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Enlarging neighborhoods of interior-point algorithms for linear programming via least values of proximity measure functions ☆

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Abstract

It is well known that a wide-neighborhood interior-point algorithm for linear programming performs much better in implementation than its small-neighborhood counterparts. In this paper, we provide a unified way to enlarge the neighborhoods of predictor–corrector interior-point algorithms for linear programming. We prove that our methods not only enlarge the neighborhoods but also retain the so-far best known iteration complexity and superlinear (or quadratic) convergence of the original interior-point algorithms. The idea of our methods is to use the global minimizers of proximity measure functions.

Keywords: Linear programming; Interior-point algorithms; Iteration complexity; Neighborhoods

1. Introduction

Consider the canonical linear programming problem:

$$\min\{c^T x:\ Ax = b,\ x \geqslant 0\},$$

where $c \in R^n$, $b \in R^m$, $A \in R^{m\times n}$ and $\operatorname{rank}(A) = m$. We assume that the problem has an interior-point, i.e., there exists a point (x^0, s^0, y^0) such that

$$Ax^0 = b, \quad A^T y^0 + s^0 = c, \quad x^0 > 0,\ s^0 > 0.$$

This assumption guarantees the existence of the central path on which most interior-point algorithms are based. There are two key factors that are closely related to the practical behavior of an interior-point algorithm, i.e., the search direction and the neighborhood of the central path. A large body of implementation shows that interior-point algorithms using wide neighborhoods perform much better than those counterparts based on small neighborhoods. Thus, how to enlarge the neighborhood of central path is an interesting and important issue for improving the efficiency of interior-point algorithms. This is why many authors have been studying the wide-neighborhood interior-point algorithms for optimization problems (see, e.g. [1,5,6,11,12,16,19,23,24]). A neighborhood of the central path is determined by

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some proximity measure function that is used to measure the distance between the point $(x, s) > 0$ and the central path. The following proximity measure functions are widely used in the literature of interior-point methods for linear programming [8,17,20,21]:

$$\delta_2(x, s, \mu) = \left\| \frac{xs}{\mu} - e \right\|_2,$$

$$\delta_\infty(x, s, \mu) = \left\| \frac{xs}{\mu} - e \right\|_\infty,$$

$$\delta_{\log}(x, s, \mu) = \frac{x^{\mathrm{T}} s}{2\mu} - \frac{n}{2} + \frac{n \log \mu}{2} - \frac{1}{2}\sum_{i=1}^{n} \log(x_i s_i),$$

$$\delta_\Phi(x, s, \mu) = \frac{1}{2}\left\| \sqrt{\frac{xs}{\mu}} - \sqrt{\frac{\mu}{xs}} \right\|_2.$$

In general cases, the parameter μ can be calculated at (x, s), and thus we can write μ as $\mu(x, s)$. For any given proximity measure function $\delta(x, s, \mu(x, s))$, the neighborhood of the central path can be defined as

$$\mathcal{N}(\tau) = \bigl\{(x, s) > 0\colon\ \delta\bigl(x, s, \mu(x, s)\bigr) \leqslant \tau\bigr\},$$

where τ is a given positive number. For most interior-point algorithms, the parameter $\mu > 0$ is set to be the duality gap $x^{\mathrm{T}} s/n$. However, we note that most proximity measure functions used in interior-point algorithms have global minimizers with respect to μ. For any given $(x, s) > 0$, let $\mu_*(x, s)$ denote the global minimizer of $\delta(x, s, \cdot)$, i.e.,

$$\delta\bigl(x, s, \mu_*(x, s)\bigr) \leqslant \delta(x, s, \mu) \quad \text{for all } \mu > 0.$$

Then the corresponding neighborhood is given by

$$\mathcal{N}^*(\tau) = \bigl\{(x, s) > 0\colon\ \delta\bigl(x, s, \mu_*(x, s)\bigr) \leqslant \tau\bigr\}.$$

Clearly, the neighborhood $\mathcal{N}^*(\tau)$ is larger than $\mathcal{N}(\tau)$ for any $\mu(x, s) \neq \mu_*(x, s)$, i.e., for a given proximity measure function $\delta(\cdot)$, the neighborhood $\mathcal{N}^*(\tau)$ is the largest neighborhood. This suggests that we may use the neighborhood $\mathcal{N}^*(\tau)$ to replace the original neighborhood $\mathcal{N}(\tau)$ of interior-point algorithms. The purpose of this paper is to adapt the predictor–corrector-type interior-point algorithms to use the new neighborhood $\mathcal{N}^*(\tau)$ so that the algorithms can work in wider neighborhoods. We first give here a brief review on predictor–corrector methods. The primal and dual predictor–corrector algorithm for linear programming was proposed by Mizuno et al. [10], also see Barnes et al. [2]. This method was later extended to complementarity problems and other optimization problems (see, e.g. [5,7,15, 16,18,19,22]). Some authors also proposed the infeasible version of this method such that the algorithm can start from an infeasible point (see, for instance, [13,14]). A more practical predictor–corrector algorithm was proposed by Mehrotra [9], which was based on the power series algorithm in [3]. Other power series (high-order) predictor–corrector algorithms can be found in [4,16,18], etc. Two neighborhoods are used in predictor–corrector algorithms: One is used in so-called predictor steps and the other is used in corrector steps. Since the original Mizuno–Todd–Ye method is actually working in small neighborhoods, several authors tried to modify this algorithm to work in some wider neighborhoods in order to achieve a faster convergence of the algorithm (see, for example, [5,6,11,12,15,16]).

