 3.1, the subtrajectory, i.e.,

$$\mathcal{T}_1 := \{(x(\theta), y(\theta)) : \theta \in (0, \delta']\},$$

is bounded.

Let δ be an arbitrary scalar with $\delta \in (0, 1)$. Without loss of generality, we assume $\delta' < \delta$. Our goal is to show the boundedness of the subtrajectory $\mathcal{T} = \{(x(\theta), y(\theta)) : \theta \in (0, \delta]\}$. Notice that $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ where

$$\mathcal{T}_2 := \{(x(\theta), y(\theta)) : \theta \in [\delta', \delta]\}.$$

It suffices to show the boundedness of the set $\mathcal{T}_2$. We show this by contradiction. Assume that $\mathcal{T}_2$ is unbounded, i.e., there exists a subsequence $\{(x(\theta_k), y(\theta_k))\} \subseteq \mathcal{T}_2$ such that $\|(x(\theta_k), y(\theta_k))\| \to \infty$ which implies that $\|x(\theta_k)\| \to \infty$. Thus, by Lemma 4.1, there exists a subsequence, denoted also by $\{x(\theta_k)\}$, such that there is an index p such that $x_p(\theta_k) \to \infty$ and $f_p(\theta_k)$ is bounded from below. Since $\theta_k \in [\delta', \delta]$ for all k, by property of $\phi(\theta)$, there must exist a scalar $\alpha > 0$ such that $\phi(\theta_k) \geq \alpha$ for all k. It follows from (13) that

$$\begin{aligned} f_p(\theta_k) &= \frac{\theta_k a_p}{(1-\theta_k)x_p(\theta_k)} - \frac{\theta_k b_p}{1-\theta_k} - \phi(\theta_k)x_p(\theta_k) \\ &\leq \frac{\delta a_p}{(1-\delta)x_p(\theta_k)} - \frac{\delta'}{1-\delta'} b_p - \alpha x_p(\theta_k) \to -\infty \end{aligned}$$

which is a contradiction since the left-hand side is bounded from below. □

The existence and the boundedness of most interior-point paths were established under Condition 1.2 or some similar versions. See, e.g., Kojima, Megiddo, and Noma (1991), Kojima, Mizuno, and Noma (1989), Kojima, Megiddo, Noma, and Yoshise (1991), Hotta and Yoshise (1999), Hotta et al. (1998), Qi and Sun (2000). In what follows, we show that Condition 1.2 also implies the existence and the boundedness of our trajectory. In fact, we can verify that Condition 1.2 implies the nonemptyness and boundedness of the solution set of CP(f).

Theorem 4.4. *Let f be a continuous function. If Condition* 1.2 *is satisfied, then the solution set of the problem* CP(f) *is nonempty and bounded, and hence the result of Theorem* 4.3 *remains valid.*

Proof. By Theorem 4.3, it is sufficient to show that Condition 1.2 implies the nonemptyness and boundedness of the solution set of CP(f). Since Kojima, Megiddo, and Noma (1991) showed that Condition 1.2 implies the boundedness of their interior-point trajectory by continuity, any accumulation point of their trajectory, as the parameter approaches to zero, is a solution to CP(f). Thus the solution set is nonempty. We now show its boundedness. By (b) of Condition 1.2, there is a point $x_0 > 0$ such that $f(x_0) > 0$. Choose a scalar r such that $0 < r < \min_{1\leq i\leq n} f_i(x_0)$. Then by the definition of $B_{++}(f)$, we have

$$\mathcal{D} := [0, re] \times [-re, re] \subseteq R^n_+ \times B_{++}(f).$$

Since $\mathcal{D}$ is a compact set, thus by (c) of Condition 1.2, the following set,

$$\mathcal{F}^{-1}(\mathcal{D}) = \{(x, y) \in R^{2n}_+ : \mathcal{F}(x, y) \in \mathcal{D}\},$$

is bounded. Since $0 \in \mathcal{D}$, it follows that

$$\mathcal{F}^{-1}(0) = \{(x, y) \in R^{2n}_+ : \mathcal{F}(x, y) = 0\},$$

which is contained in $\mathcal{F}^{-1}(\mathcal{D})$, is also bounded. Since $\mathcal{F}^{-1}(0)$ coincides with the solution set of CP(f), we have the desired result. □

We have shown that for any P_0 complementarity problem the continuation trajectory proposed in the paper always exists provided f is continuous (see Theorem 4.2), and that the trajectory is bounded under any one of the following conditions:

- Condition 4.1.
- Condition 4.2.
- The solution set $\mathrm{SOL}_{cp}(f)$ is nonempty and bounded.
- Condition 1.2 (in particular, Condition 1.1).

It is interesting to compare the above-mentioned conditions. We summarize the result as follows:

PROPOSITION 4.2. *Let $f : R^n \to R^n$ be a P_0-function and let $(a, b) \in R^n_{++} \times R^n$ be a fixed vector. Then Condition* 1.1 $\Rightarrow$ *Condition* 1.2 $\Rightarrow$ *nonemptyness and boundedness of the solution set of* CP(f) $\Rightarrow$ *Condition* 4.2 $\Rightarrow$ *Condition* 4.1. *However, Condition* 4.1 *may not imply the boundedness of solution set, and existence of a strictly feasible point. Hence, Condition* 4.1 *is strictly weaker than Condition* 1.2, *and than the nonemptyness and boundedness assumption of the solution set.*

