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Strict Feasibility Conditions in Nonlinear Complementarity Problems[1]

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Abstract. Strict feasibility plays an important role in the development of the theory and algorithms of complementarity problems. In this paper, we establish sufficient conditions to ensure strict feasibility of a nonlinear complementarity problem. Our analysis method, based on a newly introduced concept of μ-exceptional sequence, can be viewed as a unified approach for proving the existence of a strictly feasible point. Some equivalent conditions of strict feasibility are also developed for certain complementarity problems. In particular, we show that a P_*-complementarity problem is strictly feasible if and only if its solution set is nonempty and bounded.

Key Words. Complementarity problems, strict feasibility, quasimonotone maps, P_0-maps, P_*-maps.

1. Introduction

We denote by R^n_+ $[R^n_{++}]$ the space of n-dimensional real vectors with nonnegative components [positive components]. We write $x \geq 0$ $[x > 0]$, when $x \in R^n_+$ $[R^n_{++}]$. For any vector $x \in R^n$, the notation $[x]_+$ denotes the vector whose ith component is $\max\{0, x_i\}$, for $i = 1, \ldots, n$. Consider the following

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complementarity problem

$$\text{(CP)} \quad x \geq 0, \qquad f(x) \geq 0\,, \qquad x^T f(x) = 0,$$

where f is a continuous function from R^n into itself. We say that the problem is strictly feasible if there exists a vector $u > 0$ such that $f(u) > 0$. In this case, the vector u is called a strictly feasible point.

Strict feasibility has played a very important role in the development of the theory and algorithms for complementarity problems. It is well known that some interior-point algorithms and continuation methods (see for example Refs. 1–4) require the existence of some continuous interior-point paths. Since all points on these paths are strictly feasible, strict feasibility is a necessary condition for such a class of methods to solve complementarity problems. For monotone CPs, since strict feasibility implies the existence of the central path (Refs. 5–7), we conclude that the central path exists if and only if the CP is strictly feasible. For P_0-type CPs, Kojima et al. (Ref. 3) proved the existence of an interior-point path under strict feasibility and a properness condition. Strict feasibility is also closely related to several other important aspects of a CP, including the solvability and stability of the problem and the structural property of the solution set. For instance, when f is a quasi(pseudo)monotone map, or more generally, a quasi-P_* map, then the CP has a solution if it is strictly feasible; see Refs. 8–10. Recently, Chen et al. (Ref. 11) and Ravindran and Gowda (Ref. 12) proved that the nonemptiness and boundedness of the solution set of a CP of type P_0 implies the strict feasibility of the problem. However, the converse is not true for general P_0-functions. Particularly, a monotone CP is strictly feasible if and only if the solution set is nonempty and bounded (Refs. 11–13). The latter property is closely related to the stability of a CP; see Refs. 12, 14–17.

Because of the aforementioned prominent role of strict feasibility in theory and algorithms for CPs, it is worth developing criteria for strict feasibility of CPs. In this paper, we point out that the problem of judging strict feasibility can be formulated as a problem of solving a nonlinear equation. Motivated by this observation and by using the proposed concept of μ-exceptional sequence of continuous functions, we show a common property for any continuous complementarity problems. This property provides a general and very weak sufficient condition for the strict feasibility of CPs. This sufficient condition is used to develop several other criteria for checking the validity of strict feasibility. Furthermore, we develop some equivalent conditions for strict feasibility of certain CPs. One of our results reveals that the nonlinear CP with a P_*-map is strictly feasible if and only if its solution set is nonempty and bounded; hence, such a complementarity problem is stable in the Facchinei sense if it is strictly feasible. This result generalizes the one in monotone situations; see Section 4 for details.

This paper is organized as follows. In Section 2, we introduce the concept of a measure function of strict feasibility and the concept of so-called μ-exceptional sequence, which plays a key role throughout the paper; we prove a general sufficient condition for the strict feasibility of continuous complementarity problems. In Section 3, we study the strict feasibility of CPs with mappings such as quasimonotone functions and semimonotone functions. In Section 4, we focus on P_*-maps and show some equivalent conditions for strict feasibility. Final remarks are given in Section 5.

2. Definitions and General Sufficient Condition

By the continuity of the map f, it is easy to see that a CP is strictly feasible if and only if there exists a vector $x \geq 0$ such that $f(x) > 0$. Indeed, let $\mu > 0$ be a sufficiently small scalar such that

$$f(x + \mu e) > 0, \qquad \text{where } e = (1, \ldots, 1)^T;$$

then,

$$u = x + \mu e > 0$$

is a strictly feasible point. On the other hand, we note that the conditions

$$x \geq 0 \quad \text{and} \quad f(x) > 0$$

can be rewritten as

$$x \geq 0 \quad \text{and} \quad f(x) \geq \mu e, \text{ for some number } \mu > 0,$$

provided that

$$0 < \mu \leq \min_{1 \leq i \leq n} f_i(x).$$

Let ϕ_μ: $R^n \to R^n$ be given as

$$\phi_\mu(x) = x - (x^2)^{1/2} + f(x) - \mu e - \{[f(x) - \mu e]^2\}^{1/2}, \tag{1}$$

where all algebraic operations are performed componentwise. We refer to this function as a measure function of the strict feasibility of a CP. Since $v \leq (v^2)^{1/2}$, for any $v \in R^n$, it is evident that a CP is strictly feasible if and only if, for some $\mu > 0$, the equation $\phi_\mu(x) = 0$ has a solution. Actually, it is easy to see that

$$\phi_\mu(x) = 0, \qquad \text{if and only if } x \geq 0 \text{ and } f(x) \geq \mu e > 0.$$

In this paper, we use the measure function (1) to study the strict feasibility of a CP. We first introduce the concept of μ-exceptional sequence, and then use the property of homotopy invariance of topological degree (see Ref. 18) to establish an alternative theorem, and hence a general sufficient condition for the existence of a strict feasible point. The following is the definition of the concept of μ-exceptional sequence, which is inspired by the concepts of exceptional sequence and exceptional family of elements introduced respectively by Smith (Ref. 19) and Isac et al. (Refs. 20–21).

