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Extended Projection Methods for Monotone Variational Inequalities[1]

Y. B. ZHAO[2]

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Abstract. In this paper, we prove that each monotone variational inequality is equivalent to a two-mapping variational inequality problem. On the basis of this fact, a new class of iterative methods for the solution of nonlinear monotone variational inequality problems is presented. The global convergence of the proposed methods is established under the monotonicity assumption. The conditions concerning the implementability of the algorithms are also discussed. The proposed methods have a close relationship to the Douglas–Rachford operator splitting method for monotone variational inequalities.

Key Words. Monotone variational inequalities, projection methods, global convergence, two-mapping variational inequality problems.

1. Introduction

Given a closed convex subset X of R^n and a continuous mapping $F: D(F) \subseteq R^n \to R^n$, where $D(F)$ denotes the domain of definition of F and $X \subseteq D(F)$, the variational inequality problem [denoted by VI(X, F)] is to find a vector $x^* \in X$ such that

$$F(x^*)^T(x - x^*) \geq 0, \qquad \text{for all } x \in X, \tag{1}$$

where $F(x^*)^T$ denotes the transpose of the vector $F(x^*)$.

A general iterative approach (see, for example, Refs. 1–2) for solving VI(X, F) consists of generating a sequence $\{x^k\}$ such that each x^{k+1} solves the subproblem VI(X, f^k), where f^k is some approximation to $F(x)$. One

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[2]Researcher, Institute of Applied Mathematics, Chinese Academy of Sciences, Beijing, P. R. China.

of the iterative methods is the well-known symmetric projection method, which can be written as follows:

$$x^{k+1} = P_{H,X}[x^k - H^{-1}F(x^k)], \tag{2}$$

where H is a fixed, symmetric, and positive definite $n \times n$ matrix. $P_{H,X}(\cdot)$ denotes the projection operator onto the set X with respect to the H-norm (Ref. 1).

In the literature on projection-type algorithms, much attention has been paid to the global convergence of the symmetric projection method; but most of these global convergence results are established under some strong assumptions which require in general that the mapping F be strongly monotone and Lipschitz continuous (see, for example, Refs. 1–5). Recently, Marcotte and Wu (Ref. 6) proved the global convergence of the symmetric projection method for solving a nonlinear variational inequality under the cocoercive assumption, which implies the Lipschitz continuity and the monotonicity of F. This assumption on F is weaker than strong monotonicity together with Lipschitz continuity, but stronger than monotonicity. On the other hand, there exist some examples of monotone affine variational inequalities on which the symmetric projection method fails to converge (Refs. 6–7). Therefore, to assure global convergence, the symmetric projection method has the drawback that it requires other restrictive assumptions besides the monotonicity on the mapping F.

In fact, there exist a number of projection-type algorithms which are convergent for monotone VI(X, F), provided that a solution of the problem exists. The oldest one is the well-known extragradient algorithm proposed by Korpelevich in Ref. 8,

$$x^{k+1} = P_X[x^k - \beta F(P_X(x^k - \beta F(x^k)))],$$

where the constant β relies on the value of the Lipschitz constant of the mapping F. By introducing some inexact line searches, Khobotov (Ref. 9) and Marcotte (Ref. 10) proposed several improved extragradient algorithms; but global convergence results for these methods still need the Lipschitz continuity assumption on F. The modified extragradient methods by Solodov and Tseng (Ref. 11) and by Sun (Ref. 12) overcome this drawback by introducing an Armijo-type inexact line search. Moreover, the monotonicity assumption is also removed in Sun's method (Ref. 12). He and Stoer (Ref. 13) and He (Refs. 14–16) proposed a class of projection and contraction methods (PC methods) for monotone VI(X, F); these methods do not rely on line searches and Lipschitz continuity. Some of the results of He (Refs. 14–16) and He and Stoer (Ref. 13) were also obtained by Solodov and Tseng in Ref. 11. It is worth mentioning that projection-type iterative algorithms were also proposed to solve a class of quasi-variational inequalities; see, for example, Refs. 17–18.