PROOF. The first implication is well-known (Kojima, Megiddo, and Noma 1991). The second implication follows from Theorem 4.4, and the last implication is obvious. We now prove the third implication. Assume that the solution set of CP(f) is nonempty and bounded. We now show that Condition 4.2 is satisfied. Let $E(x)$ be defined as in the proof of Theorem 4.4, i.e., $E(x) = \mathscr{E}(x, 0)$ which is a P_0-function. Notice that $E^{-1}(0)$ is just the solution set which is nonempty and bounded. Let $\varepsilon > 0$ be a fixed scalar. By Lemma 2.2, there exists a $\delta > 0$ such that for any weakly univalent function $h : R^n \to R^n$ with

$$\sup_{x\in\overline{\Omega}} \|h(x) - E(x)\| < \delta, \tag{22}$$

it holds that

$$\varnothing \neq h^{-1}(0) \subseteq E^{-1}(0) + \varepsilon B. \tag{23}$$

Consider the function $h^* : R^n \times R^1_{++} \times R^n_+ \times R^n \to R^n$, i.e.,

$$h^*(x, \mu, w, v) = x + (f(x) + \mu x + v) - \sqrt{x^2 + (f(x) + \mu x + v)^2 + 2w},$$

where $\mu > 0$ and $(w, v) \in R^n_+ \times R^n$. For such fixed μ, w and v, the function $f(x) + \mu x + v$ is a P-function (in x) since f is a P_0-function. Therefore, the function $h^*(x, \mu, w, v)$ must be a P-function in x (Ravindran and Gowda 1997), and thus be weakly univalent in x. For the above given δ, by (b) of Lemma 4.2, there exists a number $\gamma > 0$ such that

$$\sup_{x\in\overline{\Omega}} \|h^*(x, \mu, w, v) - E(x)\| < \delta,$$

for all $\mu \in (0, \gamma]$ and $(w, v) \in D_\gamma := [0, \gamma e] \times [-\gamma e, \gamma e]$. Therefore, replacing h by h^* in (22) and (23), we have

$$\varnothing \neq (h^*)^{-1}_{(\mu.w,v)}(0) \subseteq E^{-1}(0) + \varepsilon B \quad \text{for all } \mu \in (0, \gamma], (w, v) \in D_\gamma,$$

where

$$(h^*)^{-1}_{(\mu.w,v)}(0) = \{x \in R^n : h^*(x, \mu, w, v) = 0\}.$$

Therefore,

$$\bigcup_{(\mu,w,v)\in(0,\gamma]\times D_\gamma} (h^*)^{-1}_{(\mu,w,v)}(0) \subseteq \mathrm{cl}(E^{-1}(0) + \varepsilon B).$$

It is not difficult to verify that

$$\left\{x : (x, y) \in \bigcup_{\mu\in(0,\gamma]} \mathscr{F}_\mu^{-1}(D_\gamma)\right\} = \bigcup_{(\mu,w,v)\in(0,\gamma]\times D_\gamma} (h^*)^{-1}_{(\mu,w,v)}(0),$$

where $\mathscr{F}_\mu^{-1}(D_\gamma) = \{(x, y) \in R^{2n}_+ : \mathscr{F}_\mu(x, y) \in D_\gamma\}$. It follows from the above two relations that the set

$$\left\{x : \ (x, y) \in \bigcup_{\mu\in(0,\gamma]} \mathscr{F}_\mu^{-1}(D_\gamma)\right\}$$

is bounded. By the definition of $\mathscr{F}_\mu(x, y)$ and continuity of f, we deduce that

$$\bigcup_{\mu\in(0,\gamma]} \mathscr{F}_\mu^{-1}(D_\gamma)$$

is bounded. Thus Condition 4.2 holds.

We now give an example to show that Condition 4.1 holds even when a strictly feasible point fails to exist. Consider the following example.

$$f(x) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 - 1 \end{pmatrix}.$$

This function is monotone. For this example, Condition 1.1 and Condition 1.2 do not hold since f has no strictly feasible point. The solution set of the CP(f) is $\{(x_1, x_2) \in R^2_+ : x_1 \geq 0, x_2 = 1\}$, which is unbounded. However, this example satisfies Condition 4.1. Indeed, for any given (a, b) where $a = (a_1, a_2)^T \in R^2_{++}$ and $b = (b_1, b_2)^T \in R^2$, we can verify that the set $\cup_{\mu\in(0,\gamma]}\mathscr{F}_\mu^{-1}(D_\mu)$ is bounded for some $\gamma \in (0, 1)$, where D_μ and $\mathscr{F}_\mu^{-1}(D_\mu)$ are given as in Condition 4.1. Let $(x, y) \geq 0$, and

$$\mathscr{F}_\mu(x, y) = \begin{pmatrix} Xy \\ y - (f(x) + \mu x) \end{pmatrix} \in D_\mu.$$

Then we have

$$\begin{aligned}
&(x_1, x_2, y_1, y_2) \geq 0, \\
&x_1 y_1 = d_1 \in \left[0, \frac{\mu}{1-\mu} a_1\right], \quad x_2 y_2 = d_2 \in \left[0, \frac{\mu}{1-\mu} a_2\right], \\
&y_1 - \mu x_1 = \bar{d}_1 \in \frac{\mu}{1-\mu}[[b]_-, [b]_+], \\
&y_2 - [(1+\mu)x_2 - 1] = \bar{d}_2 \in \frac{\mu}{1-\mu}[[b]_-, [b]_+].
\end{aligned}$$

Thus,

$$\begin{aligned}
&\frac{\mu a_1}{1-\mu} \geq x_1 y_1 = x_1(\mu x_1 + \bar{d}_1) \geq x_1\left(\mu x_1 - \frac{\mu}{1-\mu}[b_1]_-\right), \\
&\frac{\mu a_2}{1-\mu} \geq x_2 y_2 = x_2((1+\mu)x_2 - 1 + \bar{d}_2) \geq x_2\left((1+\mu)x_2 - 1 - \frac{\mu}{1-\mu}[b_2]_-\right).
\end{aligned}$$

That is,

$$x_1^2 - \frac{[b_1]_-}{1-\mu} x_1 \leq \frac{a_1}{1-\mu}, \ (1+\mu)x_2^2 - \left(\frac{\mu[b_2]_-}{1-\mu} + 1\right)x_2 \leq \frac{\mu a_2}{1-\mu},$$

which implies that (x_1, x_2) must be uniformly bounded for all $\mu \in (0, \gamma]$ where γ is a fixed scalar in (0,1). Thus, the set $\cup_{\mu\in(0,\gamma]}\mathscr{F}_\mu^{-1}(D_\mu)$ is bounded, and thus Condition 4.1 is satisfied. $\square$

While the above example has no strictly feasible point, it satisfies Condition 4.1. Hence, it follows from Theorem 4.2 that from each point $(a, b) \in R^n_{++} \times R^n$ the proposed continuation trajectory always exists and any subtrajectory is bounded. It is worth noting that for this example there exists no central path (Example 6.1).