Definition 2.1. Let $\mu > 0$ be a number, and let f be a continuous function from R^n into itself. A sequence $\{x^k\} \subseteq R^n_+$ is said to be a μ-exceptional sequence for f if $\|x^k\| \to \infty$ and there exists a positive scalar sequence $\{t^k\}$, with each $t^k \in (0, 1)$, such that

$$f_i(x_k) = -t^k x_i^k / [2(1 - t^k)] + \mu, \qquad \text{if } x_i^k > 0, \tag{2}$$

$$f_i(x^k) \geq \mu, \qquad \text{if } x_i^k = 0, \tag{3}$$

for all $i = 1, \ldots, n$.

We are now ready to prove a basic result.

Theorem 2.1. Let f be a continuous function from $R^n \to R^n$. Then, for any given scalar $\mu > 0$, there exists either a strictly feasible point for the CP or a μ-exceptional sequence for f.

Proof. Assume that the CP is not strictly feasible. We now show that f has a μ-exceptional sequence for any number $\mu > 0$. Let $\mu > 0$ be an arbitrary scalar. We consider the homotopy between the identity mapping and $\phi_\mu(x)$, that is,

$$\begin{aligned} H(x, t) &= tx + (1 - t)\phi_\mu(x) \\ &= x + (1 - t)(f(x) - \mu e) \\ &\quad - (1 - t)[(x^2)^{1/2} + \{[f(x) - \mu e]^2\}^{1/2}]. \end{aligned} \tag{4}$$

Denote

$$\mathscr{S} = \{x \in R^n : H(x, t) = 0, \text{ for some } t \in [0, 1]\}. \tag{5}$$

Under the assumption at the beginning of the proof, we assert that the set $\mathscr{S}$ must be unbounded. By contradiction, suppose that $\mathscr{S}$ is bounded. Then, there exists an open bounded set $\mathscr{D}$ such that

$$\mathscr{B}(0, 1) \cup \mathscr{S} \subset \mathscr{D},$$

where

$$\mathscr{B}(0, 1) = \{x \in R^n : \|x\| \leq 1\}$$

and

$$\mathscr{S} \cap \partial \mathscr{D} = \emptyset,$$

where $\partial\mathscr{D}$ is the boundary of $\mathscr{D}$. By the homotopy invariance theorem of degree (Theorem 2.1.2, Ref. 18), we deduce that

$$\deg(I, \mathscr{D}, 0) = \deg(\phi_\mu, \mathscr{D}, 0), \tag{6}$$

where I denotes the identity mapping. Since $\deg(I, \mathscr{D}, 0) = 1$ (Theorem 3.3.3, Ref. 18), it follows from (6) that $\phi_\mu(x) = 0$ has a solution in $\mathscr{D}$ (Theorem 2.1.1, Ref. 18). Note that each solution to the equation $\phi_\mu(x) = 0$ is a strictly feasible point for the CP. This is a contradiction. Therefore, $\mathscr{S}$ must be an unbounded set. Let $\{x^k\} \subset \mathscr{S}$ be a sequence with $\|x^k\| \to \infty$. Without loss of generality, we may assume that

$$\|x^k\| > 0, \qquad \text{for all } k.$$

It is not difficult to show that $\{x^k\}$ is a μ-exceptional sequence. Since

$$\{x^k\} \subset \mathscr{S}, \qquad \text{for each } x^k,$$

there exists a scalar $t^k \in [0, 1]$ such that

$$H(x^k, t^k) = 0.$$

From (4), we have

$$\begin{aligned} &x^k + (1 - t^k)[f(x^k) - \mu e] \\ &= (1 - t^k)\{[(x^k)^2]^{1/2} + \{[f(x^k) - \mu e]^2\}^{1/2}\}. \end{aligned} \tag{7}$$

Since $x^k \neq 0$, it follows from the above that

$$t^k \neq 1, \qquad \text{for all } k.$$

By our assumption, i.e., the CP is not strictly feasible, it follows that

$$\phi_\mu(x^k) \neq 0, \qquad \text{for all } k.$$

Thus, we deduce from (7) that $t^k \neq 0$. Therefore, in the rest of the proof, only the case $0 < t^k < 1$ needs to be considered. Taking the square of both sides of (7) yields

$$\begin{aligned} &[1 - (1 - t^k)^2]X^k x^k + 2(1 - t^k)X^k[f(x^k) - \mu e] \\ &= 2(1 - t^k)^2[\{X^k[f(x^k) - \mu e]\}^2]^{1/2}, \end{aligned} \tag{8}$$

where

$$X^k = \operatorname{diag}(x^k).$$

Again, taking the square of both sides of (8), we have

$$[1-(1-t^k)^2]^2(X^k)^3x^k+4(1-t^k)[1-(1-t^k)^2](X^k)^3[f(x^k)-\mu e]$$
$$=4(1-t^k)^2[(1-t^k)^2-1]\{X^k[f(x^k)-\mu e]\}^2.$$

Noting that $1-(1-t^k)^2\neq 0$, we have

$$(X^k)^2\{[1-(1-t^k)^2](x^k)^2+4(1-t^k)X^k[f(x^k)-\mu e]$$
$$+4(1-t^k)^2[f(x^k)-\mu e]^2\}=0. \tag{9}$$

If

$$x_i^k=0, \qquad \text{for some } i,$$

it follows from (7) and $t^k\in(0,1)$ that

$$f(x^k)-\mu e=\{[f(x^k)-\mu e]^2\}^{1/2},$$

which implies that

$$f_i(x^k)\geq\mu>0.$$

We consider now the case where

$$x_i^k\neq 0.$$

It follows from (9) that

$$[1-(1-t^k)^2](x_i^k)^2+4(1-t^k)x_i^k[f_i(x^k)-\mu]$$
$$+4(1-t^k)^2[f_i(x^k)-\mu]^2=0.$$

