JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 101, No. 2, pp. 475-495, MAY 1999

Exceptional Families and Existence Theorems for Variational Inequality Problems[1]

Y. B. Zhao,[2] J. Y. Han,[3] and H. D. Qi[4]

Communicated by Z. Q. Luo

Abstract. This paper introduces the concept of exceptional family for nonlinear variational inequality problems. Among other things, we show that the nonexistence of an exceptional family is a sufficient condition for the existence of a solution to variational inequalities. This sufficient condition is weaker than many known solution conditions and it is also necessary for pseudomonotone variational inequalities. From the results in this paper, we believe that the concept of exceptional families of variational inequalities provides a new powerful tool for the study of the existence theory for variational inequalities.

Key Words. Variational inequalities, convex programming, complementarity problems, exceptional families, existence theorems.

1. Introduction

The finite-dimensional variational inequality problem (VIP) has been studied extensively in the literature because of its successful applications in many fields such as the traffic equilibria, spatial price equilibria, prediction of interregional commodity flows, solution of Nash equilibria, and Walrasian equilibrium model. A survey of theory, algorithms, and applications of the problem can be found in the review paper by Harker and Pang (Ref. 1). The development of existence theorems has played a very important role in theory, algorithms, and applications of VIP. A large number of existence

[1]The authors are indebted to two anonymous referees for suggestions that have considerably improved this article. One of the referees points out that the proper condition in Theorem 3.4 in this paper cannot be replaced by a weakly proper condition.
[2]Researcher, Institute of Applied Mathematics, Chinese Academy of Sciences, Beijing, China.
[3]Professor, Institute of Applied Mathematics, Chinese Academy of Sciences, Beijing, China.
[4]Researcher, Institute of Computational Mathematics and Scientific/Engineering Computing, Chinese Academy of Sciences, Beijing, China.

conditions have been developed in the literature including those of Hartman and Stampacchia (Ref. 2), Cottle (Ref. 3), Eaves (Refs. 4 and 5), Karamardian (Refs. 6–9), Moré (Refs. 10 and 11), Kojima (Ref. 12), Habetler and Price (Ref. 13), and Pang (Ref. 14). Because of the diversity of results, it is not possible to list them all.

In this paper, we investigate the existence of a solution to VIP via a new analysis method that is quite different from previous methods. Our method is motivated by the pioneer works of Smith (Ref. 15) and of Isac, Bulavski, and Kalashnikov (Ref. 16). Smith (Ref. 15) introduced the concept of exceptional sequence for continuous functions, used it to study the existence of a solution to the complementarity problem, and applied his results to spatial price equilibrium. Harker (Ref. 17) gave another application of the Smith results; he presented an alternative proof of the existence of a solution to a network equilibrium problem. Independently, Isac also presented the concept of exceptional sequence under the name of opposite radial sequence (unpublished note). Recently, Isac, Bulavski, and Kalashnikov (Ref. 16) generalized the Smith concept and introduced the concept of exceptional family of elements for continuous functions, applying it to the study of explicit, implicit, and general order complementarity problems. However, it seems difficult to apply the concept of exceptional sequence (family of elements) for continuous functions to general variational inequality problems, where the feasible set is a general closed convex set instead of a closed convex cone.

In this paper, we develop a new concept of exceptional family for VIP. This concept can be viewed as a generalization of the notions introduced in Refs. 15 and 16 in connection with complementarity problems. By means of our concept, we show a deep property of VIP; that is, each VIP has either a solution or an exceptional family. Therefore, the nonexistence of an exceptional family for VIP is a sufficient condition for the existence of a solution to VIP. It is shown that this sufficient condition is weaker than many well-known sufficient conditions developed in the literature. Some new sufficient conditions that assure the nonexistence of an exceptional family are also derived. It is easy to see that, by means of an exceptional family, the proofs of these results are very simple.

In Section 2, we present some properties of VIP, introduce the concept of exceptional family, and prove an essential result. In Section 3, we discuss the conditions under which a VIP does not possess an exceptional family. Conclusions are drawn in Section 4.

2. Exceptional Families for Variational Inequality Problems

To begin, let us state the problem under investigation. A finite-dimensional variational inequality, denoted by VI(X, F), is to find a vector $x^* \in X$

such that

$$(x-x^*)^T F(x^*) \geq 0, \qquad \forall x \in X, \tag{1}$$

where X is a nonempty closed convex subset of R^n and F is a mapping from R^n into itself. It is easy to see that, when X is the nonnegative orthant, denoted by R^n_+, then VI(X, F) reduces to the following nonlinear complementarity problem (NCP(F)):

$$x \geq 0, \qquad F(x) \geq 0, \qquad x^T F(x) = 0. \tag{2}$$

In this paper, we restrict the feasible set X to the following form:

$$X = \{x \in R^n: g_i(x) \leq 0,\ i = 1, \ldots, m;\ h_j(x) = 0,\ j = 1, \ldots, l\}, \tag{3}$$

where g_i and h_j are assumed to be convex and affine real-valued differentiable functions. Let

$$g(x) = (g_1(x), \ldots, g_m(x))^T, \qquad h(x) = (h_1(x), \ldots, h_l(x))^T.$$

Then, (3) can be written as

$$X = \{x \in R^n: g(x) \leq 0,\ h(x) = 0\}. \tag{4}$$

Let $\nabla g(x)$ and $\nabla h(x)$ denote the Jacobian matrices of the mappings g and h. In addition, we assume that the Slater constraint qualification holds for X; i.e., there exists a vector $x^0 \in X$ such that

$$g(x^0) < 0, \qquad h(x^0) = 0.$$

It is known that VIP has a solution provided that the set X is bounded. Therefore, we assume throughout the paper that X is unbounded. In what follows, we aim at developing a new tool for the existence theory for VI(X, F) defined by (1) and (3). To accomplish this, we first discuss some properties of the projection operator on a convex set.

Let

$$B_\alpha = \{x \in R^n: \|x\| \leq \alpha\}$$

be the closed Euclidean ball with radius $\alpha > 0$. Unless the contrary is explicitly stated, $\|\cdot\|$ denotes the Euclidean norm. Denote the intersection between the set X and B_α by X_α, i.e.,

$$X_\alpha = X \cap B_\alpha = \{x \in R^n: g(x) \leq 0,\ h(x) = 0,\ \|x\| \leq \alpha\}. \tag{5}$$

Since X is a closed convex set, X_α is a closed convex compact set provided that $X_\alpha \neq \emptyset$. It is easy to see that, if X satisfies the Slater constraint qualification, then there is a scalar α_0 such that, for all $\alpha \geq \alpha_0$, the set X_α also satisfies the Slater constraint qualification.