In this paper, our focus is on developing a new class of methods for solving monotone VI(X, F). We show first that each monotone VI(X, F) can be formulated equivalently as another problem denoted by VI($\bar{X}$, r, G), see Section 2 for details, so that we can solve the latter problem to obtain the solution of the original problem. Along this idea, we try to extend the symmetric projection method to the equivalent problem, and we obtain a class of iterative methods (extended projection methods) that are different from the previous approaches for monotone VI(X, F). Also, the proposed methods do not require line searches and Lipschitz continuity. The algorithms are globally convergent for any affine monotone VI(X, F); in this case, the algorithms reduce to iterative schemes which include twice projection operations but they are different from extragradient methods. For a class of nonlinear functions, i.e., F monotone on $D(F)$, the proposed approaches are also well defined and globally convergent. In such a case, the proposed methods have a close relationship with several previous methods including the PC methods and the Douglas–Rachford operator splitting method (Refs. 19–20) for monotone VI(X, F). While the proposed methods are extended versions of the symmetric projection method for VI(X, F), a key advantage of them over the symmetric projection method is that, to assure global convergence, they do not require restrictive assumptions on the mapping F.

Section 2 presents some fundamental facts concerning VI(X, F) and develops the algorithm and global convergence result. The conclusions are stated in Section 3.

2. Extended Projection Methods

Let F: $D(F) \subseteq R^n \to R^n$ be a continuous mapping and set

$$G(x) = x + \beta F(x), \tag{3}$$

$$r(x) = x - P_X[x - \beta F(x)]; \tag{4}$$

here, $P_X(\cdot)$ denotes the projection operator onto the set X with respect to the Euclidean norm and $\beta > 0$ is a constant. It is well known that $r(x^*) = 0$ if and only if $x^* \in X$ solves problem VI(X, F). In what follows, we develop first several fundamental consequences which will be used to show the global convergence of the proposed methods.

Lemma 2.1. For any variational inequality problem (1), we have

$$[\beta F(x) - x + y]^T [r(x) - x + y] \geq \|r(x) - x + y\|_2^2, \tag{5}$$

for all $x \in D(F)$ and $y \in X$.

Proof. For any vectors $y \in X$ and $z \in R^n$, by the property of the projection operator P_X, we have

$$[y - P_X(z)]^T [P_X(z) - z] \geq 0.$$

Equivalently,

$$[y - P_X(z)]^T (y - z) \geq \|y - P_X(z)\|_2^2.$$

Setting

$$z = x - \beta F(x),$$

where $x \in D(F)$, and using (4), the conclusion of the lemma is straightforward. □

The following result is similar to the one obtained by He (Ref. 14). For the sake of completeness, we also give a proof of it.

Corollary 2.1. If the mapping F is monotone on the set $\bar{X}$, i.e.,

$$[F(y) - F(z)]^T (y - z) \geq 0, \qquad \text{for all } y, z \in \bar{X},$$

where $X \subseteq \bar{X} \subseteq D(F)$, then we have

$$[G(x) - G(x^*)]^T r(x) \geq \|r(x)\|_2^2, \qquad \text{for all } x^* \in X^*(F), x \in \bar{X}, \tag{6}$$

where $X^*(F)$ denotes the solution set of $\mathrm{VI}(X, F)$.

Proof. Let x^* be an arbitrary solution of problem $\mathrm{VI}(X, F)$. Setting $y = x^*$ in (5), we have

$$[\beta F(x) - x + x^*]^T [r(x) - x + x^*] \geq \|r(x) - x + x^*\|_2^2, \qquad \text{for all } x \in \bar{X}.$$

Rewriting the above inequality and rearranging terms, we have

$$[G(x) - G(x^*)]^T r(x) + \beta F(x^*)^T r(x) - \beta F(x)^T (x - x^*) \geq \|r(x)\|_2^2. \tag{7}$$

Since F is monotone on $\bar{X}$ and x^* is a solution of $\mathrm{VI}(X, F)$,

$$\begin{aligned} &F(x)^T (x - x^*) - F(x^*)^T r(x) \\ &= [F(x) - F(x^*)]^T (x - x^*) + F(x^*)^T \{P_X[x - F(x)] - x^*\} \geq 0. \end{aligned} \tag{8}$$

The desired result is obtained by adding (7) and (8). □

In order to show the next property of monotone VI(X, F), we will make use of the following definition.