Therefore,

$$f_i(x^k)-\mu=[-4(1-t^k)x_i^k\pm\{16(1-t^k)^2(x_i^k)^2$$
$$-16(1-t^k)^2[1-(1-t^k)^2](x_i^k)^2\}^{1/2}]/[8(1-t^k)^2]$$
$$=[-x_i^k\pm(1-t^k)x_i^k]/[2(1-t^k)].$$

That is, we have either

$$f_i(x^k)-\mu=-t^kx_i^k/[2(1-t^k)] \tag{10}$$

or

$$f_i(x^k)-\mu=-(2-t^k)x_i^k/[2(1-t^k)]. \tag{11}$$

We now show that the latter case in (11) is impossible to hold. Indeed, if (11) holds, then

$$[1-(1-t^k)^2](x_i^k)^2+2(1-t^k)x_i^k[f_i(x^k)-\mu]$$
$$=[1-(1-t^k)^2](x_i^k)^2+2(1-t^k)x_i^k\{-(2-t^k)x_i^k/[2(1-t^k)]\}$$

$$= [1-(1-t^k)^2-(2-t^k)](x_i^k)^2$$
$$= (t^k-1)(2-t^k)(x_i^k)^2$$
$$< 0,$$

which contradicts (8). Thus, $\{x^k\}$ satisfies (10). Furthermore, we show that $x_i^k > 0$ provided that $x_i^k \neq 0$. In fact, by (10) we have

$$x_i^k + (1-t^k)[f_i(x^k) - \mu] = (1-t^k/2)x_i^k.$$

If $x_i^k < 0$, it follows from the above that

$$x_i^k + (1-t^k)[f_i(x^k) - \mu] < 0,$$

which contradicts (7). Therefore, by Definition 2.1, the sequence $\{x^k\}$ must be a μ-exceptional sequence for f. The proof is complete. □

From the above result, if there exists a scalar $\mu > 0$ such that a continuous function does not possess a μ-exceptional sequence, then the CP is strictly feasible. As a result, it is of interest to study various conditions under which the map f has no μ-exceptional sequence. In Section 3, using Theorem 2.1, we establish several criteria to assure the strict feasibility of CPs with such maps as positively homogeneous, quasimonotone, and semimonotone maps, which include P_0-functions as special cases.

3. Some Sufficient Conditions for Strict Feasibility

We first introduce some concepts that will be used in this section. Recall that an $n \times n$ matrix M is semimonotone [strictly semimonotone] if, for any $0 \neq x \geq 0$, there exists a component $x_i > 0$ such that $(Mx)_i \geq [>]0$; see Ref. 22. The following is the nonlinear version of the semimonotone map.

Definition 3.1. See Ref. 23. A function $f: R^n \to R^n$ is said to be semimonotone [strictly semimonotone] if, for any $x \neq y$ and $x - y \geq 0$ in R^n, there exists some i such that $x_i > y_i$ and $f_i(x) \geq [>] f_i(y)$.

We say a function f is a P_0[P]-function if, for any $x \neq y$ in R^n,

$$\max_{x_i \neq y_i} (x_i - y_i)[f_i(x) - f_i(y)] \geq 0[>0]0.$$

Clearly, a P_0-function must be a semimonotone function. However, the converse is not true; see Example 3.9.2 of Ref. 22.

Definition 3.2.

(a) See Refs. 8, 24. A map is said to be quasi(pseudo)monotone if, for any x, y in R^n, $f(y)^T(x-y)>[\geq]0$ implies $f(x)^T(x-y)\geq 0$.

(b) See Ref. 25. A map $f: R^n \to R^n$ is said to be a P_*-map if there exists a scalar $\kappa \geq 0$ such that, for any $x \neq y$ in R^n, we have

$$(1+\kappa)\sum_{i\in I_+(x,y)}(x_i-y_i)[f_i(x)-f_i(y)]$$

$$+\sum_{i\in I_-(x,y)}(x_i-y_i)[f_i(x)-f_i(y)]\geq 0,$$

where

$$I_+(x,y)=\{i: (x_i-y_i)[f_i(x)-f_i(y)]>0\},$$

$$I_-(x,y)=\{1,2,\ldots,n\}\backslash I_+.$$

When

$$f=Mx+q,$$

where $M\in R^{n\times n}$ and $q\in R^n$, it is easy to see that f is a P_*-map if and only if M is a P_*-matrix, i.e., a sufficient matrix; see Refs. 1, 26.

In what follows, we establish sufficient conditions for the strict feasibility of a CP with such functions as positively homogeneous, (uniform) semimonotone functions, and quasimonotone functions. We consider first the situation where the mapping

$$G(x)=f(x)-f(0)$$

is positively homogeneous on R^n_+, that is,

$$G(lx)=lG(x), \qquad \text{for all } l>0 \text{ and } x\in R^n_+.$$

Combined with some copositive property, this class of mappings was used to develop the existence of a solution to a CP or a variational inequality; see Refs. 9, 27–29. Here, we study the strict feasibility of the CP under such an assumption.

It is worth noting that, if $G(x)=f(x)-f(0)$ is positively homogeneous on R^n_+, then there exist two nonnegative vectors $\mu^{\inf}$ and $\mu^{\sup}$ in R^n whose components are constants given by

$$0\leq \mu_i^{\inf}=\liminf_{x\in R^n_+,\|x\|\to\infty}|x_i f_i(x)|/\|x\|^2, \tag{12}$$

$$\infty>\mu_i^{\sup}=\limsup_{x\in R^n_+,\|x\|\to\infty}|x_i f_i(x)|/\|x\|^2. \tag{13}$$

Indeed,

$$\begin{aligned}
&x_i f_i(x)/\|x\|^2\\
&=\{x_i[f_i(x)-f_i(0)]+x_i f_i(0)\}/\|x\|^2\\
&=(x_i/\|x\|)[f_i(x/\|x\|)-f_i(0)]+x_i f_i(0)/\|x\|^2.
\end{aligned} \tag{14}$$

Therefore,

$$\begin{aligned}
&\limsup_{x\in R^n_+,\|x\|\to\infty} |x_i f_i(x)|/\|x\|^2\\
&\le \max_{z\in R^n,\|z\|=1} \max_{1\le i\le n} |z_j[f_j(z)-f_j(0)]|<\infty,
\end{aligned}$$

and hence the constants $\mu_i^{\inf}$ and $\mu_i^{\sup}$ exist. We now prove the following result.