When restricted to P_* complementarity problems, it turns out that Condition 4.1 can be further relaxed. In fact, for this case, we can achieve a necessary and sufficient condition for the existence and boundedness of the trajectory (see the next section for details). This prominent feature distinguishes the proposed trajectory from the central path and those continuation trajectories studied by Kojima, Megiddo, and Noma (1991), Kojima et al. (1990), and Kojima et al. (1993).

5. Limiting behavior of the trajectory. The results proved in Section 4 reveal that under certain mild conditions, the continuation trajectory generated by (8) has at least a convergence subsequence $\{(x(\theta_k), y(\theta_k))\}$, whose limit point (x^*, y^*), as $\theta_k \to 0$, is a solution to CP(f). In this section, we consider the following questions: (a) When does the entire trajectory converge? (b) In the setting of semimonotone functions, if a subsequence $\{(x(\theta_k), y(\theta_k))\} \to (x^*, y^*)$ as $(\theta_k) \to 0$, what can be said about x^*?

In this section, we show that if $\theta/\phi(\theta) \to 0$ as $\theta \to 0$, some much stronger convergence properties of the proposed trajectory can be obtained. The case of $\theta/\phi(\theta) \to 0$ as $\theta \to 0$ is easy to be satisfied. An example of such a ϕ is $\phi(\theta) = \theta^p$ where $p \in (0, 1)$.

We first show that for P_* complementarity problems this trajectory is always bounded, provided that the solution set is nonempty (not necessarily bounded). Hence, for a P_* complementarity problem, this trajectory is bounded if and only if the CP(f) has a solution. This result further improves the result of Theorem 4.2. Moreover, if the problem has a least element solution x^*, i.e., $x^* \le u^*$ for all $u^* \in \mathrm{SOL}_{cp}(f)$ (see, for example, Pang 1979), we prove that the entire trajectory is convergent for P_* complementarity problems.

THEOREM 5.1. *Let f be a continuous P_*-function from R^n into itself. Suppose that the solution set of* CP(f) *is nonempty.*

(a) *The System* (8) *has a unique solution $(x(\theta), y(\theta))$ for each $\theta \in (0, 1)$, and the solution is continuous on* (0,1). *Therefore, the homotopy continuation trajectory $\{(x(\theta), y(\theta)) : \theta \in (0, 1)\}$ generated by* (8) *always exists.*

(b) *If $\theta/\phi(\theta)$ is bounded for $\theta \in (0, 1]$, then any short part of the trajectory (subtrajectory) is bounded, that is, for any $\delta \in (0, 1)$, the set $\{(x(\theta), y(\theta)) : \theta \in (0, \delta]\}$ is bounded.*

(c) *If ϕ is chosen such that $\theta/\phi(\theta) \to 0$ as $\theta \to 0$ and $\mathrm{SOL}_{cp}(f)$ has a least element, then the entire trajectory must converge to (x^*, y^*) where x^* is the (unique) least element solution.*

PROOF. Since each P_*-function is a P_0-function, Part (a) follows immediately from Theorem 4.2. We now prove (b) and (c). Since the System (8) is equivalent to (11)–(13), we have $(x(\theta), y(\theta)) > 0$ and

$$\theta X^{-1}(\theta)a = y(\theta) = (1-\theta)(f(x(\theta)) + \phi(\theta)x(\theta)) + \theta b > 0, \tag{24}$$

where $X^{-1}(\theta) = [\mathrm{diag}(x(\theta))]^{-1}$. Let u^* be an arbitrary solution to the CP(f). For each i we have

$$\begin{aligned}(y_i(\theta) - f_i(u^*))(x_i(\theta) - u_i^*) &= y_i(\theta)x_i(\theta) - f_i(u^*)x_i(\theta) - u_i^* y_i(\theta) \\ &\le y_i(\theta)x_i(\theta) = \theta a_i. \end{aligned}\tag{25}$$

Thus,

$$(y(\theta) - f(u^*))^T (x(\theta) - u^*) \le \theta e^T a. \tag{26}$$

On the other hand, by (24) and (25), we have

$$
\begin{aligned}
&(x_i(\theta)-u_i^*)(f_i(x(\theta))-f_i(u^*))\\
&\quad= (x_i(\theta)-u_i^*)\left(\frac{y_i(\theta)}{1-\theta}-\frac{\theta b_i}{1-\theta}-\phi(\theta)x_i(\theta)-f_i(u^*)\right)\\
&\quad= \frac{(x_i(\theta)-u_i^*)(y_i(\theta)-f_i(u^*))}{1-\theta}\\
&\qquad -(x_i(\theta)-u_i^*)\left(\phi(\theta)x_i(\theta)+\frac{\theta}{1-\theta}(b_i-f_i(u^*))\right)\\
&\quad\le \frac{\theta a_i}{1-\theta}-(x_i(\theta)-u_i^*)\left(\phi(\theta)x_i(\theta)+\frac{\theta}{1-\theta}(b_i-f_i(u^*))\right)\\
&\quad= \frac{\theta(a_i+x_i(\theta)f_i(u^*))}{1-\theta}-\phi(\theta)(x_i(\theta)-u_i^*)\left(x_i(\theta)+\frac{\theta b_i}{(1-\theta)\phi(\theta)}\right)\\
&\quad\le \frac{\theta(e^Ta+f(u^*)^Tx(\theta))}{1-\theta}-\phi(\theta)\overline{M},
\end{aligned}
\tag{27}
$$

where

$$
\overline{M}=\min_{1\le i\le n}(x_i(\theta)-u_i^*)\left(x_i(\theta)+\frac{\theta b_i}{(1-\theta)\phi(\theta)}\right).
$$