Let μ_{gap} denote the duality gap $x^{\mathrm{T}} s/n$ throughout this paper. We note that $\mu_{\mathrm{gap}} = x^{\mathrm{T}} s/n$ is the global minimizer of the proximity measure function $\delta_{\log}$ (x, s, μ) with respect to μ. Therefore, it is natural for $\delta_{\log}(\cdot)$-based interior-point algorithms to use the duality gap as the updating rule for μ during the course of iteration. However, for other proximity measure functions, μ_{gap} is not a global minimizer. For example, the global minimizer (see [11]) of the function $\delta_\Phi(x, s, \mu)$ is

$$\mu_*(x, s) = \sqrt{\frac{x^{\mathrm{T}} s}{\sum_{i=1}^{n} (x_i s_i)^{-1}}}.$$

The predictor–corrector algorithms that use $\delta_\Phi(x, s, \mu)$ as the proximity measure function have been studied by several authors [11,15]. Especially, Peng et al. [11] has already studied how designing an interior-point algorithm by using the above-mentioned minimizer $\mu_*(x,s)$. In this paper, we focus our attention on the cases of 2-norm and ∞-norm neighborhoods that were commonly used in the literature of interior-point algorithms. We note that the global minimizer of the 2-norm proximity measure function $\delta_2(x,s,\mu)$ is

$$\mu_*(x,s)=\frac{(x^2)^{\mathrm T}s^2}{x^{\mathrm T}s}=\frac{\sum_{i=1}^n (x_is_i)^2}{x^{\mathrm T}s},$$

and the global minimizer (see Lemma 3.1 in this paper) of the ∞-norm proximity measure function $\delta_\infty(x,s,\mu)$ is

$$\mu_*(x,s)=\frac{\max_{1\leqslant i\leqslant n} x_is_i+\min_{1\leqslant i\leqslant n} x_is_i}{2}.$$

By using the global minimizers of the proximity measure functions $\delta_2(\cdot)$ and $\delta_\infty(\cdot)$, we provide in this paper a unified method to enlarge the neighborhood of the central path, and hence we give a new design for predictor–corrector algorithms. Also, we prove that our algorithms retain both the so-far best known iteration complexity and the quadratic convergence of the original algorithms.

This paper is organized as follows. In Section 2, we describe the algorithm based on 2-norm neighborhood, and prove that the algorithm retains the best known $\mathrm O(\sqrt n\log(x_0^{\mathrm T}s_0/\varepsilon))$-iteration complexity for small-neighborhood algorithms. In Section 3, we study the case of ∞-norm neighborhood, and prove that the algorithm retains the best known $\mathrm O(n\log(x_0^{\mathrm T}s_0/\varepsilon))$-iteration complexity for ∞-norm neighborhood algorithms. Finally, we point out that both algorithms in the paper are quadratically convergent in the sense that the duality gap sequences converge to zero quadratically.

We use the following notation throughout the paper. e denotes the vector with all components equal to 1. For any positive vectors $x,s\in R^n$ and a real number q, the symbols xs and x^q denote the vector whose components are x_is_i and x_i^q $(i=1,\dots,n)$, respectively. For any vector $x\in R^n$, we denote $\min(x)$ and $\max(x)$ denote the smallest and the largest components of x, respectively, i.e., $\min(x)=\min_{1\leqslant i\leqslant n} x_i$ and $\max(x)=\max_{1\leqslant i\leqslant n} x_i$.

2. Algorithms based on 2-norm neighborhoods

In this section, we consider the predictor–corrector algorithm based on the small neighborhood determined by the 2-norm proximity measure function:

$$\delta_2(x,s,\mu)=\left\|\frac{xs}{\mu}-e\right\|_2,$$

which was used in the original Mizuno–Todd–Ye method. As we pointed out in the last section, for a given vector $(x,s)>0$ it is easy to verify that the global minimizer of $\delta_2(x,s,\mu)$ with respect to μ is

$$\mu_*(x,s)=\frac{(x^2)^{\mathrm T}s^2}{x^{\mathrm T}s}.$$

In what follows, we denote $\mu_*(x,s)$ by μ_* for simplicity. We first give two constants τ and $\bar\tau$ in (0, 1) satisfying

$$r\bar\tau\leqslant\tau<\bar\tau\quad\text{where } r:=\frac{(1+3\bar\tau)^2}{2(1+\bar\tau)^3}. \tag{1}$$

Since $\bar\tau\in(0,1)$, it is easy to verify that $r<1$. The idea of predictor–corrector methods is very natural. Roughly speaking, the predictor–corrector method follows the central path of linear programming by alternating predictor steps and corrector steps. Starting from a point in certain neighborhood $\mathcal N(\tau)$ and moving along a predictor search director, the predictor step produces a predicted point in $\mathcal N(\bar\tau)$ that is larger than $\mathcal N(\tau)$. At the predicted point, a corrector director is calculated, and a suitable stepsize along the corrector direction yields the next iterate that is back inside the neighborhood $\mathcal N(\tau)$. In our algorithm, the predictor search direction is the same as the one in Mizuno–Todd–Ye method [10], i.e., the solution to the following system:

$$\begin{aligned}&A\Delta x=0,\\&A^{\mathrm T}\Delta y+\Delta s=0,\\&s\Delta x+x\Delta s=-xs.\end{aligned} \tag{2}$$

We denote by

$$x(\theta) = x + \theta\Delta x, \qquad s(\theta) = s + \theta\Delta s.$$

The steplength θ_* is determined by the following rule

$$\theta_* = \max\bigl\{t \in (0,1]:\ \bigl(x(\theta), s(\theta)\bigr) > 0,\ \delta_2\bigl(x(\theta), s(\theta), \mu_*(\theta)\bigr) \leqslant \bar{\tau},\ \forall\theta \in (0,t]\bigr\}, \tag{3}$$

where

$$\mu_*(\theta) := \mu_*\bigl(x(\theta), s(\theta)\bigr) = \frac{[x(\theta)^2]^{\mathrm{T}} s(\theta)^2}{x(\theta)^{\mathrm{T}} s(\theta)}.$$