Theorem 3.1. Let f be a continuous function, and let $G(x)=f(x)-f(0)$ be a positively homogeneous map on R^n_+. Assume that the following system of equations:

$$x_i G_i(x)=-\mu_i, \qquad x_i\neq 0, \qquad i=1,2,\ldots,n,$$

has no solution $(x,\mu)\in R^{2n}_+$ with $\|x\|=1$ and $\mu_i\in[\mu_i^{\inf},\mu_i^{\sup}]$, where the constant vectors $\mu^{\inf}$ and $\mu^{\sup}$ are given by (12) and (13). Then, the CP is strictly feasible.

Proof. By Theorem 2.1, it suffices to show that f has no μ-exceptional sequence for some $\mu>0$. We show the assertion by the method of contradiction. Suppose that there exists a scalar $\mu>0$ such that f has a μ-exceptional sequence $\{x^k\}$. Without loss of generality, assume that

$$x^k/\|x^k\|\to x^*.$$

Clearly,

$$x^*\in R^n_+ \quad \text{and} \quad \|x^*\|=1.$$

Then, from (14), we have

$$\begin{aligned}
\lim_{k\to\infty} x_i^k[f_i(x^k)-\mu]/\|x^k\|^2 &= \lim_{k\to\infty} x_i^k f_i(x^k)/\|x^k\|^2\\
&= x_i^* G_i(x^*).
\end{aligned} \tag{15}$$

Since

$$x_i^k[f_i(x^k)-\mu]=-t_k(x_i^k)^2/[2(1-t_k)]<0,$$

for all $x_i^k \neq 0$, it follows from (15), (12), (13) that there is a scalar $\mu^* \geq 0$, where

$$\mu_i^* \in [\mu_i^{\inf}, \mu_i^{\sup}], \qquad i = 1, \ldots, n,$$

such that

$$x_i^* G_i(x^*) = -\mu_i^*, \qquad i = 1, 2, \ldots, n, \forall x_i^* \neq 0,$$

which is a contradiction. □

The following is an immediate extension of Theorem 3.1.

Corollary 3.1. Let f be a continuous function. If $G(x) = f(x) - f(0)$ is positively homogeneous, and if

$$\max_{x_i \neq 0} x_i G_i(x) > 0, \qquad \text{for } 0 \neq x \in R_+^n, \tag{16}$$

then the CP is strictly feasible.

Particularly, if f is strictly copositive on R_+^n, i.e.,

$$x^T[f(x) - f(0)] > 0, \qquad \text{for } 0 \neq x \in R_+^n,$$

then the condition (16) holds. If f is strict semimonotone, then

$$\max_{1 \leq i \leq n} x_i[f_i(x) - f_i(0)] > 0, \qquad \text{for } 0 \neq x \in R_+^n,$$

and thus f also satisfies the condition (16). Thus, we have the following result.

Corollary 3.2. Let f be continuous, and let $G(x) = f(x) - f(0)$ be positively homogeneous on R_+^n. If one of the following conditions hold:

(a) $f(x)$ is strictly copositive on R_+^n,
(b) $f(x)$ is strictly semimonotone on R_+^n,

then the CP is strictly feasible.

For the linear function

$$f(x) = Mx + q,$$

where $M \in R^{n \times n}$ and $q \in R^n$, the positive homogeneity holds trivially. Thus, from Corollary 3.2, it follows that any linear CP with strictly copositive and strictly semimonotone matrices is strictly feasible.

We now consider the case of uniform semimonotone functions. A map $f: R^n \to R^n$ is said to be a uniform semimonotone function if, for any $x \neq y$

and $x \geq y$, there exists a constant $c > 0$ such that

$$\max_{x_i \neq y_i} (x_i - y_i)[f_i(x) - f_i(y)] \geq c\|x - y\|^2.$$

Clearly, the class of uniform semimonotone functions includes that of uniform P-functions as a special case. In the following, we show that any CP with a uniform semimonotone function is strictly feasible. The lemma below is useful.

Lemma 3.1. Let $f: R^n \to R^n$ be a continuous semimonotone function. If $\{x^k\}$ is a μ-exceptional sequence for f, then there exists a subsequence of $\{x^k\}$, denoted by $\{x^{k_j}\}$, such that the corresponding sequence $t^{k_j} \to 0$.

Proof. Let $\{x^k\}$ be a μ-exceptional sequence of f. Thus,

$$\{x^k\} \subset R^n_+ \quad \text{and} \quad \|x^k\| \to \infty.$$

Choosing a subsequence if necessary, we may assume that there exists an index set I such that

$$x_i^k \to \infty, \qquad \text{for all } i \in I,$$

and x_i^k is bounded for $i \notin I$. We construct a sequence $\{y^k\}$ as follows:

$$y^k = \begin{cases} 0, & \text{if } i \in I, \\ x_i^k, & \text{if } i \notin I. \end{cases}$$

Thus,

$$x^k \geq y^k \quad \text{and} \quad x^k \neq y^k.$$

By the semimonotonicity of f, we have

$$\max_{i \in I} (x_i^k - y_i^k)[f_i(x^k) - f_i(y^k)] \geq 0.$$

Therefore, there exists an index m and a subsequence $\{x^{k_j}\}$ such that

$$f_m(x^{k_j}) \geq f_m(y^{k_j}), \qquad \text{for all } j.$$

Since $\{y^{k_j}\}$ is bounded, $\{f_m(x^{k_j})\}$ is bounded from below. By (2), we have

$$0 > -t^{k_j} x_m^{k_j} / [2(1 - t^{k_j})] = f_m(x^{k_j}) - \mu.$$

From the fact that $x_m^{k_j} \to \infty$ and $f_m(x^{k_j})$ is bounded from below, we deduce that $t^{k_j} \to 0$. $\square$

Theorem 3.2. If $f: R^n \to R^n$ is a continuous uniform semimonotone mapping, then CP is strictly feasible.