It is well known that, for any $z\in R^n$, the projection $P_{X_\alpha}(z)$ on the set X_α is the unique solution to the following problem:

$$\min\{\|x-z\|: x\in X_\alpha\}. \tag{6}$$

The following lemma, characterizing some properties of the projection operator, plays a very important role in introducing the concept of exceptional family for VI(X, F).

Lemma 2.1. Let $x\in X_\alpha$ and $z\in R^n$. Then, $x=P_{X_\alpha}(z)$ if, and only if, there exist some scalar $\mu\geq 0$ and two vectors $\lambda=(\lambda_1, \ldots, \lambda_m)^T\in R^m_+$ and $u=(u_1, \ldots, u_l)^T\in R^l$ such that

$$2[(1+\mu)x-z]+\nabla g(x)^T\lambda+\nabla h(x)^T u=0, \tag{7}$$

$$\lambda_i g_i(x)=0, \qquad i=1, \ldots, m, \tag{8}$$

$$\mu(\|x\|^2-\alpha^2)=0. \tag{9}$$

Proof. By the property of the projection operator, $x=P_{X_\alpha}(z)$ if, and only if, x is the unique solution to the optimization problem (6), which can be formulated equivalently as the following convex program:

$$\min_{x\in R^n}\{\|x-z\|^2: g(x)\leq 0,\ h(x)=0,\ \|x\|^2\leq\alpha^2\}. \tag{10}$$

By the aforementioned assumption on X, the Slater constraint qualification holds for the above convex program. Hence, the Karush–Kuhn–Tucker conditions completely characterize the solution of this convex program. In a word, $x=P_{X_\alpha}(z)$ if, and only if, x is the Karush–Kuhn–Tucker point of (10); namely, there exist vectors $\lambda\in R^m_+$ and $u\in R^l$ and a scalar $\mu\geq 0$ such that

$$2(x-z)+\nabla g(x)^T\lambda+\nabla h(x)^T u+2\mu x=0, \tag{11}$$

$$g(x)\leq 0, \qquad h(x)=0, \qquad \|x\|^2\leq\alpha^2, \tag{12}$$

$$\lambda_i g_i(x)=0, \qquad i=1, \ldots, m, \tag{13}$$

$$\mu(\|x\|^2-\alpha^2)=0. \tag{14}$$

Note that $x\in X_\alpha$; therefore, (11) through (14) are equivalent to (7) through (9). This establishes the lemma. □

By Proposition 2.3 in Ref. 1, x^* solves problem VI(X, F) if, and only if, x^* is a fixed point to the mapping

$$H(x)=P_X(x-F(x));$$

i.e.,

$$x^* = P_X(x^* - F(x^*)). \tag{15}$$

The next result characterizes some property of $P_X(\cdot)$. In fact, replacing X_α by X in the above proof of Lemma 2.1, it is not difficult to obtain the following result.

Lemma 2.2. Let $x \in X$ and $z \in R^n$. Then, $x = P_X(z)$ if, and only if, there exist two vectors $\lambda \in R^m_+$ and $u \in R^l$ such that

$$2(x - z) + \nabla g(x)^T \lambda + \nabla h(x)^T u = 0, \tag{16}$$

$$\lambda_i g_i(x) = 0, \qquad i = 1, \ldots, m. \tag{17}$$

Let $z = x^* - F(x^*)$ and $x = x^*$. By (15), an immediate consequence of the above result is the following corollary.

Corollary 2.1. x^* is a solution to the variational inequality problem VI(X, F) if, and only if, there exist two vectors $\lambda^* \in R^m_+$ and $u^* \in R^l$ such that

$$F(x^*) = -(\nabla g(x^*)^T \lambda^* + \nabla h(x^*)^T u^*)/2, \tag{18}$$

$$\lambda_i^* g_i(x^*) = 0, \qquad i = 1, \ldots, m. \tag{19}$$

Similarly, x^α solves VI(X_α, F) if and only if

$$x^\alpha = P_{X_\alpha}(x^\alpha - F(x^\alpha)). \tag{20}$$

Replacing z and x in Lemma 2.1 by $x^\alpha - F(x^\alpha)$ and x^α, we have the following result.

Corollary 2.2. x^α solves the variational inequality problem VI(X_α, F) if, and only if, there exist two vectors $\lambda^\alpha \in R^m_+$ and $u^\alpha \in R^l$ and a scalar $\mu^\alpha \geq 0$ such that

$$F(x^\alpha) = -\mu^\alpha x^\alpha - (\nabla g(x^\alpha)^T \lambda^\alpha + \nabla h(x^\alpha)^T u^\alpha)/2, \tag{21}$$

$$\lambda_i^\alpha g_i(x^\alpha) = 0, \qquad i = 1, \ldots, m, \tag{22}$$

$$\mu^\alpha(\|x^\alpha\|^2 - \alpha^2) = 0. \tag{23}$$

Motivated by the above observation, we define the concept of exceptional family for VI(X, F) as follows.

Definition 2.1. Let F be a mapping from X into R^n; the feasible set X is defined by (3). We say that a set of points $\{x^\alpha\}_{\alpha \to \infty} \subset X$ is an exceptional family for the variational inequality VI(X, F), if $\|x^\alpha\| \to \infty$ as $\alpha \to \infty$, and for each α there exist some positive scalar $\mu^\alpha > 0$ and two vectors $\lambda^\alpha \in R^m_+$ and $u^\alpha \in R^l$ such that

$$F(x^\alpha) = -\mu^\alpha x^\alpha - (\nabla g(x^\alpha)^T \lambda^\alpha + \nabla h(x^\alpha)^T u^\alpha)/2, \tag{24}$$

$$\lambda_i^\alpha g_i(x^\alpha) = 0, \qquad i = 1, \ldots, m. \tag{25}$$

Remark 2.1. It is not difficult to see that the above concept is a generalization of that introduced by Smith (Ref. 15) and by Isac, Bulavski, and Kalashnikov (Ref. 16). Indeed, when $g(x) = -x$ and there exists no system $h(x) = 0$, (24) and (25) reduce to

$$F_i(x^\alpha) = -\mu^\alpha x_i^\alpha, \qquad \text{if } x_i^\alpha > 0, \tag{26}$$

$$F_i(x^\alpha) \geq 0, \qquad \text{if } x_i^\alpha = 0. \tag{27}$$

This is the concept introduced in Ref. 15 under the name of exceptional sequence and that introduced in Ref. 16 under the name of exceptional family of elements. It is shown (Refs. 15 and 16) that a nonlinear complementarity problem has either a solution or an exceptional family. The following Theorem 2.1 extends this result to general VIP by using the new notion of exceptional family for VI(X, F). To show the result, we make use of the following lemma, which describes a relation between the variational inequality VI(X, F) on the unbounded set X and VI(X_α, F) on $X_\alpha \subset X$.