Since f is a P_*-function, by using (26) and (27), we have

$$
\begin{aligned}
\theta e^Ta &\ge (y(\theta)-f(u^*))^T(x(\theta)-u^*)\\
&= [(1-\theta)(f(x(\theta))+\phi(\theta)x(\theta))+\theta b-f(u^*)]^T(x(\theta)-u^*)\\
&= (1-\theta)(f(x(\theta))-f(u^*))^T(x(\theta)-u^*)\\
&\quad +(1-\theta)\phi(\theta)x(\theta)^T(x(\theta)-u^*)+\theta(b-f(u^*))^T(x(\theta)-u^*)\\
&\ge -(1-\theta)\tau\sum_{i\in I_+(x(\theta),u^*)}(f_i(x(\theta))-f_i(u^*))(x_i(\theta)-u_i^*)\\
&\quad +(1-\theta)\phi(\theta)x(\theta)^T(x(\theta)-u^*)+\theta b^T(x(\theta)-u^*)-\theta f(u^*)^Tx(\theta)\\
&\ge -(1-\theta)\tau n\max_{1\le i\le n}(f_i(x(\theta))-f_i(u^*))(x_i(\theta)-u_i^*)\\
&\quad +(1-\theta)\phi(\theta)x(\theta)^T(x(\theta)-u^*)+\theta b^T(x(\theta)-u^*)-\theta f(u^*)^Tx(\theta)\\
&\ge -(1-\theta)\tau n\left(\frac{\theta(e^Ta+f(u^*)^Tx(\theta))}{1-\theta}-\phi(\theta)\overline{M}\right)\\
&\quad +(1-\theta)\phi(\theta)x(\theta)^T(x(\theta)-u^*)+\theta b^T(x(\theta)-u^*)-\theta f(u^*)^Tx(\theta)\\
&= -\theta\tau ne^Ta-\theta(1+\tau n)f(u^*)^Tx(\theta)+(1-\theta)\tau n\phi(\theta)\overline{M}\\
&\quad +(1-\theta)\phi(\theta)x(\theta)^T(x(\theta)-u^*))+\theta b^T(x(\theta)-u^*).
\end{aligned}
$$

Rearranging terms and dividing both sides by $\phi(\theta)$, we have

$$
\begin{aligned}
(1+\tau n)(\theta/\phi(\theta))e^Ta &\ge (1-\theta)\tau n\min_{1\le i\le n}(x_i(\theta)-u_i^*)\left(x_i(\theta)+\frac{\theta b_i}{(1-\theta)\phi(\theta)}\right)\\
&\quad +(1-\theta)x(\theta)^T\left(x(\theta)-u^*-\frac{\theta(1+\tau n)}{\phi(\theta)(1-\theta)}f(u^*)\right)\\
&\quad +(\theta/\phi(\theta))b^T(x(\theta)-u^*).
\end{aligned}
\tag{28}
$$

Let $\delta \in (0, 1)$ be an arbitrary scalar. Since $\theta/\phi(\theta)$ is bounded, it follows from the above inequality that the set $\{x(\theta) : \theta \in (0, \delta]\}$ is bounded, and by (24), so is $\{y(\theta) : \theta \in (0, \delta]\}$. Part (b) follows.

We now consider the case $\theta/\phi(\theta) \to 0$ as $\theta \to 0$. By the boundedness of the trajectory, there exists a convergence subsequence $\{(x(\theta_k), y(\theta_k))\}$ where $\theta_k \to 0$ such that $x(\theta_k) \to x^*$ which is a solution to CP(f). Taking the limit, it follows from (28) that there exists an index i_0 such that

$$0 \geq \tau n x^*_{i_0}(x^*_{i_0} - u^*_{i_0}) + (x^*)^T(x^* - u^*) \text{ for all } u^* \in \mathrm{SOL}_{cp}(f).$$

If CP(f) has a least element solution $\bar{x}^*$, substituting u^* by $\bar{x}^*$ in the above we deduce that $x^* = \bar{x}^*$. Since $\bar{x}^*$ is unique, the entire trajectory must converge to this least element of the solution set. Thus we have Part (c). $\square$

For the monotone cases, we have the following consequence of Theorem 5.1.

THEOREM 5.2. *Let f be a continuous monotone map from R^n into R^n. Suppose that the solution set of* CP(f) *is nonempty. Let ϕ be given such that $\theta/\phi(\theta) \to 0$ as $\theta \to 0$. Then the entire trajectory*, $\{(x(\theta), y(\theta)) : \theta \in (0, 1)\}$, *generated by the System* (8) *converges*, *as* $\theta \to 0$, *to* (x^*, y^*) *where x^* is a least* 2*-norm solution*, *i.e.*, $\|x^*\| \leq \|u^*\|$ *for all* $u^* \in \mathrm{SOL}_{cp}(f)$.

PROOF. Since each monotone map is a P_*-map with the constant $\tau = 0$, from Theorem 5.1, the trajectory in question always exists and is bounded. In this case, the Inequality (28) reduces to

$$\frac{\theta(e^T a)}{\phi(\theta)} \geq (1-\theta)x(\theta)^T\left(x(\theta) - u^* - \frac{\theta}{\phi(\theta)(1-\theta)}f(u^*)\right) - \frac{\theta b^T(x(\theta) - u^*)}{\phi(\theta)}. \tag{29}$$

Suppose that $\{x(\theta_k)\}$, where $\theta_k \to 0$, is an arbitrary convergent subsequence with the limiting point x^*, i.e., $x(\theta_k) \to x^*$. Since $\theta_k/\phi(\theta_k) \to 0$ as $\theta_k \to 0$, we have from (29) that

$$(x^*)^T(x^* - u^*) \leq 0.$$

Since u^* is an arbitrary element in $\mathrm{SOL}_{cp}(f)$, the above inequality implies that x^* is unique, and thus the entire trajectory must converge to this solution. It follows from the above inequality that

$$\|x^*\|^2 \leq (x^*)^T u^* \leq \|x^*\|\|u^*\|,$$

that is,

$$\|x^*\|(\|x^*\| - \|u^*\|) \leq 0,$$

which implies that x^* is a least 2-norm solution. $\square$

It is worth stressing the prominent features of the above results. First, it does not require the strict feasible condition. Second, it does not need any properness conditions such as Condition 4.1, Part (c) of Condition 1.2 and those used in Monteiro and Pang (1996). For a P_* complementarity problem, the existence and boundedness of the proposed trajectory is equivalent to the nonemptyness of the solution set. For a monotone problem, the entire trajectory converges to a least 2-norm solution if and only if the solution set is nonempty.