Definition 2.1. Let F_1 and F_2 be two functions from the set D into R^n, where D is a closed subset of R^n. The two-mapping variational inequality problem, denoted by VI(D, F_1, F_2), is to find a vector $x^* \in D$ such that

$$F_1(x^*)^T[F_2(x) - F_2(x^*)] \geq 0, \qquad \text{for all } x \in D.$$

It is evident that VI(X, F) is a special case of problem VI(D, F_1, F_2). In fact, when F_2 is the identity mapping and $X = D$, problem VI(D, F_1, F_2) reduces to VI(X, F). Therefore, VI(D, F_1, F_2) can be viewed as the generalization of VI(X, F).

The following theorem shows that each monotone variational inequality problem is equivalent to a two-mapping variational inequality problem.

Theorem 2.1. Assume that the mapping F is monotone on the set $\bar{X}$, where $X \subseteq \bar{X} \subseteq D(F)$. Let $X^*(F)$ and $\bar{X}^*(r, G)$ denote the solution sets of VI(X, F) and the two-mapping variational inequality VI($\bar{X}$, r, G), respectively. Suppose that the set $X^*(F)$ is nonempty. Then, we have

$$X^*(F) = \bar{X}^*(r, G). \tag{9}$$

That is, solving the monotone problem VI(X, F) is equivalent to finding a vector $x^* \in \bar{X}$ such that

$$r(x^*)^T[G(x) - G(x^*)] \geq 0, \qquad \text{for all } x \in \bar{X}. \tag{10}$$

Proof. Let $x^* \in X^*(F)$. Then, $r(x^*) = 0$, which implies that (10) holds trivially, hence $x^* \in \bar{X}^*(r, G)$. Conversely, suppose that $u^* \in \bar{X}^*(r, G)$; by Definition 2.1, we have

$$r(u^*)^T[G(x) - G(u^*)] \geq 0, \qquad \text{for all } x \in \bar{X}.$$

In particular, setting $x = x^*(x^* \in X^*(F))$ in the above inequality, we have

$$r(u^*)^T[G(x^*) - G(u^*)] \geq 0.$$

Since F is a monotone mapping on the set $\bar{X}$, by Corollary 2.1 and the above inequality, we obtain

$$\|r(u^*)\|_2^2 \leq r(u^*)^T[G(u^*) - G(x^*)] \leq 0.$$

Therefore, $r(u^*) = 0$, which implies that $u^* \in X^*(F)$. □

Motivated by the above observation, we can solve the two-mapping variational inequality problem to obtain a solution of the original problem. The general iterative method for the two-mapping variational inequality $\mathrm{VI}(D, F_1, F_2)$ can be defined as follows: Given x^k and the $n \times n$ matrix $A(x^k)$, let x^{k+1} solve the subproblem $\mathrm{VI}(D, F_1^k, F_2)$; i.e., let x^{k+1} be such that

$$F_1^k(x^{k+1})^T[F_2(x) - F_2(x^{k+1})] \geq 0, \qquad \text{for all } x \in D, \tag{11}$$

where

$$F_1^k(x) = F_1(x^k) + A(x^k)[F_2(x) - F_2(x^k)]. \tag{12}$$

If $A(x^k) \equiv H$, a fixed symmetric, positive definite matrix, the above algorithm can be called the extended projection method. This method can be viewed as a generalization of the symmetric projection method for problem $\mathrm{VI}(X, F)$. This paper does not discuss how to solve a general two-mapping problem $\mathrm{VI}(D, F_1, F_2)$. Our attention is focused on the extended projection method for the special two-mapping problem $\mathrm{VI}(\bar{X}, r, G)$ which has the same solution set as $\mathrm{VI}(X, F)$; i.e., we find a solution of $\mathrm{VI}(X, F)$ via solving its equivalent problem $\mathrm{VI}(\bar{X}, r, G)$.