Lemma 2.3. See Ref. 18. Let F be a continuous mapping from X into R^n. Then, VI(X, F) has a solution if, and only if, there exists some $\alpha > 0$ such that $x^\alpha \in X_\alpha$ is a solution to VI(X_α, F) with $\|x^\alpha\| < \alpha$.

Theorem 2.1. Suppose that F is a continuous mapping from X into R^n. Then, the nonlinear variational inequality VI(X, F), has either a solution or an exceptional family.

Proof. Suppose that problem VI(X, F) has no solution. We show that VI(X, F) has an exceptional family. In fact, since VI(X, F) has no solution, from Lemma 2.3, there exists no $\alpha > 0$ such that $x^\alpha \in X_\alpha$ is a solution to VI(X_α, F) and $\|x^\alpha\| < \alpha$. Since X_α is a closed convex compact set, problem VI(X_α, F) has at least one solution by Theorem 3.1 in Ref. 1. Therefore, there exists a sequence $\{x^\alpha\}_{\alpha\to\infty}$ with the following property: For each α, x^α solves problem VI(X_α, F) and $\|x^\alpha\| = \alpha$. We show that $\{x^\alpha\}_{\alpha\to\infty}$ is an exceptional family for VI(X, F).

Since x^α solves VI(X_α, F), by Corollary 2.2, there exists two vectors $\lambda^\alpha \in R^m_+$ and $u^\alpha \in R^l$ and a scalar $\mu^\alpha \geq 0$ such that (21) and (22) hold. By

Definition 2.1, it suffices to show that $\mu^\alpha > 0$ holds for all α. If $\mu^\alpha = 0$ for some $\alpha > 0$, then (21) and (22) reduce to

$$F(x^\alpha) = -(\nabla g(x^\alpha)^T \lambda^\alpha + \nabla h(x^\alpha)^T u^\alpha)/2,$$

$$\lambda_i^\alpha g_i(x^\alpha) = 0, \qquad i = 1, \ldots, m.$$

By Corollary 2.1, x^α is a solution to the variational inequality problem VI(X, F). This is in contradiction with our assumption at the beginning of the proof. □

The following corollary is an immediate consequence of the above result.

Corollary 2.3. Let F be a continuous function from X into R^n. If the variational inequality VI(X, F) has no exceptional family, then VI(X, F) has at least one solution.

The above corollary presents a new sufficient condition for the existence of a solution to VI(X, F). The concept of exceptional family makes it possible for us to investigate the existence property of VI(X, F) via studying the nonexistence of an exceptional family for VI(X, F). From the above results, it is of interest and important to know when VI(X, F) has no exceptional family. The next section is devoted to discussing this question. Our results show that this sufficient condition is weaker than many well-known conditions developed in the literature.

3. Conditions for Nonexistence of an Exceptional Family

Theorem 3.1. Let F be a continuous mapping from X into R^n. If there exists some $x^0 \in X$ such that the set

$$X(x^0) = \{x \in X : (x - x^0)^T F(x) < 0\}$$

is bounded (possibly empty), then the variational inequality problem VI(X, F) has no exceptional family.

Proof. Since $g_i(x)$, $i = 1, \ldots, m$ and $h_j(x)$, $j = 1, \ldots, l$, are convex and affine differentiable functions, the following inequalities hold:

$$g(x^0) \geq g(x) + \nabla g(x)(x^0 - x), \qquad \forall x \in X, \tag{28a}$$

$$h(x^0) = h(x) + \nabla h(x)(x^0 - x), \qquad \forall x \in X. \tag{28b}$$

We now assume that problem VI(X, F) has an exceptional family $\{x^\alpha\}_{\alpha\to\infty}\subset X$. We show that the set $X(x^0)$ is unbounded. By Definition 2.1, we have that $\|x^\alpha\|\to\infty$, as $\alpha\to\infty$, and that, for each α, there exist a scalar $\mu^\alpha>0$ and two vectors $\lambda^\alpha\in R^m_+$ and $u^\alpha\in R^l$ such that (24) and (25) hold. Noting that $g(x^0)\le 0$ and $h(x^0)=0$, by (28) and (25), we have

$$(\lambda^\alpha)^T\nabla g(x^\alpha)(x^0-x^\alpha)\le(\lambda^\alpha)^T(g(x^0)-g(x^\alpha))=(\lambda^\alpha)^Tg(x^0)\le 0, \quad (29)$$

$$(u^\alpha)^T\nabla h(x^\alpha)(x^0-x^\alpha)=(u^\alpha)^T(h(x^0)-h(x^\alpha))=0. \quad (30)$$

Therefore, by (24), (29), and (30), we have

$$\begin{aligned}&(x^\alpha-x^0)^TF(x^\alpha)\\&=(x^\alpha-x^0)^T[-\mu^\alpha x^\alpha-(\nabla g(x^\alpha)^T\lambda^\alpha+\nabla h(x^\alpha)^Tu^\alpha)/2]\\&=-\mu^\alpha(\|x^\alpha\|^2-(x^\alpha)^Tx^0)+(x^0-x^\alpha)^T(\nabla g(x^\alpha)^T\lambda^\alpha+\nabla h(x^\alpha)^Tu^\alpha)/2\\&\le-\mu^\alpha\|x^\alpha\|(\|x^\alpha\|-\|x^0\|). \quad (31)\end{aligned}$$

Since $\mu^\alpha>0$ and $\|x^\alpha\|\to\infty$, it follows that there exists a scalar α_0 such that, for all $\alpha\ge\alpha_0$, we have

$$(x^\alpha-x^0)^TF(x^\alpha)<0,$$

which implies that the unbounded sequence $\{x^\alpha\}_{\alpha\ge\alpha_0}$ is contained in the set $X(x^0)$. The proof is complete. □

From the above proof, the conclusion of the above theorem remains valid if the condition of the theorem is replaced by "the set $X(x^0)=\{x\in X: (x-x^0)^TF(x)\le 0\}$ is bounded." Since all the solutions are contained in the set $X(x^0)$, the solution set of VI(X, F) is compact in this case. By Theorems 2.1 and 3.1, we rederived the existence theorem due to Harker and Pang (Ref. 1, Theorem 3.3).