Proof. Assume that $\mu > 0$ is a number such that f has a μ-exceptional sequence $\{x^k\}$. In the following, we derive a contradiction. We note that

$$\{x^k\} \subset R^n_+, \qquad \text{with } \|x^k\| \to \infty.$$

Passing through a subsequence, we may assume that there exists an index i_0 such that

$$\begin{aligned}&(x^k_{i_0} - 0)[f_{i_0}(x^k) - f_{i_0}(0)] \\ &= \max_{1 \le i \le n} (x^k_i - 0)[f_i(x^k) - f_i(0)] \ge c\|x^k\|^2,\end{aligned}$$

for all k. Thus,

$$-x^k_{i_0} f_{i_0}(0) + \max_{1 \le i \le n} x^k_i f_i(x^k) \ge x^k_{i_0}[f_{i_0}(x^k) - f_{i_0}(0)] \ge c\|x^k\|^2.$$

Therefore,

$$\max_{1 \le i \le n} x^k_i f_i(x^k)/\|x^k\|^2 \ge c + x^k_{i_0} f_{i_0}(0)/\|x^k\|^2 \to c. \tag{17}$$

However, from (2) and Lemma 3.1, passing through a subsequence, we have

$$\begin{aligned}&x^k_i f_i(x^k)/\|x^k\|^2 \\ &= -t^k (x^k)^2/[2(1 - t^k)\|x^k\|^2] + \mu x^k/\|x^k\|^2 \to 0,\end{aligned} \tag{18}$$

for all i such that $x^k_i > 0$. There is a contradiction between (17) and (18). Therefore, f has no μ-exceptional sequence for any $\mu > 0$. By Theorem 2.1, the CP is strictly feasible. □

For general semimonotone functions, including P_0-functions, Zhao and Li (Ref. 23) actually showed the following result.

Theorem 3.3. Let f be a continuous semimonotone function. Suppose that f satisfies the following properness condition: For any sequence $\{x^k\} \subset R^n_{++}$ such that $\|x^k\| \to \infty$, $[-f(x^k)]_+/\|x^k\| \to 0$, and the sequence $\{f_i(x^k)\}$ is bounded from below for each index i with $x^k_i \to \infty$, and such that there exists at least one index i_0 with $x^k_{i_0} \to \infty$ such that $\{f_{i_0}(x^k)\}$ is bounded, it holds that

$$\max_{1 \le i \le n} x^{k_l}_i f_i(x^{k_l}) \to \infty,$$

for some subsequence $\{x^{k_l}\}$. Then, the CP is strictly feasible.

It should be pointed out that the properness condition in the above theorem is weaker than several previous conditions in the literature such as the R_0-property (Refs. 30–31) and the conditions in Theorem 4.1 of Ref. 32 and Theorem 1 of Ref. 33.

For quasimonotone mappings (Refs. 8–9, 24), it is known that strict feasibility is sufficient for the solvability of CPs and variational inequalities. Zhao and Isac (Ref. 10) generalized the concept of quasimonotonicity to the so-called quasi-P_* map; they showed that the CP with a quasi-P_* map has a solution if it is strictly feasible. In what follows, we develop a sufficient condition to assure the strict feasibility of the CP with a quasimonotone map. We will make use of the following properness condition.

Condition 3.1. Properness. For any sequence $\{x^k\} \subset R^n_+$ satisfying

$$\|x^k\| \to \infty \quad \text{and} \quad \lim_{k\to\infty} [-f(x^k)]_+ / \|x^k\| = 0,$$

the following holds for some subsequence $\{x^{k_j}\}$:

$$\max_{1\le i\le n} x_i^{k_j} f_i(x^{k_j}) / \|x^{k_j}\| \to \infty.$$

The following is the main result concerning quasimonotone CPs.

Theorem 3.4. Let f be a quasimonotone map satisfying Condition 3.1 and the Lipschitz condition; i.e., there exists a constant such that

$$\|f(x) - f(y)\| \le L\|x - y\|, \qquad \text{for all } x, y \in R^n_+.$$

If there exists a point $u \in R^n$ such that $f(u) > 0$, then the CP is strictly feasible.

Proof. From Theorem 2.1, it is sufficient to show that there exists no μ-exceptional sequence for f for any number $\mu > 0$. By a contradiction, assume that there exists a scalar $\mu > 0$ such that f has a μ-exceptional sequence $\{x^k\}$. From the fact that

$$\{x^k\} \subseteq R^n_+, \qquad \|x^k\| \to \infty, \qquad f(u) > 0,$$

we deduce that

$$f(u)^T (x^k - u) > 0,$$

for all sufficiently large k. Since f is quasimonotone, the above inequality implies that

$$f(x^k)^T (x^k - u) \ge 0, \tag{19}$$

for all sufficiently large k. Let

$$\alpha_k = t_k / [2(1 - t_k)].$$

By (2), we have

$$\begin{aligned}
&f(x^k)^T(x^k-u)\\
&=\sum_{x_i^k>0}(-\alpha^k x_i^k+\mu)(x_i^k-u_i)+\sum_{x_i^k=0}-u_i f_i(x^k)\\
&\leq\sum_{x_i^k>0}\{-\alpha^k[(x_i^k)^2-(u_i+\mu/\alpha^k)x_i^k]-\mu u_i\}+\sum_{x_i^k=0}|u_i||f_i(x^k)|.
\end{aligned}$$

Since f is Lipschitz continuous, we have

$$\sum_{x_i^k=0}|u_i|\cdot|f_i(x^k)|\leq\|u\|[\|f(0)\|+L\|x^k\|].$$

Combining the above two inequalities yields

$$\begin{aligned}
&f(x^k)^T(x^k-u)/\|x^k\|^2\\
&\leq-\alpha^k\sum_{x_i^k>0}[(x_i^k)^2-(u_i+\mu/\alpha^k)x_i^k]/\|x^k\|^2-\sum_{x_i^k>0}\mu u_i/\|x^k\|^2\\
&\quad+\|u\|[\|f(0)\|+L\|x^k\|]/\|x^k\|^2. \qquad (20)
\end{aligned}$$

Choosing a subsequence if necessary, we consider two cases.