Kojima, Megiddo, and Noma (1991) showed that if Condition 1.2 holds and f is an affine P_0-function, i.e., $f = Mx + q$, where M is an $n \times n$ P_0-matrix, then the whole interior-point trajectory studied by them is convergent. Moreover, for positive semidefinite matrix M, if the strictly feasible condition holds, Kojima et al. (1990) showed that the entire interior-point trajectory studied by them converges to a solution of CP(f) which is a maximum complementary solution. The trajectory studied in this paper is convergent even when f is a nonlinear monotone map and the strictly feasible condition fails to hold.

Theorems 5.1 and 5.2 answer the first question presented at the beginning of this section. They also partially answer the second question. For P_*-functions, we have proved that the limiting point x^* of the trajectory generated by (8) is the least element of the solution set provided such an element exists. For monotone problems, Theorem 5.2 states that the limiting point x^* is a least 2-norm solution. We now study the property of the limiting point x^* of the trajectory in more general cases. Our result reveals that x^* is at least a weak Pareto minimal solution if f is a semimonotone map. The following is the definition of weak Pareto minimal solution (see, e.g. Definition 2 in Sznajder and Gowda 1998, Definition 2.1 and an equivalent description of the concept in Luc 1989).

DEFINITION 5.1. (Sznajder and Gowda 1998, Luc 1989) Let x^* be an element of a nonempty set S. We say that x^* is a weak Pareto minimal element if $(x^* - R^n_{++}) \cap S = \varnothing$. In other words, x^* is weakly Pareto minimal element of S if there is no element s of S satisfying the inequality $s < x^*$.

We have the following result.

THEOREM 5.3. *Let f be a continuous semimonotone function from R^n into itself, and let $(x(\theta), y(\theta))$ be a solution to the System* (8) *for each $\theta \in (0, 1)$. If the subsequence $x(\theta_k) \to x^*$, where $\theta_k \to 0$ and $\theta_k/\phi(\theta_k) \to 0$, then x^* is a weak Pareto minimal element of the solution set of CP(f).*

PROOF. We assume that x^* is not a weak Pareto minimum. Then by the definition, there exists a solution u^* such that $u^* < x^*$. Since $x(\theta_k) \to x^*$, we must have that $x(\theta_k) > u^*$ for all sufficiently large k. Since f is a semimonotone function, there exist an index p and a subsequence of $\{x(\theta_k)\}$, denoted also by $\{x(\theta_k)\}$, such that $x_p(\theta_k) > u^*_p$ and $f_p(x(\theta_k)) \geq f_p(u^*)$ for all sufficiently large k. Thus

$$A^k_p := (x_p(\theta_k) - u^*_p)(f_p(x(\theta_k)) - f_p(u^*)) \geq 0$$

for all sufficiently large k. Since $x_p(\theta_k) > 0$ and $y_p(\theta_k) > 0$ and u^* is a solution to CP(f), by the same proof of (25), we have

$$(y_p(\theta_k) - f_p(u^*))(x_p(\theta_k) - u^*_p) \leq y_p(\theta_k)x_p(\theta_k) = \theta_k a_p.$$

On the other hand, we have

$$\begin{aligned}
&(y_p(\theta_k) - f_p(u^*))(x_p(\theta_k) - u^*_p) \\
&\quad = [(1-\theta_k)(f_p(x(\theta_k)) + \phi(\theta_k)x_p(\theta_k)) + \theta_k b_p - f_p(u^*)](x_p(\theta_k) - u^*_p) \\
&\quad = (1-\theta_k)A^k_p + (1-\theta_k)\phi(\theta_k)x_p(\theta_k)(x_p(\theta_k) - u^*_p) \\
&\qquad + \theta_k(b_p - f_p(u^*))(x_p(\theta_k) - u^*_p) \\
&\quad \geq (1-\theta_k)\phi(\theta_k)\left(x_p(\theta_k) + \frac{\theta_k(b_p - f_p(u^*))}{(1-\theta_k)\phi(\theta_k)}\right)(x_p(\theta_k) - u^*_p).
\end{aligned}$$

Combining the above two inequalities yields

$$\frac{\theta_k a_p}{\phi(\theta_k)} \geq (1-\theta_k)\left(x_p(\theta_k) + \frac{\theta_k(b_p - f_p(u^*))}{(1-\theta_k)\phi(\theta_k)}\right)(x_p(\theta_k) - u^*_p).$$

Taking the limit, by using the fact $\theta_k \to 0$ and $\theta_k/\phi(\theta_k) \to 0$, we have

$$x^*_p(x^*_p - u^*_p) \leq 0,$$

which contradicts the relation $0 \leq u^* < x^*$. $\square$

As we have mentioned in §3, the Tikhonov regularization trajectory can be viewed as an extreme variant of the trajectory generated by Homotopy (5). We close this section by considering this extreme variant. It is known that for P_0-function, the System (4) has a unique solution $x(\mu)$ which is continuous in $\mu \in (0, \infty)$. See Facchinei (1998), Facchinei and Kanzow (1999), Ravindran and Gowda (1997), Sznajder and Gowda (1998), Gowda and Tawhid (1999), and Facchinei and Pang (1998). The set $\{x(\mu) : \mu \in (0, \infty)\}$ forms the Tikhonov regularization trajectory. Motivated by the proof of Theorems 5.2 and 5.3, we prove the following result which generalizes and improves several known results concerning the Tikhonov regularization methods for CP(f).