It is not difficult to see that Theorem 3.1 can be stated equivalently as follows.

Corollary 3.1. Let $F: X\to R^n$ be a continuous function. If there exists a vector $x^0\in X$ such that, for each sequence $\{x^\alpha\}_{\alpha>0}\subset X$ with the property $\|x^\alpha\|\to\infty$, F satisfies the condition

$$(x^\alpha-x^0)^TF(x^\alpha)\ge 0, \qquad \text{for some } \alpha \text{ with the property } \|x^\alpha\|>\|x^0\|,$$

then VI(X, F) has no exceptional family.

Definition 3.1. We say that $F: R^n \to R^n$ satisfies the Karamardian condition on X if there exists a nonempty bounded subset D of X such that, for every $x \in X \setminus D$, there is a $y \in D$ such that

$$(x-y)^T F(x) \geq 0.$$

The following result shows that the Karamardian condition implies the nonexistence of an exceptional family.

Theorem 3.2. Let F be a continuous mapping from $R^n \to R^n$. If F satisfies the Karamardian condition on X, then the variational inequality VI(X, F) has no exceptional family.

Proof. Suppose that VI(X, F) has an exceptional family $\{x^\alpha\}_{\alpha \to \infty} \subset X$. We show that, for every bounded set $D \subset X$, there exists a scalar $\alpha > 0$ such that $x^\alpha \in X \setminus D$ and

$$(x^\alpha - y)^T F(x^\alpha) < 0, \qquad \forall y \in D.$$

That is, the function F does not satisfy the Karamardian condition on X.

Let D be an arbitrary bounded subset of X. For each x^α and $y \in D$, replacing x^0 by y and via the same proof of Theorem 3.1, we have

$$(x^\alpha - y)^T F(x^\alpha) \leq -\mu^\alpha \|x^\alpha\| (\|x^\alpha\| - \|y\|). \tag{32}$$

Since D is a bounded set, there exists a positive scalar $\beta > 0$ such that

$$\|y\| \leq \beta, \qquad \forall y \in D.$$

Since $\{x^\alpha\}_{\alpha \to \infty}$ is an exceptional family of VI(X, F),

$$\|x^\alpha\| \to \infty, \qquad \text{as } \alpha \to \infty.$$

Therefore, there exists some $x^\alpha \in X \setminus D$ such that $\|x^\alpha\| > \beta$. For this x^α, (32) implies that

$$(x^\alpha - y)^T F(x^\alpha) < 0.$$

The proof is complete. □

Definition 3.2. See Refs. 9, 19. A map F is said to be:

(a) pseudomonotone on the set X if, for every distinct pair $x, y \in X$, we have that

$$(y-x)^T F(x) \geq 0 \quad \text{implies} \quad (y-x)^T F(y) \geq 0;$$

(b) quasimonotone on X if, for every distinct pair $x, y \in X$, we have that

$$(y-x)^T F(x) > 0 \quad \text{implies} \quad (y-x)^T F(y) \geq 0. \tag{33}$$

It should be noted that, among the monotone concepts, the quasimonotone property is the weakest. A pseudomonotone map is quasimonotone, but the converse is not true; for example, when $X = R$, $F(x) = x^4$ is quasimonotone, but not pseudomonotone. Other examples can be found in Ref. 19. Under a pseudomonotonicity assumption on F, Karamardian (Ref. 9) developed an existence result for complementarity problems where X is a solid closed convex cone in R^n. This result was extended to a Hilbert space by Cottle and Yao (Ref. 23) and was stated for a variational inequality by Harker and Pang (Ref. 1, Theorem 3.4). Furthermore, Hadjisavvas and Schaible (Ref. 20) generalized this result to a quasimonotone variational inequality in a Banach space. The next theorem proves that the Hadjisavvas and Schaible condition, restricted to VI(X, F), implies the condition "without an exceptional family." Let X^* denote the dual cone of X, i.e.,

$$X^* = \{ y \in R^n : y^T x \geq 0, \text{ for all } x \in X \}.$$

Let int($\cdot$) denote the interior of a set.

Theorem 3.3. Let F be a continuous mapping from X into R^n. Suppose that F is quasimonotone on the feasible set X. If there exists a vector $x^0 \in X$ such that $F(x^0) \in \text{int}(X^*)$, then problem VI($X, F$) has no exceptional family.

Proof. Let

$$\text{con}(X) = \{ rx : r \in R_+, x \in X \};$$

i.e., con(X) is the cone generalized by X. We denote by cl[con(X)] the closure of the set con(X). We now show that the condition $F(x^0) \in \text{int}(X^*)$ supports the following two assertions:

(A1) for all nonzero vector $x \in \text{cl}[\text{con}(X)]$, we have $x^T F(x^0) > 0$;
(A2) the set

$$L(x^0, X) = \{ x \in X : x^T F(x^0) \leq (x^0)^T F(x^0) \}$$

is bounded.

It is easy to see that

$$X^* = (\text{cl}[\text{con}(X)])^*;$$

hence,

$$F(x^0) \in \text{int}(\text{cl}[\text{con}(X)])^*.$$

Let

$$0 \neq \hat{x} \in \text{cl}[\text{con}(X)].$$

We have that

$$\hat{x}^T F(x^0) \geq 0, \qquad F(x^0) - \beta \hat{x} \in (\text{cl}[\text{con}(X)])^*,$$

for sufficiently small $\beta > 0$. Assume that

$$\hat{x}^T F(x^0) = 0.$$

Then,

$$0 \leq (F(x^0) - \beta \hat{x})^T \hat{x} = F(x^0)^T \hat{x} - \beta \|\hat{x}\|^2 < 0,$$

a contradiction. Hence, assertion (A1) holds. Now, we assume that the set $L(x^0, X)$ is unbounded. Then, there exists a sequence $\{x^k\} \subset L(x^0, X)$ with the property $\|x^k\| \to \infty$. Without loss of generality, we suppose that $[x^k/\|x^k\| \to \bar{x}$. Since

$$x^k/\|x^k\| \in \text{cl}[\text{con}(X)],$$

we have that

$$\bar{x} \in \text{cl}[\text{con}(X)].$$

From assertion (A1) and $\|\bar{x}\| = 1$, we have

$$0 < F(x^0)^T \bar{x} = \lim_{k \to \infty} F(x^0)^T x^k/\|x^k\| \leq \lim_{k \to \infty} F(x^0)^T x^0/\|x^k\| = 0,$$

a contradiction. Hence, assertion (A2) also holds.