Case 1. There exists a constant $\delta\in(0,1)$ such that

$$\delta\leq t_k<1, \qquad \text{for all } k.$$

In this case,

$$\alpha^k\geq\delta/[2(1-\delta)], \qquad \text{for all } k.$$

Without loss of generality, we assume that $x^k/\|x^k\|\to x^*$. Then, it is easy to see that

$$\sum_{x_i^k>0}[(x_i^k)^2-(u_i+\mu/\alpha^k)x_i^k]/\|x^k\|^2\to\|x^*\|=1.$$

Thus, from (20), taking $k\to\infty$, we have

$$f(x^k)^T(x^k-u)/\|x^k\|^2\leq-\delta\|x^*\|/[2(1-\delta)]<0,$$

which contradicts (19).

Case 2. $t_k\to 0$. Thus, $\alpha_k\to 0$. If $x_i^k=0$, we have from (3) that $f_i(x^k)\geq\mu$, which implies $[-f_i(x^k)]_+=0$. If $x_i^k>0$, we have

$$f_i(x^k)/\|x^k\|=-\alpha^k x_i^k/\|x^k\|+\mu/\|x^k\|\to 0.$$

Thus,

$$[-f(x^k)]_+/\|x^k\|\to 0.$$

By Condition 3.1, there exists a subsequence $\{x^{k_j}\}$ such that

$$\max_{1\le i\le n} x_i^{k_j} f_i(x^{k_j})/\|x^{k_j}\| \to \infty. \tag{21}$$

On the other hand, we have

$$x_i^{k_j} f_i(x^{k_j})/\|z^{k_j}\| = -\alpha_{k_j} x_i^{k_j}/\|x^{k_j}\| + x_i^{k_j}\mu/\|x^{k_j}\| \le \mu, \tag{22}$$

for all $x_i^{k_j} > 0$. There exists a contradiction between (21) and (22). □

4. Equivalent Conditions of Strict Feasibility for Certain CPs

The focus of this section is on the class of P_*-complementarity problems; several equivalent conditions of strict feasibility are developed for this class of problems. We show first that, if f is a P_*-map satisfying the so called co-P_0 property, then the CP is strictly feasible if and only if there exists a point $u \in R^n$ (possibly $u_i < 0$, for some component) such that $f(u) > 0$. To show this result, we need the following lemma.

Lemma 4.1. See Ref. 10. A mapping $f: R^n \to R^n$ is a P_*-map if and only if there exists a constant $\tau \ge 0$ such that

$$(1+\tau) \max_{1\le i\le n} (x_i - y_i)[f_i(x) - f_i(y)] + \min_{1\le i\le n} (x_i - y_i)[f_i(x) - f_i(y)] \ge 0, \tag{23}$$

for all x, y in R^n.

The above result shows that the class of P_*-maps can be defined equivalently by (23). The following is the concept of co-P_0 function.

Definition 4.1. A map is said to be a co-P_0 function if there exists a constant $c > 0$ such that

$$\max_{1\le i\le n} (x_i - y_i)[f_i(x) - f_i(y)] \ge c\|f(x) - f(y)\|^2,$$

for all x, y in R^n.

Clearly, each co-P_0 function is a P_0-function. However, the converse is not true. It is easy to see that a Lipschitz continuous uniform P-function is a co-P_0 function, and each cocoercive map (see, Refs. 34–35) is also a co-P_0 function.

Theorem 4.1. Assume that f is a continuous P_*-function and co-P_0 function. Then, the following two conditions are equivalent:

(a) There is a point $u \in R^n$ such that $f(u) > 0$.
(b) There is a point $x > 0$ such that $f(x) > 0$.

Proof. Clearly, (b) $\Rightarrow$ (a). It is sufficient to prove that (a) $\Rightarrow$ (b). Let u be such a point satisfying (a). We show that the CP is strictly feasible. Let $\mu > 0$ be a positive scalar such that

$$\mu < \min_{1 \le i \le n} f(u).$$

We show that f has no μ-exceptional sequence. Assume the contrary, that there exists a μ-exceptional sequence $\{x^k\}$ for f. We derive a contradiction. In fact, since $\{x^k\} \subset R^n_+$ and $\|x^k\| \to \infty$, there exists an index p and a subsequence $\{x^{k_j}\}$ such that $x_p^{k_j} \to \infty$ as $j \to \infty$. Since $f_p(u) > \mu$ and

$$f_p(x^{k_j}) = -t^{k_j} x_p^{k_j}/[2(1-t^{k_j})] + \mu < \mu,$$

we deduce that

$$(x_p^{k_j} - u_p)[f_p(x^{k_j}) - f_p(u)] < (x_p^{k_j} - u_p)[\mu - f_p(u)] \to -\infty. \tag{24}$$

There exists a subsequence of $\{x^{k_j}\}$, denoted also by $\{x^{k_j}\}$, such that there exist indices m, q such that

$$(x_m^{k_j} - u_m)[f_m(x^{k_j}) - f_m(u)] = \max_{1 \le i \le n} (x_i^{k_j} - u_i)[f_i(x^{k_j}) - f_i(u)],$$

$$(x_q^{k_j} - u_q)[f_q(x^{k_j}) - f_q(u)] = \min_{1 \le i \le n} (x_i^{k_j} - u_i)[f_i(x^{k_j}) - f_i(u)].$$

It follows from (24) that

$$(x_q^{k_j} - u_q)[f_q(x^{k_j}) - f_q(u)] \to -\infty. \tag{25}$$

Since f is a P_*-map, by Lemma 4.1, there exists a constant $\tau \ge 0$ such that

$$(1+\tau)(x_m^{k_j} - u_m)[f_m(x^{k_j}) - f_m(u)] + (x_q^{k_j} - u_q)[f_q(x^{k_j}) - f_q(u)] \ge 0. \tag{26}$$