THEOREM 5.4. (a) *Let f be a continuous P_*-function from R^n into R^n. If* $\mathrm{SOL}_{cp}(f) \neq \varnothing$, *then the entire Tikhonov regularization trajectory, i.e.,* $\{x(\mu) : \mu \in (0, \infty)\}$, *is bounded. If* CP($f$) *has a least element solution x^*, i.e., $x^* \leq u^*$ for all $u^* \in$* $\mathrm{SOL}_{cp}(f)$, *then the entire trajectory converges to this solution as $\mu \to 0$.*

(b) *If $f : R^n \to R^n$ is a continuous monotone function and* $\mathrm{SOL}_{cp}(f) \neq \varnothing$, *then the entire Tikhonov regularization trajectory, i.e.,* $\{x(\mu) : \mu \in (0, \infty)\}$, *is bounded and converges to a least* 2*-norm solution as $\mu \to 0$.*

(c) *If $f : R^n \to R^n$ is a continuous semimonotone function, then for any sequence $x(\mu_k) \to x^*$ as $\mu_k \to 0$, where $x(\mu_k)$ is a solution to the System* (4) *for each μ_k, x^* is a weak Pareto minimal solution of* CP(f).

PROOF. The ideas of the proof of Parts (a)–(c) are analogous to that of Theorems 5.1, 5.2 and 5.3, respectively. Here, we only prove the Part (a). Parts (b) and (c) can be easily proved. Let u^* be an arbitrary solution of CP(f). Since

$$x(\mu) \geq 0, y(\mu) = f(x(\mu)) + \mu x(\mu) \geq 0, \quad x(\mu)^T y(\mu) = 0,$$

we have that

$$(y_i(\mu) - f_i(u^*))(x_i(\mu) - u_i^*) \leq y_i(\mu) x_i(\mu) = 0. \tag{30}$$

Thus,

$$(y(\mu) - f(u^*))^T (x(\mu) - u^*) \leq 0. \tag{31}$$

It follows from (30) that

$$\begin{aligned}(f_i(x(\mu)) - f_i(u^*))(x_i(\mu) - u_i^*) &= (y_i(\mu) - \mu x_i(\mu) - f_i(u^*))(x_i(\mu) - u_i^*) \\ &\leq -\mu x_i(\mu)(x_i(\mu) - u_i^*)\end{aligned} \tag{32}$$

for all $i = 1, \ldots, n$. By using (31) and (32) and noting that f is a P_*-function, we obtain

$$\begin{aligned}0 &\geq (y(\mu) - f(u^*))^T (x(\mu) - u^*) \\ &= (f(x(\mu)) - f(u^*))^T (x(\mu) - u^*) + \mu x(\mu)^T (x(\mu) - u^*) \\ &\geq -\tau \sum_{i \in I_+(x(\mu), u^*)} (f_i(x(\mu)) - f_i(u^*))(x_i(\mu) - u_i^*) + \mu x(\mu)^T (x(\mu) - u^*) \\ &\geq -\tau n \max_{1 \leq i \leq n} (f_i(x(\mu)) - f_i(u^*))(x_i(\mu) - u_i^*) + \mu x(\mu)^T (x(\mu) - u^*) \\ &\geq -\tau n \max_{1 \leq i \leq n} [-\mu x_i(\mu)(x_i(\mu) - u_i^*)] + \mu x(\mu)^T (x(\mu) - u^*) \\ &= \tau n \min_{1 \leq i \leq n} \mu x_i(\mu)(x_i(\mu) - u_i^*) + \mu x(\mu)^T (x(\mu) - u^*).\end{aligned}$$

Thus,

$$0 \geq \tau n \min_{1\leq i\leq n} x_i(\mu)(x_i(\mu) - u_i^*) + x(\mu)^T(x(\mu) - u^*) \tag{33}$$

which implies that the set $\{x(\mu) : \mu \in (0, \infty)\}$ is bounded. We assume that $\{x(\mu_k)\}$ is a subsequence and $x(\mu_k) \to x^*$ as $\mu_k \to 0$. From (33), we deduce that

$$0 \geq \tau n \min_{1\leq i\leq n} x_i^*(x_i^* - u_i^*) + (x^*)^T(x^* - u^*).$$

If u^* is a least element solution in the sense that $u^* \leq v^*$ for all $v^* \in \mathrm{SOL}_{cp}(f)$ (Pang 1979), it follows from the above inequality that $x^* = u^*$. Since the least element solution is unique, the entire trajectory $\{x(\mu) :\ \mu \in (0, \infty)\}$ must converge to this solution as $\mu \to 0$. $\square$

For a differentiable P_0-function f, Facchinei (1998) showed that if $\mathrm{SOL}_{cp}(f)$ is nonempty and bounded the Tikhonov regularization trajectory $\{x(\mu) : \mu \in (0, \bar{\mu}]\}$ is bounded for any fixed $\bar{\mu}$, and he gave an example to show that it is not possible to remove the boundedness assumption of the solution set in his result without destroying the boundedness of the regularization subtrajectory. Here, we significantly improved Facchinei's result in the setting of P_* complementarity problems, and showed that the boundedness assumption of the solution set can be removed, and the entire Tikhonov regularization trajectory $\{x(\mu) : \mu \in (0, \infty)\}$, rather than just a subtrajectory, is bounded. Since P_* problems include the monotone ones as special cases, the above (a) and (b) of Theorem 5.4 can be viewed as a generalization of the results of Subramanian (1988) and Sznajder and Gowda (1998) for the monotone linear complementarity problems. Part (c) of Theorem 5.4 extends the result of Theorem 3 in Sznajder and Gowda (1998) concerning P_0-functions to general semimonotone functions.

6. Examples and a framework of path-following method. We have shown that for P_0-complementarity problems it only needs a mild condition to ensure the existence and boundedness of the proposed homotopy continuation trajectory. This feature of the homotopy continuation trajectory enables us to design a path-following method (continuation method) to solve a very general class of complementarity problems even when a strictly feasible point fails to exist (in this case, if it is solvable, the P_0 complementarity problem has an unbounded solution set). We first give two examples to show that the proposed trajectory does exist and is bounded in general situations.