Since the set $L(x^0, X)$ is bounded, for each sequence $\{x^\alpha\}_{\alpha > 0} \subset X$ with the property $\|x^\alpha\| \to \infty$ as $\alpha \to \infty$, there is an index $\alpha_0 > 0$ such that, for all $\alpha \geq \alpha_0$, we have

$$(x^\alpha)^T F(x^0) > (x^0)^T F(x^0).$$

By the quasimonotonicity of F, we have

$$(x^\alpha - x^0)^T F(x^\alpha) \geq 0,$$

which implies that problem VI(X, F) has no exceptional family by using Corollary 3.1. □

Definition 3.3.

(D1) A point x^0 is said to be strictly feasible to VI(X, F) if $x^0 \in X$ and $F(x^0) \in \text{int}(X^*)$.

(D2) A mapping $F: X \to R^n$ is said to be proper at the point $x^0 \in X$ if the set

$$L(x^0, X) = \{x \in X: (x - x^0)^T F(x^0) \leq 0\} \tag{34}$$

is bounded.

From the proof of Theorem 3.3, it is shown that the strictly feasibility condition implies that the set $L(x^0, X)$ is bounded; i.e., F is proper at x^0. In fact, from the end of the proof of Theorem 3.3, we have shown the next result.

Theorem 3.4. Let $F: X \to R^n$ be a continuous and quasimonotone function. If there exists a point $x^0 \in X$ such that F is proper at x^0, then problem VI(X, F) has no exceptional family.

Definition 3.4. The mapping $F: X \to R^n$ is said to be weakly proper at the point $x^0 \in X$ if, for each sequence $\{x^\alpha\} \subset X$ with the property $\|x^\alpha\| \to \infty$ as $\alpha \to \infty$, there exists some α such that

$$(x^\alpha - x^0)^T F(x^0) \geq 0, \qquad \|x^\alpha\| > \|x^0\|. \tag{35}$$

The properness condition implies the weakly properness condition, but the converse is, in general, not true. In fact, when $x^0 \in X$ such that $F(x^0) = 0$, then

$$(x^\alpha - x^0)^T F(x^0) \geq 0, \qquad \text{for all } \|x^\alpha\| > \|x^0\|;$$

therefore, F is weakly proper at x^0. However, in this case, $L(x^0, X) = X$, which is unbounded. The following example shows that the properness condition in Theorem 3.4 cannot be replaced by the weak properness condition.

Example 3.1. Consider VI(X, F) with

$$F = x^4, \qquad X = \{x \in R: x \leq 0\};$$

F is quasimonotone and continuous on X, F is weakly proper at $x^0 = 0$. However, it is easy to show that VI(X, F) has an exceptional family $\{x^\alpha\}$, where

$$x^\alpha = -\alpha, \qquad \alpha \geq 0, \qquad \alpha \to \infty.$$

We have the following result concerning pseudomonotone problems under the weak properness assumption.

Theorem 3.5. Let F be a continuous pseudomonotone mapping from X into R^n. If there exists a point $x^0 \in X$ such that F is weakly proper at x^0, then VI(X, F) has no exceptional family.

Proof. Since F is weakly proper at x^0, for each sequence $\{x^\alpha\} \subset X$ with the property $\|x^\alpha\| \to \infty$ as $\alpha \to \infty$, there exists some α such that $\|x^\alpha\| > \|x^0\|$ and

$$(x^\alpha - x^0)^T F(x^0) \geq 0.$$

By the pseudomonotonicity of F, we have

$$(x^\alpha - x^0)^T F(x^\alpha) \geq 0,$$

which implies that VI(X, F) has no exceptional family by Corollary 3.1. □

While the strict feasibility condition implies the weak properness condition, it should be noted that feasibility [namely, $x^0 \in X$ such that $F(x^0) \in X^*$] does not. Indeed, even for a monotone mapping F, feasibility cannot assure the existence of a solution to VI(X, F). Megiddo (Ref. 21) gave the following example to show that this is the case.

Consider the function F: $R^2 \to R^2$ defined by

$$F(x_1, x_2) = (2x_1 x_2 - 2x_2 + 1, \ -(x_1 - 1)^2)^T \tag{36}$$

and the set $X = R^2_+$. F is monotone on R^2_+ and the set of feasible solutions is

$$\{x: x_1 = 1, x_2 \geq 0\},$$

but the complementarity problem has no solution, since $F_1(x) = 1$.

Now, we point out that the above example does not satisfy the aforementioned weak properness condition (35). It suffices to show that, for any $x^0 = (x^0_1, x^0_2) \in R^2_+$, there exists a sequence $\{x^\alpha\} \subset R^2_+$ such that

$$(x^\alpha - x^0)^T F(x^0) < 0, \qquad \text{for every } \alpha \text{ with the property } \|x^\alpha\| > \|x^0\|;$$

i.e., to show that the function (36) has no weakly proper point x^0. Actually,

$$(x^\alpha - x^0)^T F(x^0) = (x^\alpha_1 - x^0_1)(2x^0_1 x^0_2 - 2x^0_2 + 1) - (x^\alpha_2 - x^0_2)(x^0_1 - 1)^2.$$

If $x_1^0 \neq 1$, let

$$x^\alpha = (x_1^0, x_2^0 + \alpha), \qquad \forall \alpha > 0.$$

Then,

$$(x^\alpha - x^0)^T F(x^0) = -\alpha(x_1^0 - 1)^2 < 0, \qquad \text{for all } \alpha > 0.$$

If $x_1^0 = 1$, let

$$x^\alpha = (1/2, x_2^0 + \alpha), \qquad \forall \alpha > 0.$$

Then,

$$(x^\alpha - x^0)^T F(x^0) = -(1/2) < 0.$$

Hence, this example does not satisfy our weak properness condition. In fact, the weak properness condition is also a necessary condition for a pseudomonotone variational inequality to possess a solution (see Corollary 3.4).