The main computing work of the extended projection method is to solve the subproblem $\mathrm{VI}(D, F_1^k, F_2)$ defined by (11)–(12). We now point out that the subproblem can be written as an affine variational inequality and a nonlinear equation. Suppose that the range

$$F_2(D) = \{F_2(x) : x \in D\}$$

is a closed set in R^n and $f_1^k(y)$ is an affine mapping from $F_2(D)$ into R^n, where

$$f_1^k(y) = F_1(x^k) + A(x^k)[y - F_2(x^k)]. \tag{13}$$

From (11)–(12), it is evident that we may find first a vector $y^* \in F_2(D)$ such that

$$f_1^k(y^*)^T(y - y^*) \geq 0, \qquad \text{for all } y \in F_2(D). \tag{14}$$

Next, we solve the nonlinear equation

$$F_2(x) = y^*. \tag{15}$$

Let

$$x^{k+1} \in \{x : F_2(x) = y^*\};$$

then, x^{k+1} is the solution of subproblem (11)–(12). It is not necessary to compute the set $F_2(D)$, which is quite difficult to compute and in general is not a closed convex set. In fact, we can replace $F_2(D)$ by a closed convex set K; see the following algorithm for details.

Let $A(x^k) \equiv H/\alpha$, where α is a fixed positive number, and let

$$G(X) = \{G(x) : x \in X\}$$

be the range of the mapping G on the feasible set X. From the above observation, replacing the mappings F_1 and F_2 by $r(x)$ and $G(x)$, respectively, we obtain the following basic extended projection method.

Algorithm 2.1.

Step 0. Let $x^0 \in D(F)$, and let K be a closed convex set such that $G(X) \subseteq K$; set $k = 0$.

Step 1. Given x^k, let $y^* \in K$ be such that

$$[r(x^k) + (H/\alpha)(y^* - G(x^k))]^T(y - y^*) \geq 0, \quad \text{for all } y \in K. \tag{16}$$

Let $x^{k+1} \in G^{-1}(K)$ be a solution to $G(x) = y^*$.

Step 2. If x^{k+1} satisfies a prescribed stopping rule, terminate. Otherwise, return to Step 1 with k replaced by $k + 1$.

Remark 2.1. If F is monotone on the set $G^{-1}(K)$, the mapping

$$G(x) = x + \beta F(x)$$

is strongly monotone on the same set; actually,

$$[G(u) - G(v)]^T(u - v) \geq \|u - v\|_2^2.$$

Therefore, the mapping $G: G^{-1}(K) \to K$ is injective and the inverse mapping G^{-1} exists and is Lipschitz continuous on the set K (Ref. 21, Theorem 4.3.8). Therefore, the equation $G(x) = y^*$ has a unique solution in the closed set $G^{-1}(K)$.

In what follows, we establish a global convergence result for the above algorithm.

Theorem 2.2. Let K be a closed convex set satisfying $G(X) \subseteq K \subseteq G(D(F))$. Suppose that F is monotone on the set $G^{-1}(K)$. Let H be an arbitrary symmetric, positive definite $n \times n$ matrix; let $r(x)$ and $G(x)$ be defined by (3) and (4). Let the positive number α be strictly less than $2\lambda_{\min}(H)$, where $\lambda_{\min}(H)$ denotes the minimum eigenvalue of the matrix H. Suppose that the solution set $X^*(F) \neq \varnothing$. Then, the sequence $\{x^k\}$ generated by Algorithm 2.1 converges to a solution of $\mathrm{VI}(X, F)$.

Proof. By the construction of Algorithm 2.1, x^{k+1} satisfies the following inequality:

$$[\alpha r(x^k)+H(G(x^{k+1})-G(x^k))]^T[G(x)-G(x^{k+1})]\geq 0, \tag{17}$$

for all $x\in G^{-1}(K)$. Let x^* be a solution of problem VI(X, F). Setting $x=x^*$ in (17), rearranging the terms, and using Corollary 2.1, we have