EXAMPLE 6.1: Consider the monotone LCP(M, q), where

$$M = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad q = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

The solution set $\mathrm{SOL}_{cp}(f) = \{x \in R^n : x_1 \geq 0, x_2 = 1\}$ is unbounded. Clearly, this problem has no strictly feasible point. Thus, this problem has no central path, i.e., the following system,

$$X(Mx + q) = \mu e,\ x > 0,\ Mx + q > 0,$$

has no solution for each $\mu > 0$. However, from any $(a, b) \in R^2_{++} \times R^2$, we can verify that the proposed continuation trajectory exists and converges to the least 2-norm solution. Indeed, let $a = (a_1, a_2)^T \in R^2_{++}$ and $b = (b_1, b_2)^T \in R^2$. The System (7) can be written as

$$x > 0,\ (1-\theta)(Mx + q + \phi(\theta)x) + \theta b > 0,$$
$$X[(1-\theta)(Mx + q + \phi(\theta)x) + \theta b] = \theta a,$$

i.e.,

$$x=(x_1,x_2)>0,\ \ x_1(\theta b_1+(1-\theta)\phi(\theta)x_1)=\theta a_1,$$
$$x_2(\theta b_2+(1-\theta)(1+\phi(\theta))x_2)=\theta a_2.$$

This system has a unique solution $x(\theta)$, i.e.,

$$x_1(\theta)=\frac{\sqrt{\theta^2 b_1^2+4(1-\theta)\theta\phi(\theta)a_1}-\theta b_1}{2(1-\theta)\phi(\theta)}>0,$$
$$x_2(\theta)=\frac{\sqrt{(1-\theta b_2)^2+4(1-\theta)\theta(1+\phi(\theta))a_2}-(1-\theta b_2)}{2(1-\theta)(1+\phi(\theta))}>0.$$

Thus $\{x(\theta):\theta\in(0,1)\}$ forms a continuous trajectory. Let $\theta\to 0$ and $\theta/\phi(\theta)\to 0$. Then it is easy to see that $(x_1(\theta),x_2(\theta))\to(0,1)$ which is the least 2-norm solution of the CP.

It is worth noting that for our trajectory, the vector b, can be chosen as an arbitrary point in R^2. However, we can easily check that the trajectory of Kojima, Megiddo, and Noma (1991) does not exist if there exists a component $b_i\le 0$.

EXAMPLE 6.2. Let $f=Mx+q$, where

$$M=\begin{pmatrix}0 & -2\\ 0 & 0\end{pmatrix},\quad q=\begin{pmatrix}0\\0\end{pmatrix}$$

This matrix is a P_0-matrix, but not a P_*-matrix. The solution set $\mathrm{SOL}_{cp}(f)=\{x\in R: x_1\ge 0, x_2=0\}$ is an unbounded set. Clearly, this problem has no strictly feasible point. We show that the trajectory studied by Kojima, Megiddo, and Noma (1991) does not exist. Let $(a,b)\in R^2_{++}\times R^2$ where $a=(a_1,a_2)^T$ and $b=(b_1,b_2)^T$. Consider the system

$$H(x,y,\theta)=0,(x,y)>0,\theta\in(0,1),$$

where H is given by (6). For this example, the above system can be written as

$$(x_1,x_2)>0,\ \ -2(1-\theta)x_2+\theta b_1>0,\theta b_2>0,$$
$$\theta b_1-2(1-\theta)x_2)x_1=\theta a_1,\theta b_2 x_2=\theta a_2.$$

There are three possible cases:

Case 1: $b_2\le 0$. The above system has no solution.

Case 2: $b_1\le 0$. The above system also has no solution.

Case 3: $b_1>0$ and $b_2>0$. Then $x_2=b_2^{-1}a>0$, and hence,

$$\theta b_1-2(1-\theta)x_2=\theta b_1-2(1-\theta)b_2^{-1}a_2<0,$$

for all sufficiently small $\theta>0$. Thus the above system has no solution for all sufficiently small θ. In summary, from any starting point $(a,b)\in R^2_{++}\times R^2$, the trajectory of Kojima, Megiddo, and Noma (1991) does not exist. However, since f is a P_0-function, by Theorem 4.2, the continuous trajectory generated by System (8) always exists for each given point $(a,b)\in R^2_{++}\times R^2$. We now verify this fact. Choose $\phi(\theta)$ such that $\theta/\phi(\theta)\to 0$ as $\theta\to 0$. The System (8) can be written as

$$x>0,\ \ y=(1-\theta)(f(x)+\phi(\theta)x)+\theta b)>0,\ \ Xy=\theta a.$$

i.e.,

$$(x_1, x_2) > 0, \tag{34}$$

$$-2(1-\theta)x_2 + (1-\theta)\phi(\theta)x_1 + \theta b_1 > 0, \tag{35}$$

$$(1-\theta)\phi(\theta)x_2 + \theta b_2 > 0, \tag{36}$$

$$(\theta b_1 - 2(1-\theta)x_2)x_1 + (1-\theta)\phi(\theta)x_1^2 = \theta a_1, \tag{37}$$

$$(1-\theta)\phi(\theta)x_2^2 + \theta b_2 x_2 = \theta a_2. \tag{38}$$

The Systems (34), (37), and (38) have a unique solution, i.e.,

$$x_1(\theta) = \frac{\sqrt{(\theta b_1 - 2(1-\theta)x_2(\theta))^2 + 4\theta\phi(\theta)(1-\theta)a_1} - (\theta b_1 - 2(1-\theta)x_2(\theta))}{2(1-\theta)\phi(\theta)} > 0, \tag{39}$$

$$x_2(\theta) = \frac{\sqrt{(\theta b_1)^2 + 4\theta\phi(\theta)(1-\theta)a_2} - \theta b_2}{2(1-\theta)\phi(\theta)} > 0. \tag{40}$$