Definition 3.5. See Ref. 22. A mapping $F: X \to R^n$ is said to be:

(a) a P-function on X if

$$\max_{1 \le i \le n} [F_i(x) - F_i(y)](x_i - y_i) > 0, \qquad \forall x, y \in X,\ x \neq y; \tag{37}$$

(b) a uniform P-function on X if there exists a scalar $c > 0$ such that

$$\max_{1 \le i \le n} [F_i(x) - F_i(y)](x_i - y_i) \ge c\|x - y\|^2, \qquad \forall x, y \in X. \tag{38}$$

For nonlinear P-functions, VI(X, F) is possibly unsolvable. Moré (Ref. 10) gave an example to show that this is the case. Therefore a P-function cannot assure the nonexistence of an exceptional family. In the next section, among other things we show that a uniform P-function implies the nonexistence of an exceptional family if some conditions are imposed on $g(x)$ and $h(x)$; see Theorem 3.6.

Let $\|\cdot\|_\infty$ denote the max norm, i.e.,

$$\|x\|_\infty = \max_{1 \le i \le n} |x_i|.$$

We now introduce the concept of uniform diagonally dominant function.

Definition 3.6. $F: X \to R^n$ is said to be a uniform diagonally dominant function with respect to X if, for any distinct x, y in X and any index k with $|x_k - y_k| = \|x - y\|_\infty$, there exists a positive scalar c such that

$$(x_k - y_k)(F_k(x) - F_k(y)) \ge c\|x - y\|_\infty^2. \tag{39}$$

It is not difficult to show that the above concept is a generalization of the linear map $F(x) = Mx$, where M is a strictly diagonally dominant matrix with positive diagonal entries (Ref. 24).

Let g_i, $i = 1, \ldots, m$, be a convex function from R into R and let h_j, $j = 1, \ldots, l$, be a linear function from R into R. Given the indexes k_i and j_i satisfying

$$1 \leq k_1 < k_2 < \cdots < k_m \leq n, \qquad 1 \leq j_1 < j_2 < \cdots < j_l \leq n,$$

for any $x = (x_1, \ldots, x_n)^T \in R^n$, let $g(x)$: $R^n \to R^m$, $h(x)$: $R^n \to R^l$ be defined as follows

$$\begin{aligned} g(x) &= (g_1(x_{k_1}), \ldots, g_m(x_{k_m}))^T, \\ h(x) &= (h_1(x_{j_1}), \ldots, h_l(x_{j_l}))^T. \end{aligned} \tag{40}$$

Then, it is easy to see that

$$K = \{x \in R^n : g(x) \leq 0, h(x) = 0\} \tag{41}$$

is a rectangular set in R^n, that is, the Cartesian product of n intervals $[t_1, t_2]$ in R, where t_1 or t_2 can be chosen as ∞. When $m = n$ and $g(x) = -x$, K is the nonnegative orthant. Conversely, it is easy to see that each rectangular set in R^n can be formulated in the form of (40) and (41).

Lemma 3.1. Let g: $R^n \to R^m$ and h: $R^n \to R^l$ be given as (40) and (41). Then, for any $\lambda \in R^m$, $u \in R^l$, $z \in R^n$, $x \in R^n$, we have

$$z_p(\nabla g(x)^T \lambda)_p = \begin{cases} 0, & \text{if } p \neq k_i (i = 1, \ldots, m), \\ \lambda_i (\nabla g(x) z)_i, & \text{if } p = k_i, \end{cases} \tag{42}$$

$$z_p(\nabla h(x)^T u)_p = \begin{cases} 0, & \text{if } p \neq j_i (i = 1, \ldots, l), \\ u_i (\nabla h(x) z)_i, & \text{if } p = j_i, \end{cases} \tag{43}$$

for $p = 1, \ldots, n$.

Proof. Let g_i' denote the derivative of g_i. It is obvious that

$$(\nabla g(x)^T \lambda)_p = \begin{cases} 0, & \text{if } p \neq k_i, \\ g_i'(x_{k_i}) \lambda_i, & \text{if } p = k_i, \end{cases}$$

for $p = 1, \ldots, n$, and

$$\nabla g(x) z = (g_1'(x_{k_1}) z_{k_1}, g_2'(x_{k_2}) z_{k_2}, \ldots, g_m'(x_{k_m}) z_{k_m})^T.$$

Therefore,

$$z_p(\nabla g(x)^T\lambda)_p = \begin{cases} 0, & \text{if } p \neq k_i, \\ z_{k_i} g_i'(x_{k_i})\lambda_i, & \text{if } p = k_i, \end{cases}$$

$$= \begin{cases} 0, & \text{if } p \neq k_i, \\ \lambda_i(\nabla g(x)z)_i, & \text{if } p = k_i. \end{cases}$$

Equation (43) can be shown by the same proof as above. □

Theorem 3.6. Let $F: X \to R^n$ be a continuous function and let X be a rectangular set in R^n. Let one of the following conditions hold:

(C1) F is a uniform diagonally dominant function with respect to X;
(C2) F is a uniform P-function with respect to X and $0 \in X$.

Then, the variational inequality VI(X, F) has no exceptional family.

Proof. Since each rectangular set can be represented in the form of (40) and (41), without loss of generality, we assume that X is given by (40) and (41).

(C1) Let y be a fixed vector in X. Suppose that there exists an exceptional family $\{x^\alpha\}_{\alpha\to\infty}$. By Definition 2.1, for each α, there are two vectors $\lambda^\alpha \in R^m_+$ and $u^\alpha \in R^l$ and a scalar $\mu^\alpha > 0$ such that (24) and (25) hold. For such α, by using (28) and (25), we have

$$(\lambda^\alpha)_i[\nabla g(x^\alpha)(x^\alpha - y)]_i \geq (\lambda^\alpha)_i(g_i(x^\alpha) - g_i(y)) \geq 0, \tag{44}$$

for all $i = 1, \ldots, m$, and

$$(u^\alpha)_i[\nabla h(x^\alpha)(x^\alpha - y)]_i = (u^\alpha)_i(h_i(x^\alpha) - h_i(y)) = 0, \tag{45}$$

for all $i = 1, \ldots, l$. By (44), (45), and Lemma 3.1, we deduce that

$$(x^\alpha - y)_i(\nabla g(x^\alpha)^T\lambda^\alpha)_i \geq 0, \qquad \forall i = 1, \ldots, n, \tag{46}$$

$$(x^\alpha - y)_i(\nabla h(x^\alpha)^T u^\alpha)_i = 0, \qquad \forall i = 1, \ldots, n. \tag{47}$$