$$\begin{aligned}
0&\leq \alpha r(x^k)^T[G(x^*)-G(x^{k+1})]\\
&\quad+[G(x^{k+1})-G(x^k)]^TH[G(x^*)-G(x^{k+1})]\\
&=\alpha r(x^k)^T[G(x^*)-G(x^k)]+\alpha r(x^k)^T[G(x^k)-G(x^{k+1})]\\
&\quad+[G(x^{k+1})-G(x^k)]^TH[G(x^*)-G(x^{k+1})]\\
&\leq -\alpha\|r(x^k)\|_2^2+\alpha r(x^k)^T[G(x^k)-G(x^{k+1})]\\
&\quad+[G(x^{k+1})-G(x^k)]^TH[G(x^*)-G(x^{k+1})]\\
&=-\alpha[\|r(x^k)+(G(x^{k+1})-G(x^k))/2\|_2^2-\|G(x^{k+1})-G(x^k)\|_2^2/4]\\
&\quad+[G(x^{k+1})-G(x^k)]^TH[G(x^*)-G(x^{k+1})]\\
&\leq(\alpha/4)\|G(x^{k+1})-G(x^k)\|_2^2\\
&\quad+[H^{1/2}(G(x^{k+1})-G(x^k))]^T[H^{1/2}(G(x^*)-G(x^{k+1}))]\\
&=(\alpha/4)\|G(x^{k+1})-G(x^k)\|_2^2\\
&\quad+(1/2)[\|H^{1/2}(G(x^*)-G(x^k))\|_2^2\\
&\qquad-\|H^{1/2}(G(x^{k+1})-G(x^k))\|_2^2-\|H^{1/2}(G(x^*)-G(x^{k+1}))\|_2^2]\\
&\leq(\alpha/4)\|H^{-1/2}\|_2^2\|G(x^{k+1})-G(x^k)\|_H^2\\
&\quad+(1/2)[\|G(x^*)-G(x^k)\|_H^2\\
&\qquad-\|G(x^{k+1})-G(x^k)\|_H^2-\|G(x^*)-G(x^{k+1})\|_H^2].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|G(x^{k+1})-G(x^*)\|_H^2&\leq\|G(x^k)-G(x^*)\|_H^2\\
&\quad-[1-\alpha/2\|H^{-1/2}\|_2^2]\|G(x^{k+1})-G(x^k)\|_H^2.
\end{aligned}\tag{18}$$

Since α is strictly less than $2\lambda_{\min}(H)$, the sequence $\{\|G(x^k)-G(x^*)\|_H^2\}$ is nonincreasing. There exists a nonnegative number m such that

$$\lim_{k\to\infty}\|G(x^k)-G(x^*)\|_H=m\geq 0, \tag{19}$$

and hence,

$$\lim_{k\to\infty}\|G(x^{k+1})-G(x^k)\|_H=0. \tag{20}$$

Noting that G is an injective mapping and G^{-1} is Lipschitz continuous on the set K (see Remark 2.1), there exists a constant $L>0$ such that

$$\begin{aligned}\|x^k - x^*\|_2^2 &= \|G^{-1}[G(x^k)] - G^{-1}[G(x^*)]\|_2^2\\ &\leq L^2\|G(x^k) - G(x^*)\|_2^2\\ &\leq L^2\|H^{-1/2}\|_2^2\|G(x^k) - G(x^*)\|_H^2 \to L^2m^2\|H^{-1/2}\|_2^2.\end{aligned}$$

Thus, x^k is bounded. Let $\bar{x}$ be a cluster point of $\{x^k\}$; there exists a subsequence $x^{k_j} \to \bar{x}$. It is evident that $\bar{x} \in G^{-1}(K)$, since $G^{-1}(K)$ is closed and $x^k \in G^{-1}(K)$ for all $k \geq 1$.

Replacing x^k by x^{k_j} in (17), taking the limit $k_j \to \infty$, and using (20), we have

$$\alpha r(\bar{x})^T[G(x) - G(\bar{x})] \geq 0, \qquad \text{for all } x \in G^{-1}(K).$$

Hence, in the case $\bar{X} = G^{-1}(K) \supseteq X$, $\bar{x}$ solves VI(X, F) by Theorem 2.1. Notice that (19) holds for any solution x^*; replacing x^* by $\bar{x}$ in (19), we see that the entire sequence $G(x^k) \to G(\bar{x})$. Hence,

$$\lim_{k\to\infty} x^k = \lim_{k\to\infty} G^{-1}[G(x^k)] = G^{-1}[G(\bar{x})] = \bar{x}.$$