Different from traditional methods, our corrector search direction is the solution to the system

$$\begin{aligned} &A\Delta x = 0,\\ &A^{\mathrm{T}}\Delta y + \Delta s = 0,\\ &s\Delta x + x\Delta s = xs - \frac{x^2 s^2}{\mu_*}. \end{aligned} \tag{4}$$

A characteristic of system (4) is that the duality gap remains invariant along the direction $(\Delta x, \Delta s)$. Actually, since $\Delta x^{\mathrm{T}}\Delta s = 0$, we have

$$\begin{aligned} (x+\theta\Delta x)^{\mathrm{T}}(s+\theta\Delta s) &= x^{\mathrm{T}}s + \theta\bigl(x^{\mathrm{T}}\Delta s + s^{\mathrm{T}}\Delta x\bigr) + \theta^2\Delta x^{\mathrm{T}}\Delta s\\ &= x^{\mathrm{T}}s + \theta\bigl(x^{\mathrm{T}}\Delta s + s^{\mathrm{T}}\Delta x\bigr)\\ &= x^{\mathrm{T}}s + \theta\left(x^{\mathrm{T}}s - \frac{(x^2)^{\mathrm{T}}s^2}{\mu_*}\right)\\ &= x^{\mathrm{T}}s. \end{aligned}$$

The last equality above follows from the fact $\mu_* = (x^2)^{\mathrm{T}}s^2/x^{\mathrm{T}}s$. We now describe the algorithm as follows.

Algorithm 2.1. Given two positive numbers τ and $\bar{\tau}$ in (0,1) satisfying (1), and an initial feasible point $(x^0, s^0) > 0$ such that $\delta_2(x^0, s^0, \mu_*^0) \leqslant \tau$ where $\mu_*^0 = \frac{((x^0)^2)^{\mathrm{T}}(s^0)^2}{(s^0)^{\mathrm{T}}s^0}$. Set $k := 0$, and do the following steps:

Step 1 (Predictor). Solve system (2) with $(x,s) := (x^k, s^k)$ for the search direction $(\Delta x^k, \Delta s^k)$. Compute the steplength θ_*^k according to (3), and set

$$\bigl(\bar{x}^k, \bar{s}^k\bigr) = \bigl(x^k + \theta_*^k\Delta x^k, s^k + \theta_*^k\Delta s^k\bigr), \quad \text{and} \quad \bar{\mu}_*^k = \frac{[(\bar{x}^k)^2]^{\mathrm{T}}(\bar{s}^k)^2}{(\bar{x}^k)^{\mathrm{T}}\bar{s}^k}.$$

Step 2 (Corrector). Solve system (4) with $(x, s, \mu_*) = (\bar{x}^k, \bar{s}^k, \bar{\mu}_*^k)$ for the corrector direction $(\Delta\bar{x}^k, \Delta\bar{s}^k)$. Set

$$\bigl(x^{k+1}, s^{k+1}\bigr) = \bigl(\bar{x}^k + \alpha^k\Delta\bar{x}^k, \bar{s}^k + \alpha^k\Delta\bar{s}^k\bigr),$$

where

$$\alpha^k = \frac{\min(\bar{x}^k\bar{s}^k)\bigl(1 - \sqrt{1 - 2(1-r)\frac{\bar{\mu}_*^k \max(\bar{x}^k\bar{s}^k)}{\min(\bar{x}^k\bar{s}^k)^2}}\bigr)}{\max(\bar{x}^k\bar{s}^k)}.$$

Set $k := k+1$. Repeat the above steps until certain stopping criterion, for instance $\mu_{\mathrm{gap}}^k = (x^k)^{\mathrm{T}}s^k/n \leqslant \varepsilon$, is satisfied.

At each step if $(x^k, s^k) > 0$, the positivity of $(\bar{x}^k, \bar{s}^k)$ follows directly from the choice of θ_*^k. Later, we will see that the positivity of (x^{k+1}, s^{k+1}) follows from the choice of α^k (see the proof of Theorem 2.1). Thus, Algorithm 2.1 is well-defined. We now aim at proving that the algorithm has an $\mathrm{O}(\sqrt{n}\log\frac{(x^0)^{\mathrm{T}}s^0}{\varepsilon})$-iteration complexity. First, we have several technical results.

Lemma 2.1. [10] *Let* $(x, s) > 0$ *be given. For any given* $u \in R^n$*, if* $(\widetilde{\Delta x}, \widetilde{\Delta s})$ *is the solution to the system*

$$A\widetilde{\Delta x} = 0,$$
$$A^{\mathrm{T}}\widetilde{\Delta y} + \widetilde{\Delta s} = 0,$$
$$s\widetilde{\Delta x} + x\widetilde{\Delta s} = u,$$

then $(\widetilde{\Delta x}, \widetilde{\Delta s})$ *satisfies the following inequalities*

$$\sum_{i\in I_+} \widetilde{\Delta x}_i\widetilde{\Delta s}_i = \sum_{i\in I_-} |\widetilde{\Delta x}_i\widetilde{\Delta s}_i| \leqslant \frac{1}{4}\left\|(xs)^{-1/2}u\right\|_2^2,$$

where $I_+ = \{i\colon\ \widetilde{\Delta x}_i\widetilde{\Delta s}_i > 0\}$ *and* $I_- = \{i\colon\ \widetilde{\Delta x}_i\widetilde{\Delta s}_i < 0\}$*. Hence,* $\|\widetilde{\Delta x}\widetilde{\Delta s}\|_\infty \leqslant \frac{1}{4}\|(xs)^{-1/2}u\|_2^2$.