Since $\{x^\alpha\}$ is an infinite sequence, there exists a subsequence $\{x^{\alpha_j}\}_{\alpha_j\to\infty}$ such that, for some fixed index k,

$$|x_k^{\alpha_j} - y_k| = \|x^{\alpha_j} - y\|_\infty, \qquad \forall \alpha_j.$$

Noticing that F is a uniform diagonally dominant function, we have

$$(x^{\alpha_j} - y)_k(F_k(x^{\alpha_j}) - F_k(y)) \geq c\|x^{\alpha_j} - y\|_\infty^2, \qquad \forall \alpha_j. \tag{48}$$

By using (24), (46), and (47), we have

$$\begin{aligned}&[F_k(x^{\alpha_j})-F_k(y)](x^{\alpha_j}-y)_k\\&=F_k(x^{\alpha_j})(x^{\alpha_j}-y)_k-F_k(y)(x^{\alpha_j}-y)_k\\&=[-\mu^{\alpha_j}x_k^{\alpha_j}-(\nabla g(x^{\alpha_j})^T\lambda^{\alpha_j}+\nabla h(x^{\alpha_j})^T u^{\alpha_j})_k/2]\\&\quad(x^{\alpha_j}-y)_k-F_k(y)(x^{\alpha_j}-y)_k\\&\le-\mu^{\alpha_j}x_k^{\alpha_j}(x^{\alpha_j}-y)_k-F_k(y)(x^{\alpha_j}-y)_k.\end{aligned}$$

Combining (48) and the above inequality yields

$$[-\mu^{\alpha_j}x_k^{\alpha_j}(x^{\alpha_j}-y)_k-F_k(y)(x^{\alpha_j}-y)_k]/\|x^{\alpha_j}-y\|_\infty^2\ge c.$$

However since $\|x^{\alpha_j}\|\to\infty$, the above inequality cannot hold, and we obtain a contradiction.

(C2) Suppose that there exists an exceptional family $\{x^\alpha\}_{\alpha\to\infty}$. Let $y=0$; there is a subsequence $\{x^{\alpha_j}\}$ such that, for some fixed index k, we have

$$[F_k(x^{\alpha_j})-F_k(0)]x_k^{\alpha_j}=\max_{1\le i\le n}[F_i(x^{\alpha_j})-F_i(0)]x_i^{\alpha_j},\qquad\forall\alpha_j.$$

From (38), we have

$$[F_k(x^{\alpha_j})-F_k(0)]x_k^{\alpha_j}\ge c\|x^{\alpha_j}\|^2.$$

Then, we derive a contradiction by the same argument as in (C1). We do not repeat this verification. □

The theorems established above are general enough to include as special cases nonlinear complementarity problems (NCP). Specializing the consequence of Theorem 3.6 to NCP(F), we have the following corollary.

Corollary 3.2. If F is a uniform P-function or a uniform diagonally dominant function, then NCP(F) has no exceptional family.

A function $F: X\to R^n$ is said to be coercive with respect to X if there exists some $x^0\in X$ such that

$$\lim_{x\in X,\|x\|\to\infty}[(F(x)-F(x^0))^T(x-x^0)]/\|x-x^0\|=+\infty. \tag{49}$$

Coercivity has played a very important role in the existence theory of VI(X, F); see Refs. 1, 2, 11, and 12. It is well-known (Ref. 1, Theorem 3.3) that VI(X, F) has a nonempty compact solution set if F is coercive with respect to X. In what follows, we will show that the sufficient condition "nonexistence of an exceptional family" for VI(X, F) is weaker than the coercivity condition. The result below can also be viewed as a generalization of Proposition 4.7 in Ref. 15 and Proposition 4 in Ref. 16.

Theorem 3.7. If F is a continuous and coercive function with respect to X, then there exists no exceptional family for the variational inequality problem VI(X, F).

Proof. Let $x^0 \in X$ satisfy (49). Assume that there is an exceptional family $\{x^\alpha\}_{\alpha\to\infty}$ for problem VI(X, F). Let $\lambda^\alpha \in R^m_+$ and $u^\alpha \in R^l$ and the scalar $\mu^\alpha > 0$ be defined as in Definition 2.1. From the proof of Theorem 3.1, we have that

$$(x^\alpha - x^0)^T F(x^\alpha) \leq -\mu^\alpha(\|x^\alpha\|^2 - (x^\alpha)^T x^0).$$

Also,

$$(x^\alpha - x^0)^T F(x^\alpha) \leq -\mu^\alpha[\|x^\alpha - x^0\|^2 + (x^\alpha - x^0)^T x^0].$$

Then, for sufficiently large α, we have

$$\begin{aligned}
&[F(x^\alpha) - F(x^0)]^T(x^\alpha - x^0)/\|x^\alpha - x^0\| \\
&\leq -\mu^\alpha[\|x^\alpha - x^0\|^2 + (x^\alpha - x^0)^T x^0]/\|x^\alpha - x^0\| \\
&\quad - F(x^0)^T(x^\alpha - x^0)/\|x^\alpha - x^0\| \\
&\leq -\mu^\alpha[\|x^\alpha - x^0\| - \|x^0\|] - (x^\alpha - x^0)^T F(x^0)/\|x^\alpha - x^0\| \\
&\leq \|F(x^0)\|.
\end{aligned}$$

This is a contradiction, since f is coercive with respect to X. □

For complementarity problems, Isac, Bulavski, and Kalashnikov (Ref. 16) gave an example to show that the condition "nonexistence of an exceptional family" cannot imply the coercivity condition.

Corollary 3.3. Assume that F satisfies one of the following conditions:

(C1) F is strongly monotone over X; i.e., there is a scalar $\alpha > 0$ such that

$$(F(x) - F(y))^T(x - y) \geq \alpha\|x - y\|^2, \qquad \forall x, y \in X;$$

(C2) $0 \in X$ and F is strongly copositive over X; i.e., there exists some scalar $\alpha > 0$ such that

$$(F(x) - F(0))^T x \geq \alpha\|x\|^2, \qquad \forall x \in X.$$

Then, VI(X, F) has no exceptional family.

Proof. If F satisfies one of the conditions (1) and (2), then F is coercive with respect to X. The corollary is an immediate consequence of Theorem 3.7. □

It should be noted that, while the condition "without an exceptional family" is a very weak sufficient condition for the existence of a solution to VI(X, F), in general, it is not necessary. Smith (Ref. 15) gave an example to show that NCP(F) can have both a solution and an exceptional family. We end this paper by pointing out that the condition "without an exceptional family" is also necessary for pseudomonotone VI(X, F) to have a solution and we prove that it is equivalent to the weak properness condition.