This establishes the theorem. □

Remark 2.2. In some special cases, $G(X)$ is a closed convex set and can be obtained precisely; thus, we can set $K = G(X)$ in the above algorithm. To demonstrate the fact, we consider the case where X is a rectangle, that is, the Cartesian product of n arbitrary intervals such as $[a, b]$, $[a, \infty)$, $(-\infty, b]$, $(-\infty, +\infty)$. So, X can be represented by $\prod_{i=1}^n X_i$, where X_i is an interval in R. Clearly, the nonnegative orthant $X = R_+^n$ is also a rectangle. If X_1 and X_2 are two intervals in R, we denote by $X_1 \vee X_2$ the smallest interval that contains X_1 and X_2. For instance,

$$[2, 3] \vee [4, 5] = [2, 5].$$

The following Proposition 2.1 points out that the range $G(X)$ is convex and can be represented explicitly under suitable assumptions.

Let

$$G(x) = (G_1(x), \ldots, G_n(x))^T \in R^n,$$

where $G_i: R^n \to R$, $i = 1, \ldots, n$. We denote by $G_i(X_1, \ldots, X_n)$ the natural interval extension of $G_i(x)$ on the rectangle $X = \prod_{i=1}^n X_i$ (Ref. 22); that is, $G_i(X_1, \ldots, X_n)$ is an interval which is obtained by replacing each component x_i by the domain X_i and evaluating the expression using interval arithmetic operators. If f is a continuous, monotone function from R to R, then for any $[a, b]$,

$$f([a, b]) = f(a) \vee f(b).$$

The following proposition generalizes this simple fact to an n-dimensional monotone mapping $G(x)$.

Proposition 2.1. Let $X = \prod_{i=1}^{n} X_i$ be a rectangle, and let $G(x) = (G_1(x), \ldots, G_n(x))^T \in R^n$ be monotone on X. Suppose that $X_i = [a_i, b_i]$, $i = 1, \ldots, n$, where $a_i = -\infty$ or $b_i = +\infty$ is permitted. Assume that, for each fixed scalar $\alpha_i \in X_i$, the $(n-1)$-dimensional function $G_i(x_1, \ldots, x_{i-1}, \alpha_i, x_{i+1}, \ldots, x_n)$, $i = 1, \ldots, n$, is a rational function with respect to the variables x_j, $j = 1, \ldots, n$, $j \neq i$, and that each variable x_j, $j \neq i$, occurs in the expression at most once and of one order. Then, the range $G(X) = \prod_{i=1}^{n} G_i(X)$ is also a rectangle and, for each i,

$$\begin{aligned} G_i(X) = {} & G_i(X_1, \ldots, X_{i-1}, a_i, X_{i+1}, \ldots, X_n) \\ & \vee G_i(X_1, \ldots, X_{i-1}, b_i, X_{i+1}, \ldots, X_n). \end{aligned}$$

The above result is an immediate consequence of Lemmas 3.7 and 3.8 in Ref. 22.

In most cases, the range $G(X)$ is difficult to obtain even if X is a rectangle. Therefore, a convex inclusion K of the range $G(X)$ is needed in Algorithm 2.1. In the last three decades, a great deal of work has been done on the problem of computing good convex inclusions of the range of a function over a rectangle. A comprehensive survey of the theory and computing methods for the inclusion of the range of a function is given in Ref. 22.

Remark 2.3. If F is monotone over R^n, $G(x) = x + \beta F(x)$ is a homeomorphic mapping from R^n to R^n by the continuity and strong monotonicity of G (Ref. 21, Theorem 6.4.4). Therefore, $G(D(F)) = R^n$. Thus, for an arbitrary closed convex inclusion $K \supseteq G(X)$, Algorithm 2.1 is always well defined and globally convergent. In particular, let $K = R^n$. It is easy to see that Algorithm 2.1 is reduced to the following iterative scheme:

$$x^{k+1} = G(x^k) - \alpha H^{-1} r(x^k) - \beta F(x^{k+1}). \tag{21}$$

Moreover, let $H = \alpha I$, where I is the identity matrix. Then, the iterative scheme (21) reduces to the Douglas–Rachford operator splitting method for monotone VI(X, F) (Refs. 19–20), i.e.,

$$x^{k+1} = P_X[x^k - \beta F(x^k)] + \beta F(x^k) - \beta F(x^{k+1}). \tag{22}$$