For the sake of convenience, we suppress the iteration index k and denote (x^k, s^k), $(\bar{x}^k, \bar{s}^k)$ by (x, s) and $(\bar{x}, \bar{s})$, respectively, if there is no confusion arising. Similarly, denote $(\Delta x^k, \Delta s^k)$ and $(\Delta\bar{x}^k, \Delta\bar{s}^k)$ by $(\Delta x, \Delta s)$ and $(\Delta\bar{x}, \Delta\bar{s})$, respectively. We also denote by $p = \frac{xs}{\mu_*}$ and $w = \frac{\Delta x\Delta s}{\mu_*}$.

Consider the interval $[0, \hat{\eta})$ where $\hat{\eta} = \min_{w_i<0} \frac{p_i}{|w_i|}$. Clearly, there exists a unique $\hat{\theta} \in (0, 1)$ such that the transformation $\eta = \theta^2/(1-\theta)$ is a bijection from $[0, \hat{\theta})$ to $[0, \hat{\eta})$. For any given $(x, s) > 0$, since $\mu_* = (x^2)^{\mathrm{T}}s^2/x^{\mathrm{T}}s$, it is easy to verify that

$$\left[\delta_2(x, s, \mu_*)\right]^2 = n - \frac{x^{\mathrm{T}}s}{\mu_*} = n - \frac{[x^{\mathrm{T}}s]^2}{(x^2)^{\mathrm{T}}s^2}, \tag{5}$$

and

$$\sum_{i=1}^n p_i = \sum_{i=1}^n p_i^2, \tag{6}$$

where $p_i = x_is_i/\mu_*$ for $i = 1, \ldots, n$. We recall that

$$\big(x(\theta), s(\theta)\big) = (x + \theta\Delta x, s + \theta\Delta s) \quad \text{and} \quad \mu_*(\theta) = \frac{(x(\theta)^2)^{\mathrm{T}}s(\theta)^2}{x(\theta)^{\mathrm{T}}s(\theta)}.$$

From system (2), it is evident that

$$x(\theta)^{\mathrm{T}}s(\theta) = (1-\theta)x^{\mathrm{T}}s \quad \text{and} \quad x_i(\theta)s_i(\theta) = (1-\theta)x_is_i + \theta^2\Delta x_i\Delta s_i. \tag{7}$$

We now prove the next result.

Lemma 2.2. *Let* $\eta = \theta^2/(1-\theta)$*. Then for any* $\eta \in [0, \hat{\eta})$*, we have*

$$\left[\delta_2\big(x(\theta), s(\theta), \mu_*(\theta)\big)\right]^2 - \left[\delta_2(x, s, \mu_*)\right]^2 \leqslant f(\eta)$$

where

$$f(\eta) = \frac{1}{2}\eta\big(1 + \eta\|w\|_\infty\big)\sum_{i=1}^n \frac{1}{(p_i + \eta w_i)^2}.$$

Proof. By (5), (6) and (7), we have

$$\begin{aligned}
\left[\delta_2\big(x(\theta), s(\theta), \mu_*(\theta)\big)\right]^2 - \left[\delta_2(x, s, \mu_*)\right]^2 &= \left(n - \frac{x(\theta)^{\mathrm{T}}s(\theta)}{\mu_*(\theta)}\right) - \left(n - \frac{x^{\mathrm{T}}s}{\mu_*}\right)\\
&= \frac{x^{\mathrm{T}}s}{\mu_*} - \frac{[x(\theta)^{\mathrm{T}}s(\theta)]^2}{(x(\theta)^2)^{\mathrm{T}}s(\theta)^2}\\
&= \frac{x^{\mathrm{T}}s}{\mu_*} - \frac{[(1-\theta)x^{\mathrm{T}}s]^2}{\sum_{i=1}^n[(1-\theta)x_is_i + \theta^2\Delta x_i\Delta s_i]^2}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{x^{\mathrm{T}} s}{\mu_*} - \frac{(x^{\mathrm{T}} s)^2}{\sum_{i=1}^n (x_i s_i + \eta \Delta x_i \Delta s_i)^2} \\
&= \sum_{i=1}^n p_i \left(1 - \frac{\sum_{i=1}^n p_i}{\sum_{i=1}^n (p_i + \eta w_i)^2}\right) \\
&= \frac{(\sum_{i=1}^n p_i) \sum_{i=1}^n (2\eta p_i w_i + \eta^2 w_i^2)}{\sum_{i=1}^n (p_i + \eta w_i)^2}. \qquad (8)
\end{aligned}
$$

Since $x_i \Delta s_i + s_i \Delta x_i = -x_i s_i$ for all $i = 1, \ldots, n$, we have

$$x_i^2 (\Delta s_i)^2 + s_i^2 (\Delta x_i)^2 + 2 x_i s_i \Delta x_i \Delta s_i = x_i^2 s_i^2$$

for all $i = 1, \ldots, n$. Thus, $4 x_i s_i \Delta x_i \Delta s_i \leqslant x_i^2 s_i^2$ which implies that

$$\sum_{i=1}^n x_i s_i \Delta x_i \Delta s_i \leqslant \frac{1}{4} \left(x^2\right)^{\mathrm{T}} s^2.$$

Note that equality (5) implies that $x^{\mathrm{T}} s / \mu_* \leqslant n$, i.e., $\sum_{i=1}^n p_i \leqslant n$. Dividing both sides of the inequality above by μ_*^2, we have

$$\sum_{i=1}^n p_i w_i \leqslant \frac{(x^2)^{\mathrm{T}} s^2}{4\mu_*^2} = \frac{x^{\mathrm{T}} s}{4\mu_*} \leqslant \frac{n}{4}. \qquad (9)$$