By (25)–(26), we deduce that

$$(x_m^{k_j} - u_m)[f_m(x^{k_j}) - f_m(u)] \to \infty. \tag{27}$$

We consider the following two cases:

Case 1. $x_m^{k_j} > 0$. Noting that $\mu < f_m(u)$ and by using (2), we have

$$\begin{aligned}&(x_m^{k_j} - u_m)[f_m(x^{k_j}) - f_m(u)]\\ &= (x_m^{k_j} - u_m)\{-t^{k_j} x_m^{k_j}/[2(1-t^{k_j})] - [f_m(u) - \mu]\}.\end{aligned} \tag{28}$$

From the above equation and (27), we deduce that

$$x_m^{k_j} - u_m < 0, \qquad \text{for all } j;$$

thus,

$$0 < x_m^{k_j} < u_m;$$

i.e., $\{x_m^{k_j}\}$ is bounded. It follows from (27)–(28) that $t^{k_j} \to 1$. However, since f is a P_0-function, by Lemma 3.1, there exists a subsequence of $\{t^{k_j}\}$, denoted also by $\{t^{k_j}\}$, such that $t^{k_j} \to 0$. This is a contradiction.

Case 2. $x_m^{k_j} = 0$. In this case,

$$\begin{aligned} c\|f(x^{k_j}) - f(u)\|^2 &\leq (x_m^{k_j} - u_m)[f_m(x^{k_j}) - f_m(u)] \\ &= -u_m[f_m(x^{k_j}) - f_m(u)] \\ &\leq \|u\|\|f(x^{k_j}) - f(u)\|. \end{aligned}$$

Thus,

$$\|f(x^{k_j}) - f(u)\| \leq \|u\|/c.$$

Therefore, $\|f(x^{k_j})\|$ is bounded; hence, (27) is impossible to hold. We have a contradiction. The desired result follows from Theorem 2.1. □

The following corollary follows immediately from the above result.

Corollary 4.1. Let f be a cocoercive mapping on R^n; that is, there exists a constant $\kappa \geq 0$ such that

$$(x-y)^T[f(x) - f(y)] \geq \kappa\|f(x) - f(y)\|^2, \qquad \text{for all } x, y \in R^n.$$

Then, the CP is strictly feasible if and only if there exists a point $u \in R^n$ such that $f(u) > 0$.

The class of cocoercive mappings was studied also by Marcotte et al. (Refs. 34–35). They showed that a cocoercive map has many interesting properties and plays an important role in the convergence of projection iterative schemes for monotone variational inequalities. Corollary 4.1 shows a new property for the class of cocoercive maps. The following example shows that the result of Corollary 4.1 is not valid in general for a monotone mapping.

Example 4.1. Let

$$f = Mx + q,$$

where

$$M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad q = (-1, 1)^T.$$

This function is monotone on R^2. It is easy to see that there is a point $u \in R^n$ such that $f(u) > 0$; however, f has no strict feasible point.

It is known that, for a monotone mapping, the CP is strictly feasible if and only if the solution set is nonempty and bounded. Karamardian (Theorem 4.1, Ref. 8) proved that, for a monotone CP, strict feasibility implies the nonemptiness and boundedness of the solution set. McLinden (Ref. 13) proved that the converse is true in the setting of maximal monotone mappings (Theorem 4, Ref. 8). Chen, Chen, and Kanzow (Proposition 3.12, Ref. 11) used the Fischer function and the mountain pass theorem to show that the converse is also true for continuously differentiable P_0-functions. Ravindran and Gowda (Corollary 5, Ref. 12) proved the same result by degree theory, but they required only the continuity of the mappings. For a general CP of type P_0, however, strict feasibility does not imply the nonemptiness and boundedness of the solution set. An interesting problem arises: What class of functions beyond monotone functions can ensure the equivalence of strict feasibility and nonemptiness and boundedness of the solution set? The following result offers a positive answer to this question.

Theorem 4.2. Let f be a continuous P_*-map. Then, a CP is strictly feasible if and only if the solution set is nonempty and bounded.

Proof. Since each P_*-map is a P_0-function, the nonemptiness and boundedness of the solution set implies strict feasibility; see Corollary 5 in Ref. 12 and Proposition 3.12 in Ref. 11. It is sufficient to show that strict feasibility implies the nonemptiness and boundedness of the solution set. In Ref. 25, Zhao and Han showed that, for P_*-maps, the solution set of a CP is nonempty if it is strictly feasible. Thus, it suffices to show the boundedness of solution set when the CP is strictly feasible. Let u be a strict feasible point, i.e.,

$$u > 0 \quad \text{and} \quad f(u) > 0.$$

By contradiction, let $\{x^k\}$ be a solution sequence and $\|x^k\| \to \infty$. Noting that

$$X^k f(x^k) = 0, \qquad \text{for all } k, \text{ where } X^k = \operatorname{diag}(x^k),$$

we have that, for all i,

$$\begin{aligned}
&(x_i^k - u_i)[f_i(x^k) - f_i(u)] \\
&= x_i^k f_i(x^k) - x_i^k f_i(u) - u_i f_i(x^k) + u_i f_i(u) \\
&= -x_i^k f_i(u) - u_i f_i(x^k) + u_i f_i(u) \\
&\leq -x_i^k f_i(u) + u_i f_i(u).
\end{aligned} \tag{29}$$

Since $\|x^k\| \to \infty$, there exists a subsequence $\{x^{k_j}\}$ and indices m, p such that, for all j,

$$(x_m^{k_j} - u_m)[f_m(x^{k_j}) - f_m(u)] = \max_{1 \le i \le n} (x_i^{k_j} - u_i^{k_j})[f_i(x^{k_j}) - f_i(u)],$$

and $x_p^{k_j} \to \infty$. It follows from $x_p^{k_j} \to \infty$ so that

$$(x_p^{k_j} - u_p)[f_p(x^{k_j}) - f_p(u)] \le -x_p^{k_j} f_p(u) + u_p f_p(u) \to -\infty. \tag{30}$$

We have from (30) that

$$\begin{aligned} &\min_{1 \le i \le n} (x_i^{k_j} - u_i^{k_j})[f_i(x^{k_j}) - f_i(u^{k_j})] \\ &\le (x_p^{k_j} - u_p)[f_p(x^{k_j}) - f_p(u)] \to -\infty. \end{aligned}$$

Since f is a P_*-map, we deduce from the above relation and Lemma 4.1 that

$$(x_m^k - u_m)[f_m(x^k) - f_m(u)] \to \infty.$$

However, (29) implies that

$$\begin{aligned} &(x_m^k - u_m)[f_m(x^k) - f_m(u)] \\ &\le -x_m^k f_m(u) + u_m f_m(u) \le u_m f_m(u). \end{aligned}$$

This is a contradiction. Thus, we have shown that, for P_*-maps, the solution set is nonempty and bounded if it is strictly feasible. □

Since the class of P_*-maps encompasses the class of monotone maps as its subset, the above result is a generalized version of the result for monotone cases; see Refs. 11–13.