Theorem 3.8. Let $F: X \to R^n$ be a continuous pseudomonotone mapping. Then one and only one of the following alternatives holds:

(A1) VI(X, F) has a solution;
(A2) VI(X, F) has an exceptional family.

Proof. If (A1) holds, let x^* be a solution to VI(X, F). Then,

$$(x-x^*)^T F(x^*) \geq 0, \qquad \forall x \in X.$$

Since F is pseudomonotone, it follows that

$$(x-x^*)^T F(x) \geq 0, \qquad \forall x \in X.$$

Let $x^0 = x^*$. Then, the above relation implies that the condition of Corollary 3.1 holds; hence, VI(X, F) has no exceptional family; i.e., (A2) does not hold.

Conversely, if (A1) does not hold, then (A2) holds through Theorem 2.1. □

The following result establishes two new equivalent conditions for a pseudomonotone variational inequality VI(X, F) to have a solution. These conditions are quite different from the equivalent conditions developed by Cottle and Yao (Ref. 23), but restricted to VI(X, F).

Corollary 3.4. Let $F: X \to R^n$ be a continuous pseudomonotone mapping. Then, the following three conditions are equivalent:

(C1) VI(X, F) has a solution;
(C2) There exists a point $x^0 \in X$ such that F is weakly proper at x^0;
(C3) VI(X, F) has no exceptional family.

Proof. Suppose that VI(X, F) has a solution, denoted by x^*. Obviously, F is weakly proper at x^*. Hence, condition (C1) implies (C2). Condition (C2), implying (C3), is the consequence of Theorem 3.5. Condition (C3), implying (C1), follows from Corollary 2.1. □

4. Conclusions

The concept of an exceptional family for VI(X, F) introduced in this paper provides a new method for investigating the existence of a solution to VI(X, F). The nonexistence of an exceptional family implies the existence of a solution to VI(X, F). This new sufficient condition is shown to be weaker than many known sufficient conditions developed in the literature. We think that the concept of an exceptional family for VI(X, F) is a new interesting research direction in variational inequality problems.

References

1. HARKER, P. T., and PANG, J. S., *Finite-Dimensional Variational Inequality and Nonlinear Complementarity Problems: A Survey of Theory, Algorithms, and Applications*, Mathematical Programming, Vol. 48, pp. 161–220, 1990.
2. HARTMAN, P., and STAMPACCHIA, G., *On Some Nonlinear Elliptic Differentiable Functional Equation*, Acta Mathematica, Vol. 115, pp. 71–310, 1966.
3. COTTLE, R. W., *Nonlinear Programs with Positively Bounded Jacobians*, SIAM Journal on Applied Mathematics, Vol. 14, pp. 147–158, 1966.
4. EAVES, B. C., *On the Basic Theorem of Complementarity Problems*, Mathematical Programming, Vol. 1, pp. 68–75, 1971.
5. EAVES, B. C., *The Linear Complementarity Problem*, Management Science, Vol. 17, pp. 612–634, 1971.
6. KARAMARDIAN, S., *The Complementarity Problem*, Mathematical Programming, Vol. 2, pp. 107–129, 1972.
7. KARAMARDIAN, S., *An Existence Theorem for the Complementarity Problem*, Journal of Optimization Theory and Applications, Vol. 19, pp. 227–232, 1976.
8. KARAMARDIAN, S., *Generalized Complementarity Problems*, Journal of Optimization Theory and Applications, Vol. 8, pp. 161–167, 1971.
9. KARAMARDIAN, S., *Complementarity Problems over Cones with Monotone and Pseudomonotone Maps*, Journal of Optimization Theory and Applications, Vol. 18, pp. 445–454, 1976.
10. MORÉ, J. J., *Classes of Functions and Feasibility Conditions in Nonlinear Complementarity Problems*, Mathematical Programming, Vol. 6, pp. 327–338, 1974.
11. MORÉ, J. J., *Coercivity Conditions in Nonlinear Complementarity Problems*, SIAM Review, Vol. 16, pp. 1–16, 1974.

12. KOJIMA, M., *A Unification of Existence Theorems of the Nonlinear Complementarity Problem*, Mathematical Programming, Vol. 9, pp. 257-277, 1975.
13. HABETLER, G. J., and PRICE, A. L., *Existence Theorem for Generalized Nonlinear Complementarity Problems*, Journal of Optimization Theory and Applications, Vol. 4, pp. 223-239, 1971.
14. PANG, J. S., *Least-Element Complementarity Theory*, Ph.D. Dissertation, Department of Operations Research, Stanford University, Stanford, California, 1976.
15. SMITH, T. E., *A Solution Condition for Complementarity Problems with an Application to Spatial Price Equilibrium*, Applied Mathematics and Computation, Vol. 15, pp. 61-69, 1984.
16. ISAC, G., BULAVSKI, V., and KALASHNIKOV, V., *Exceptional Families, Topological Degree, and Complementarity Problems*, Journal of Global Optimization, Vol. 10, pp. 207-225, 1997.
17. HARKER, P. T., *A Note on the Existence of Traffic Equilibria*, Applied Mathematics and Computation, Vol. 18, pp. 277-283, 1986.
18. KINDERLEHRER, D., and STAMPACCHIA, G., *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, New York, 1980.
19. KARAMARDIAN, S., and SCHAIBLE, S., *Seven Kinds of Monotone Maps*, Journal of Optimization Theory and Applications, Vol. 66, pp. 37-46, 1990.
20. HADJISAVVAS, N., and SCHAIBLE, S., *Quasimonotone Variational Inequalities in Banach Space*, Journal of Optimization Theory and Applications, Vol. 90, pp. 95-111, 1996.
21. MEGIDDO, N., *A Monotone Complementarity Problem with Feasible Solutions but No Complementarity Solutions*, Mathematical Programming, Vol. 12, pp. 131-132, 1977.
22. MORÉ, J. J., and RHEINBOLDT, W. C., *On P- and S-Functions and Related Class of n-Dimensional Linear Mappings*, Linear Algebra and Its Applications, Vol. 6, pp. 45-68, 1973.
23. COTTLE, R. W., and YAO, J. C., *Pseudomonotone Complementarity Problems in Hilbert Space*, Journal of Optimization Theory and Applications, Vol. 75, pp. 281-295, 1992.
24. COTTLE, R. W., PANG, J. S., and STONE, K. E., *The Linear Complementarity Problem*, Academic Press, New York, New York, 1992.