Since $\Delta x^{\mathrm{T}} \Delta s = 0$, we have $\sum_{i=1}^n w_i = 0$, which implies that

$$\sum_{i \in I_+} w_i = \sum_{i \in I_-} |w_i|,$$

where $I_+ = \{i\colon\ w_i > 0\}$ and $I_- = \{i\colon\ w_i < 0\}$. Setting $u = -xs$ in Lemma 2.1 and by using (9), we can easily see that

$$\sum_{i \in I_+} w_i \leqslant \frac{x^{\mathrm{T}} s}{4\mu_*} \leqslant \frac{n}{4} \quad \text{and} \quad \|w\|_\infty \leqslant \frac{x^{\mathrm{T}} s}{4\mu_*} \leqslant \frac{n}{4}. \qquad (10)$$

Therefore, by (9) and (10), we have

$$
\begin{aligned}
\sum_{i=1}^n \left(2 p_i w_i + \eta w_i^2\right) &\leqslant 2 \sum_{i=1}^n p_i w_i + \eta \|w\|_\infty \sum_{i=1}^n |w_i| \\
&\leqslant \frac{n}{2} + 2\eta \|w\|_\infty \sum_{i \in I_+}^n w_i \\
&\leqslant \frac{n}{2}\left(1 + \eta \|w\|_\infty\right). \qquad (11)
\end{aligned}
$$

By harmonic inequality, we have

$$\sum_{i=1}^n (p_i + \eta w_i)^2 \sum_{i=1}^n \frac{1}{(p_i + \eta w_i)^2} \geqslant n^2. \qquad (12)$$

Thus, by (8), (11) and (12), we have

$$
\begin{aligned}
\left[\delta_2\left(x(\theta), s(\theta), \mu_*(\theta)\right)\right]^2 - \left[\delta(x, s, \mu_*)\right]^2 &\leqslant \frac{\eta (\sum_{i=1}^n p_i) [\frac{n}{2}(1 + \eta \|w\|_\infty)]}{n^2} \sum_{i=1}^n \frac{1}{(p_i + \eta w_i)^2} \\
&\leqslant \frac{\eta n^2 (1 + \eta \|w\|_\infty)}{2n^2} \sum_{i=1}^n \frac{1}{(p_i + \eta w_i)^2} \\
&= \frac{1}{2} \eta \left(1 + \eta \|w\|_\infty\right) \sum_{i=1}^n \frac{1}{(p_i + \eta w_i)^2} \\
&=: f(\eta).
\end{aligned}
$$

The second inequality above follows from the fact $\sum_{i=1}^{n} p_i \leqslant n$. $\square$

Let θ_* be the steplength given by (3), and let $\eta_* = \frac{(\theta_*)^2}{1-\theta_*}$. Therefore,

$$\theta_* = \frac{2\eta_*}{\eta_* + \sqrt{(\eta_*)^2 + 4\eta_*}}. \tag{13}$$

To prove the iteration complexity of Algorithm 2.1, we need an estimate of the lower bound of θ_*. From (13), it is sufficient to estimate the lower bound of η_* since the function $\zeta(t) = \frac{2t}{t+\sqrt{t^2+4t}}$ is an increasing function in $(0, \infty)$.

Lemma 2.3. *Let θ_* be the steplength in predictor step, determined by* (3)*, and let $\eta_* = \frac{(\theta_*)^2}{1-\theta_*}$. Then we have*

$$\eta_* \geqslant \min\left\{\frac{\beta \min(p)}{\|w\|_\infty}, \frac{2(1-\beta)^2 \min(p)^2(\bar{\tau}^2 - \tau^2)}{n(1+\beta\min(p))}\right\},$$

where $\beta \in (0, 1)$ is a given number independent of n, for instance, $\beta = 0.9$.

Proof. We first note that $f(\eta)$ satisfies the properties: $f(0) = 0$ and $f(\eta) \to \infty$ as $\eta \to \hat{\eta}$. Thus, there exists an $\tilde{\eta}$ such that

$$f(\tilde{\eta}) = \bar{\tau}^2 - \tau^2 > 0.$$

Let $\tilde{\eta}$ be the smallest solution to the above equation. If $\tilde{\eta} \leqslant \frac{\beta\min(p)}{\|w\|_\infty}$, then we have

$$\begin{aligned}
\bar{\tau}^2 - \tau^2 = f(\tilde{\eta}) &= \frac{1}{2}\tilde{\eta}\big(1 + \tilde{\eta}\|w\|_\infty\big)\sum_{i=1}^{n}\frac{1}{(p_i + \tilde{\eta}w_i)^2} \\
&\leqslant \frac{1}{2}\tilde{\eta}\big(1 + \beta\min(p)\big)\sum_{i=1}^{n}\frac{1}{(\min(p) - \tilde{\eta}\|w\|_\infty)^2} \\
&\leqslant \frac{1}{2}\tilde{\eta}\big(1 + \beta\min(p)\big)\frac{n}{(1-\beta)^2\min(p)^2},
\end{aligned}$$

i.e.,

$$\tilde{\eta} \geqslant \frac{2(1-\beta)^2\min(p)^2(\bar{\tau}^2 - \tau^2)}{n(1+\beta\min(p))}.$$

Since either $\tilde{\eta} \geqslant \frac{\beta\min(p)}{\|w\|_\infty}$ or $\tilde{\eta} \leqslant \frac{\beta\min(p)}{\|w\|_\infty}$, we conclude that

$$\tilde{\eta} \geqslant \min\left\{\frac{\beta \min(p)}{\|w\|_\infty}, \frac{2(1-\beta)^2 \min(p)^2(\bar{\tau}^2 - \tau^2)}{n(1+\beta\min(p))}\right\}.$$