The nonemptiness and boundedness of the solution set is also closely related to the stability of the CP; see for example Refs. 12–17, 22, 36. The following concept is due to Facchinei (Ref. 16).

Definition 4.2. See Ref. 16. Let SOL(f) be the solution set of the CP, where $f: R^n \to R^n$ is assumed to be continuous. We say that the CP is stable if, for every positive γ, a $\delta(\gamma) > 0$ exists such that, for every continuous function $g: R^n \to R^n$ with

$$\|f(x) - g(x)\| < \delta(\gamma)(1 + \|x\|), \qquad \forall x \in \mathrm{SOL}(f) + \mathscr{B}(0, \gamma),$$

where

$$\mathscr{B}(0, \gamma) = \{x \in R^n : \|x\| < \gamma\},$$

the CP with g has a solution in SOL(f) + $\mathscr{B}(0, \gamma)$.

The following result is proved in Ref. 16.

Lemma 4.2. See Ref. 16. Let f be a continuously differentiable P_0-function, and let SOL(f) be nonempty. Then, the CP is stable if and only if SOL(f) is bounded.

Since P_*-maps are contained in the class of P_0-functions, from Theorem 4.1 and Theorem 4.2, we have the following stability result.

Theorem 4.3. Let f be a P_*-map, and let SOL(f) be nonempty. Then, the CP is stable if and only if the CP is strictly feasible. Moreover, if f is a P_*-map satisfying the co-P_0 property, then the CP is stable if and only if there exists a vector $u \in R^n$ such that $f(u) > 0$.

In summary, for P_*-maps, the first three conditions below are equivalent. Moreover, if a P_*-map satisfying the co-P_0 property, the following four conditions are equivalent:

(C1) The CP is strictly feasible.
(C2) The solution set of the CP is nonempty and bounded.
(C3) The solution set of the CP is stable.
(C4) There exists a point $u \in R^n$ such that $f(u) > 0$.

We end this paper by pointing out an application of the measure function of strict feasibility defined by (1) in computing a strictly feasible point. It is well known that some feasible interior-point algorithms for CPs require that the initial point be a strictly feasible point. It is worth noting that determining a strictly feasible point is equivalent to locating a solution of the measure function of strict feasibility given by (1) for some given $\mu > 0$. However, $\phi_\mu(x)$ is not continuously differentiable at certain points even if f is. To overcome this drawback of $\phi_\mu(x)$, we consider another version of the measure function of strict feasibility, that is,

$$\mathcal{F}_\mu(x) = \{(x - \mu e) - [(x - \mu e)^2]^{1/2}\}^2 + [[f(x) - \mu e] - \{[f(x) - \mu e]^2\}^{1/2}]^2.$$

Clearly, there exists a $\mu > 0$ such that $\mathcal{F}_\mu(x^*) = 0$ if and only if x^* is a strictly feasible point of the CP satisfying

$$x^* \geq \mu e \quad \text{and} \quad f(x^*) \geq \mu e.$$

It is easy to see that, if f is continuously differentiable, then $\mathcal{F}_\mu(x)$ is also continuously differentiable. Thus, some algorithms for C^1 smooth equations can be utilized to solve the above equation. In fact, it suffices to solve this equation approximately because the solution satisfying

$$\min\{x^*, f(x^*)\} \geq \mu e,$$

and hence any approximate solution close to x^*, is also strictly feasible. On the other hand, if we consider the following problem:

$$\min_{x \in R^n} \|\mathscr{F}_\mu(x)\|^2,$$

then finding a strictly feasible point is equivalent to computing an approximate global solution of the above problem.

5. Final Remarks

While strict feasibility has been assumed always in many theoretical and algorithmic development in complementarity problems, to our knowledge, there is no specific paper devoted to the development of criteria checking the validity of strict feasibility. In this paper, we introduce the concept of μ-exceptional sequence for continuous functions. Using this concept and the homotopy invariance theorem of degree, we show a useful alternative result for nonlinear complementarity problems, which claims that any complementarity problem with a continuous function is either strictly feasible or has a μ-exceptional sequence. Therefore, conditions to identify maps f that have no μ-exceptional sequence can provide sufficient conditions to guarantee the strict feasibility of CPs. The analysis method presented in this paper can be viewed as a general approach in investigating conditions for the strict feasibility of a CP. Moreover, we have shown that the CP with a P_*-map is strictly feasible if and only if its solution set is nonempty and bounded, and hence if and only if the CP is stable. This result extends the one in monotone CPs. We also indicate that a measure function of strict feasibility can be used to compute a strictly feasible point.

It is worth mentioning that the feasibility problem of CPs, i.e., $x \geq 0$ and $f(x) \geq 0$, was studied recently by Isac (Ref. 37), who established an alternative result for the existence of a feasible point of CP. Also, Isac (Ref. 37) presented an open problem, that is, whether or not an appropriate notion of exceptional family of elements can be used to study the strict feasibility of CPs (see also Ref. 10). It is evident that the concept of μ-exceptional sequence, tailed to the need of the strict feasibility problem, can be viewed as a modified version of the exceptional family of elements. Thus, the results presented in this paper actually give a positive answer to the open problem posed in Ref. 37.

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