On the other hand, introducing the steplength ρ_k, from (21) we have

$$x^{k+1} = G(x^k) - \alpha \rho_k H^{-1} r(x^k) - \beta F(x^{k+1}), \tag{23}$$

where

$$\rho_k = \|r(x^k)\|^2 / r(x^k)^T H^{-1} r(x^k).$$

The iteration (23) characterizes the PC-type methods for linear and nonlinear variational inequalities, which have been studied extensively, for instance, in Refs. 13–16.

Remark 2.4. For the affine variational inequality problem,

$$x^* \in X, \qquad (x - x^*)^T (Mx^* + q) \geq 0, \quad \text{for all } x \in X, \tag{24}$$

where $q \in R^n$ and M is a $n \times n$ positive semidefinite matrix. We can set $K = G(X)$ and Algorithm 2.1 can be written as a simple iterative scheme. Actually, since

$$G(x) = x + \beta(Mx + q)$$

is an affine mapping, the range $G(X)$ is a closed convex set, and the solution of the equation $G(x) = y^*$ is given by

$$x^{k+1} = (\beta M + I)^{-1}(y^* - \beta q). \tag{25}$$

Let

$$P^* = (I + \beta M)^T H (I + \beta M),$$

and let $K = G(X)$; it is easy to show that Algorithm 2.1 is reduced to the following iteration:

$$x^{k+1} = P_{P^*,X}[x^k - \alpha(H(I + \beta M))^{-1} r(x^k)], \tag{26}$$

where $P_{P^*,X}$ is the projection on X with respect to the P^*-norm.

In particular, if $X = R^n$, we have

$$x^{k+1} = x^k - \alpha[H(I + \beta M)]^{-1}(Mx^k + q), \tag{27}$$

which is an iterative scheme for the linear equation $Mx + q = 0$.

Remark 2.5. For the affine problem (24), there exists another equivalent two-mapping problem VI(X, R, W) on which the aforementioned methods and related results can be applied. Actually, replacing $F(x)$ by $Mx + q$ in (6), we have (see also Ref. 14)

$$(x - x^*)^T (I + \beta M^T) r(x) \geq \|r(x)\|_2^2, \qquad x^* \in X^*(F).$$

Therefore,

$$(x - x^*)^T (I + \beta M^T) r(x) \geq (1/c) \|(I + M^T) r(x)\|_2^2,$$

where

$$c \geq \|I+M^T\|_2^2.$$

Let

$$W(x)=cx \quad \text{and} \quad R(x)=(I+M^T)r(x);$$

then, the above inequality can be written as

$$[W(x)-W(x^*)]^T R(x) \geq \|R(x)\|_2^2.$$

It is obvious that, besides VI(X, r, G), problem VI(X, R, W) is also equivalent to problem (24). Replacing $G(x)$ and $r(x)$ by $W(x)$ and $R(x)$, respectively, we obtain the following iteration:

$$x^{k+1}=P_{H,X}[x^k-\alpha c^{-1}H^{-1}(I+\beta M^T)r(x^k)]. \tag{28}$$

Such an iterative scheme is similar to projection-type methods for affine variational inequalities (Refs. 14–16).

3. Conclusions

Theorem 2.1 established in this paper characterizes an inherent property for any monotone VI(X, F), that is, each monotone VI(X, F) is equivalent to a two-mapping variational inequality problem. Developing algorithms to solve the equivalent problem efficiently may provide alternative methods that are different from the previous approaches for monotone VI(X, F). The proposed algorithm can be viewed as such a method for the equivalent problem. It should be pointed out that, except for the aforementioned several cases (Remark 2.2 through Remark 2.5), our algorithms need further improvement with respect to implementability. We believe that it is worthwhile to find more efficient algorithms for the equivalent problem.

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