To prove the desired result, it suffices to show that $\eta_* \geqslant \tilde{\eta}$. Actually, since $f(0) = 0$ and since $\tilde{\eta}$ is the smallest solution to the equation $f(t) = \bar{\tau}^2 - \tau^2 > 0$ in the interval $(0, \hat{\eta})$, we deduce that

$$f(\eta) \leqslant \bar{\tau}^2 - \tau^2 \quad \text{for all } \eta \in [0, \tilde{\eta}].$$

Thus, for any $\theta \in [0, \tilde{\theta}]$ where $\tilde{\theta}$ satisfies that $(\tilde{\theta})^2/(1-\tilde{\theta}) = \tilde{\eta}$, it follows from Lemma 2.2 that

$$\big[\delta_2\big(x(\theta), s(\theta), \mu_*(\theta)\big)\big]^2 \leqslant \bar{\tau}^2 + \big[\delta_2(x, s, \mu_*)\big]^2 - \tau^2 \leqslant \bar{\tau}^2.$$

The last inequality follows from that $\delta_2(x, s, \mu_*) \leqslant \tau$. Thus, for any $\theta \in [0, \tilde{\theta}]$, we have

$$\delta_2(x(\theta), s(\theta), \mu_*(\theta)) \leqslant \bar{\tau} < 1.$$

which also implies that $(x(\theta), s(\theta)) > 0$. By definition of θ_*, we conclude that $\theta_* \geqslant \tilde{\theta}$, and hence $\eta_* \geqslant \tilde{\eta}$. The proof is complete. $\square$

We are ready to prove the following result.

Theorem 2.1. *Let (x^k, s^k, μ_*^k) be generated by Algorithm 2.1. Then the following properties hold.*

(i) $\delta_2(x^k, s^k, \mu_*^k) \leqslant \tau$ *for all* $k \geqslant 0$; $\delta_2(\bar{x}^k, \bar{s}^k, \bar{\mu}_*^k) \leqslant \bar{\tau}$ *for all* $k \geqslant 0$.
(ii) *The duality gap* $\mu_{\text{gap}}^{k+1} = (1-\theta_*^k)\mu_{\text{gap}}^k$ *for all* $k \geqslant 0$.
(iii) *There exists a constant* $\sigma \in (0,1)$ *independent of n such that* $\theta_*^k \geqslant \frac{\sigma}{\sqrt{n}}$ *for all* $k \geqslant 0$, *which implies that Algorithm* 2.1 *has an* $\mathrm{O}(\sqrt{n}\log\frac{(x^0)^{\mathrm{T}}s^0}{\varepsilon})$*-iteration complexity.*

Proof. Result (i) can be proved by deduction. We assume that $\delta_2(x^k, s^k, \mu_*^k) \leqslant \tau$. By the choice of θ_*^k, the inequality $\delta_2(\bar{x}^k, \bar{s}^k, \bar{\mu}_*^k) \leqslant \bar{\tau}$ holds trivially. It suffices to prove that $\delta_2(x^{k+1}, s^{k+1}, \mu_*^{k+1}) \leqslant \tau$. Note that for any constant $\bar{\tau} \in (0,1)$ we have

$$r := \frac{(1+3\bar{\tau})^2}{2(1+\bar{\tau})^3} < 1.$$

Since $\delta_2(\bar{x}^k, \bar{s}^k, \bar{\mu}_*^k) \leqslant \bar{\tau}$, we have $(1-\bar{\tau})\bar{\mu}_*^k \leqslant \bar{x}_i^k \bar{s}_i^k \leqslant (1+\bar{\tau})\bar{\mu}_*^k$. Thus,

$$\begin{aligned}\frac{[\min(\bar{x}^k\bar{s}^k)]^2}{2\max(\bar{x}^k\bar{s}^k)\bar{\mu}_*^k}\left(1-\left(\frac{\bar{\tau}}{1+\bar{\tau}}\right)^2\right) &\geqslant \frac{(1-\bar{\tau})^2}{2(1+\bar{\tau})}\left(1-\left(\frac{\bar{\tau}}{1+\bar{\tau}}\right)^2\right)\\ &= \frac{(1+2\bar{\tau})(1-\bar{\tau})^2}{2(1+\bar{\tau})^3}\\ &= 1-r.\end{aligned}$$

The above inequality can be written as

$$(1+\bar{\tau})\left(1-\sqrt{1-2(1-r)\frac{\bar{\mu}_*^k\max(\bar{x}^k\bar{s}^k)}{[\min(\bar{x}^k\bar{s}^k)]^2}}\right) \leqslant 1.$$

Therefore,

$$\begin{aligned}\alpha^k &= \frac{\min(\bar{x}^k\bar{s}^k)\big(1-\sqrt{1-2(1-r)\frac{\bar{\mu}_*^k\max(\bar{x}^k\bar{s}^k)}{[\min(\bar{x}^k\bar{s}^k)]^2}}\big)}{\max(\bar{x}^k\bar{s}^k)}\\ &\leqslant \frac{\bar{\mu}_*^k(1+\bar{\tau})\big(1-\sqrt{1-2(1-r)\frac{\bar{\mu}_*^k\max(\bar{x}^k\bar{s}^k)}{[\min(\bar{x}^k\bar{s}^k)]^2}}\big)}{\max(\bar{x}^k\bar{s}^k)}\\ &\leqslant \frac{\bar{\mu}_*^k}{\max(\bar{x}^k\bar{s}^k)}.\end{aligned}$$

Noting that for any $0 < t \leqslant \frac{\bar{\mu}_*^k}{\max(\bar{x}^k\bar{s}^k)}$,

$$\left\|1-t\frac{\bar{x}^k\bar{s}^k}{\bar{\mu}_*^k}\right\|_\infty = \max_{1\leqslant i\leqslant n}\left|1-t\frac{\bar{x}_i^k\bar{s}_i^k}{\bar{\mu}_*^k}\right| = 1-t\frac{\min(\bar{x}^k\bar{s}^k)}{\bar{\mu}_*^k}.$$