Clearly, $(x_1(\theta), x_2(\theta))$ also satisfies (35) and (36). Thus $(x_1(\theta), x_2(\theta))$ given by (39) and (40) is the unique solution to the Systems (34)–(38). Therefore, $\{x(\theta) : \theta \in (0, 1)\}$ forms a continuous trajectory. Since $\theta/\phi(\theta) \to 0$, we deduce that $x_2(\theta) \to 0$ as $\theta \to 0$. If $\theta/\phi^3(\theta) \to 0$, then $x_2(\theta)/\phi(\theta) \to 0$, and thus $(x_1(\theta), x_2(\theta)) \to (0, 0)$ which is the least 2-norm solution. If $\theta/\phi^3(\theta) \le c$, where c is a positive constant, then $x_2(\theta)/\phi(\theta)$ is bounded. Therefore, it is easy to see that the set $\{(x_1(\theta), x_2(\theta)) : \theta \in (0, \delta]\}$ is bounded for any fixed $\delta \in (0, 1)$, and any accumulation point of the set is a weakly Pareto minimal solution. If $\theta/\phi^3(\theta) \to \infty$, then $(x_1(\theta), x_2(\theta)) \to (\infty, 0)$ which can also be viewed as a weakly Pareto minimal solution.

Based on the results established in the paper, we may develop a continuation method for CP(f) by tracing the proposed trajectory. Such a method is expected to solve a broad class of complementarity problems without the requirement of the strictly feasible condition or boundedness assumption of the solution set. Here we provide only a framework of the algorithm without convergence analysis. Let a be an arbitrary point in R^n_{++}. Let $x^0 > 0$ and $b = y^0 = (x^0)^{-1}a$, $\theta_0 = 1$. Then $\overline{\mathscr{H}}(x^0, y^0, \theta_0) = (a, b)^T$. Thus the starting point can be chosen as (x^0, y^0, θ_0). Assume $(x^k, y^k, \theta_k) > 0$ where $0 < \theta_k \le 1$ is the current point. We attempt to obtain the next iterate by solving approximately the following system,

$$\begin{pmatrix} \theta \\ \overline{\mathscr{H}}(x, y, \theta) \end{pmatrix} = \gamma_k \begin{pmatrix} \theta_k \\ \overline{\mathscr{H}}(x^k, y^k, \theta_k) \end{pmatrix},$$

where $0 < \gamma_k < 1$ is a scalar. The Newton direction $(\Delta\theta, \Delta x, \Delta y) \in R^{1+2n}$ to the above system should satisfy the following equation

$$\begin{pmatrix} 1 & 0 & 0 \\ -a & Y^k & X^k \\ z^k & -(1-\theta_k)(f'(x^k) + \phi(\theta_k)I) & I \end{pmatrix} \begin{pmatrix} \Delta\theta \\ \Delta x \\ \Delta y \end{pmatrix} = (1-\gamma_k) \begin{pmatrix} \theta_k \\ \overline{\mathscr{H}}(x^k, y^k, \theta_k) \end{pmatrix},$$

where

$$z^k = f(x^k) + \phi(\theta_k)x^k - (1-\theta_k)\phi'(\theta_k)x^k - b.$$

Notice that $\Delta\theta = (1-\gamma_k)\theta_k$. We can specify a framework of the algorithm as follows.

ALGORITHM: (a): Select a starting point $(x^0, y^0, \theta_0) > 0$.

(b): At the current iterate, $(x^k, y^k, \theta_k) > 0$, solve the following system

$$\begin{pmatrix} Y^k & X^k \\ -(1-\theta_k)(f'(x^k)+\phi(\theta_k)I) & I \end{pmatrix}\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$$
$$= (1-\gamma_k)\begin{pmatrix} 2\theta_k a - X^k y^k \\ (1-\theta_k)(f(x^k)+\phi(\theta_k)x^k)+\theta_k b - y^k - z^k \end{pmatrix}.$$

(c): Choose suitable step parameters α_k and β_k such that

$$x^{k+1} = x^k + \alpha_k \Delta x, \quad y^{k+1} = y^k + \beta_k \Delta y$$

and $(x^{k+1}, y^{k+1}) > 0$. Let $\theta_{k+1} = (1-\gamma_k)\theta_k$.

(d): Update γ_k and go back to (b).

The main feature of the above algorithm is that the newton direction $(\Delta x, \Delta y)$ is determined by a system which is quite different from the previous ones in the literature. Some interesting topics are the global and local convergence and polynomial iteration complexity of the above algorithm.

7. Conclusions. For most interior-point and non-interior-point continuation methods, either the existence or the boundedness of the trajectory in question requires some relatively restrictive assumptions such as Condition 1.1 or Condition 1.2. Because these methods strongly depend on the existence of a strictly feasible point that is equivalent to the nonemptyness and boundedness of the solution set in the case of P_* complementarity problems (Zhao and Li 2000), they possibly fail to solve the problems with an unbounded solution set even for the monotone cases. However, the continuation trajectory proposed in this paper always exists for P_0 problems without any additional assumption, and the boundedness of it needs no strictly feasible condition as shown by Theorems 4.2, 5.1 and 5.2. Particularly, for P_* problems we prove that the existence and the boundedness of the proposed trajectory is equivalent to the solvability of CP(f). Moreover, if f is monotone, the entire trajectory converges to a least 2-norm solution whenever the solution set is nonempty. As a by-product, a new property (Theorem 5.4) is elicited for Tikhonov regularization trajectory. The results presented in this paper have provided us with a theoretical basis for constructing a new path-following method to solve a CP(f). This method is expected to solve a general class of complementarity problems which is broader than those to which most existing path-following methods can be applied.

Acknowledgments. The research presented in this paper was partially supported by research Grants Council of Hong Kong grant CUHK358/96P. We would like to thank two anonymous referees, and Associate Editor and Area Editor Professor, J. S. Pang, for their helpful comments and suggestions that led to the improvement of the paper.

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