In particular, setting $t=\alpha^k$, we have

$$\left\|1-\alpha^k\frac{\bar{x}^k\bar{s}^k}{\bar{\mu}_*^k}\right\|_\infty = 1-\alpha^k\frac{\min(\bar{x}^k\bar{s}^k)}{\bar{\mu}_*^k}. \tag{14}$$

On the other hand, by Lemma 2.1, we have

$$\begin{aligned}
\|\bar{w}^k\|_2 &\leqslant \frac{1}{4\bar{\mu}_*^k}\left\|\left(\bar{x}^k\bar{s}^k\right)^{-\frac{1}{2}}\left(\bar{x}^k\bar{s}^k-\frac{(\bar{x}^k)^2(\bar{s}^k)^2}{\bar{\mu}_*^k}\right)\right\|_2^2 \\
&\leqslant \frac{1}{2\bar{\mu}_*^k}\left\|\left(\bar{x}^k\bar{s}^k\right)^{\frac{1}{2}}\left(e-\frac{\bar{x}^k\bar{s}^k}{\bar{\mu}_*^k}\right)\right\|_2^2 \\
&\leqslant \frac{\max(\bar{x}^k\bar{s}^k)}{2\bar{\mu}_*^k}\left\|e-\frac{\bar{x}^k\bar{s}^k}{\bar{\mu}_*^k}\right\|_2^2. \qquad (15)
\end{aligned}$$

We also note that α^k is the solution to the following quadratic equation with respect to t:

$$1-t\frac{\min(\bar{x}^k\bar{s}^k)}{\bar{\mu}_*^k}+t^2\frac{\max(\bar{x}^k\bar{s}^k)}{2\bar{\mu}_*^k}=r.$$

Therefore, by (14) and (15) and noting that $\|\bar{x}^k\bar{s}^k/\bar{\mu}_*^k-e\|\leqslant\bar{\tau}$, we have

$$\begin{aligned}
\left\|\frac{x^{k+1}s^{k+1}}{\bar{\mu}_*^k}-e\right\|_2 &= \left\|\frac{\bar{x}^k\bar{s}^k+\alpha^k(\bar{x}^k\bar{s}^k-(\bar{x}^k)^2(\bar{s}^k)^2/\bar{\mu}_*)+(\alpha^k)^2\Delta\bar{x}^k\Delta\bar{s}^k}{\bar{\mu}_*^k}-e\right\|_2 \\
&= \left\|\left(\frac{\bar{x}^k\bar{s}^k}{\bar{\mu}_*^k}-e\right)\left(e-\alpha^k\frac{\bar{x}^k\bar{s}^k}{\bar{\mu}_*^k}\right)+\left(\alpha^k\right)^2\bar{w}^k\right\|_2 \\
&\leqslant \left\|\left(\frac{\bar{x}^k\bar{s}^k}{\bar{\mu}_*^k}-e\right)\left(e-\alpha^k\frac{\bar{x}^k\bar{s}^k}{\bar{\mu}_*^k}\right)\right\|_2+\left(\alpha^k\right)^2\|\bar{w}^k\|_2 \\
&\leqslant \left\|e-\alpha^k\frac{\bar{x}^k\bar{s}^k}{\bar{\mu}_*^k}\right\|_\infty\left\|\frac{\bar{x}^k\bar{s}^k}{\bar{\mu}_*^k}-e\right\|_2+\left(\alpha^k\right)^2\|\bar{w}^k\|_2 \\
&= \left(1-\alpha^k\frac{\min(\bar{x}^k\bar{s}^k)}{\bar{\mu}_*^k}\right)\left\|\frac{\bar{x}^k\bar{s}^k}{\bar{\mu}_*^k}-e\right\|_2+\left(\alpha^k\right)^2\|\bar{w}^k\|_2 \\
&\leqslant \left[1-\alpha^k\frac{\min(\bar{x}^k\bar{s}^k)}{\bar{\mu}_*^k}+\frac{(\alpha^k)^2\max(\bar{x}^k\bar{s}^k)}{2\bar{\mu}_*^k}\left\|\frac{\bar{x}^k\bar{s}^k}{\bar{\mu}_*^k}-e\right\|_2\right]\left\|\frac{\bar{x}^k\bar{s}^k}{\bar{\mu}_*^k}-e\right\|_2 \\
&\leqslant \left[1-\alpha^k\frac{\min(\bar{x}^k\bar{s}^k)}{\bar{\mu}_*^k}+\frac{(\alpha^k)^2\max(\bar{x}^k\bar{s}^k)}{2\bar{\mu}_*^k}\right]\left\|\frac{\bar{x}^k\bar{s}^k}{\bar{\mu}_*^k}-e\right\|_2 \\
&\leqslant r\bar{\tau} \\
&\leqslant \tau.
\end{aligned}$$

Since $\tau<1$, the inequality above also implies that $(x^{k+1},s^{k+1})>0$. Since μ_*^{k+1} is the global minimizer of the function $\delta_2(x^{k+1},s^{k+1},\mu)$ with respect to $\mu>0$, we conclude that

$$\delta_2\left(x^{k+1},s^{k+1},\mu_*^{k+1}\right)=\left\|\frac{x^{k+1}s^{k+1}}{\mu_*^{k+1}}-e\right\|_2\leqslant\left\|\frac{x^{k+1}s^{k+1}}{\bar{\mu}_*^k}-e\right\|_2\leqslant\tau.$$

Result (i) is proved.

Note that $\bar{\mu}_{\text{gap}}^k=(1-\theta_*^k)\mu_{\text{gap}}^k$ and

$$\begin{aligned}
\left(x^{k+1}\right)^{\mathrm{T}}s^{k+1} &= \left(\bar{x}^k+\alpha^k\Delta\bar{x}^k\right)^{\mathrm{T}}\left(\bar{s}^k+\alpha^k\Delta\bar{s}^k\right) \\
&= \left(\bar{x}^k\right)^{\mathrm{T}}\bar{s}^k+\alpha^k e^{\mathrm{T}}\left(\bar{x}^k\bar{s}^k-\frac{(\bar{x}^k)^2(\bar{s}^k)^2}{\bar{\mu}_*^k}\right)+\left(\alpha^k\right)^2\left(\Delta\bar{x}^k\right)^{\mathrm{T}}\Delta\bar{s}^k \\
&= \left(\bar{x}^k\right)^{\mathrm{T}}\bar{s}^k.
\end{aligned}$$

It follows that

$$\mu_{\text{gap}}^{k+1}=\bar{\mu}_{\text{gap}}^k=\left(1-\theta_*^k\right)\mu_{\text{gap}}^k.$$

Result (ii) follows.

We now prove (iii). Since $\sum_{i=1}^{n} p_i \leqslant n$ and $\delta_2(x, s, \mu_*) \leqslant \tau$, it follows that $1 \geqslant \min(p) \geqslant 1-\tau$. Note that $\|w\|_\infty \leqslant n/4$. From Lemma 2.3, we conclude that $\eta_*^k \geqslant \frac{\rho}{n}$, where

$$\rho = \min\left\{4\beta(1-\tau), \frac{2(1-\beta)^2(1-\tau)^2(\bar{\tau}^2-\tau^2)}{1+\beta}\right\}.$$

Since the function $\xi(t) := \frac{2t}{t+\sqrt{t^2+4t}}$ is an increasing function on $(0, \infty)$, we obtain

$$\theta_*^k = \frac{2\eta_*^k}{\eta_*^k + \sqrt{(\eta_*^k)^2 + 4\eta_*^k}} \geqslant \frac{2(\rho/n)}{\rho/n + \sqrt{(\rho/n)^2 + 4\rho/n}} \geqslant \frac{\sigma}{\sqrt{n}}$$

where $0 < \sigma = \frac{2}{1+\sqrt{1+4/\rho}} < 1$. As a result, the iteration complexity of the algorithm is $\mathrm{O}(\sqrt{n}\log\frac{(x^0)^{\mathrm{T}}s^0}{\varepsilon})$. □

3. Algorithms based on ∞-norm neighborhoods

Since the original Mizuno–Todd–Ye method is actually working in small neighborhoods of the central path, many investigators tried to adapt this method to work in some wider neighborhoods such as the ∞-norm neighborhood in order to achieve a faster convergence of the algorithms. See, for example, [5,11,12,15,18]. In this section, we consider the predictor–corrector algorithm working in ∞-norm neighborhoods. We discuss how to further enlarge the ∞-norm neighborhood by using the global minimizer of the ∞-norm proximity measure function, and prove that the proposed algorithm retains the best known iteration complexity for ∞-norm-neighborhood interior-point algorithms. First, we have the following result.

Lemma 3.1. *For any given $(x, s) > 0$, the unique global minimizer of the function $\delta_\infty(x, s, \mu) = \|\frac{xs}{\mu} - e\|_\infty$ with respect to μ is given by*

$$\mu_\infty = \frac{\max(xs) + \min(xs)}{2},$$

and the least value of $\delta_\infty(x, s, \cdot)$ is given by

$$\delta_\infty(x, s, \mu_\infty) = \frac{\max(xs) - \min(xs)}{\max(xs) + \min(xs)}.$$

Proof. For any given vector $(x, s) > 0$ and scalar $t > 0$, it is easy to verify that

$$\begin{aligned}\delta_\infty(x, s, t) &= \max_{1\leqslant i\leqslant n}\left|\frac{x_i s_i}{t} - 1\right| \\ &= \max\left\{\left|\frac{\max(xs)}{t} - 1\right|, \left|\frac{\min(xs)}{t} - 1\right|\right\} \\ &= \max\left\{\frac{\max(xs)}{t} - 1,\ 1 - \frac{\min(xs)}{t}\right\}.\end{aligned}$$

When $0 < t \leqslant \frac{\max(xs)+\min(xs)}{2}$, it follows that

$$\delta_\infty(x, s, t) = \frac{\max(xs)}{t} - 1,$$

which is decreasing in the interval $(0, \frac{\max(xs)+\min(xs)}{2}]$. When $t \geqslant \frac{\max(xs)+\min(xs)}{2}$, it follows that

$$\delta_\infty(x, s, t) = 1 - \frac{\min(xs)}{t},$$

which is increasing in the interval $[\frac{\max(xs)+\min(xs)}{2}, \infty)$. Thus for any given $(x, s) > 0$, the global minimizer of $\delta(x, s, \cdot)$ in $(0, \infty)$ is $t_* = \frac{\max(xs)+\min(xs)}{2}$, at which the least value of the proximity measure function is given by

$$\delta_\infty(x, s, t_*) = \frac{\max(xs)}{t_*} - 1 = 1 - \frac{\min(xs)}{t_*} = \frac{\max(xs) - \min(xs)}{\max(xs) + \min(xs)}.$$

The proof is complete. □

In this section, all notation is similar to those in Section 2 except for μ_* being replaced by μ_∞. For instance, we still use θ_* to denote the stepsize in predictor steps, and $(\Delta x, \Delta s)$ is the search direction in the predictor step. θ_* is given by

$$\theta_* = \max\bigl\{t \in (0,1]\colon\ x(\theta) > 0,\ s(\theta) > 0,\ \delta_\infty\bigl(x(\theta), s(\theta), \mu_\infty(\theta)\bigr) \leqslant \bar{\tau},\ \forall\theta \in (0,t]\bigr\}, \tag{16}$$

where $x(\theta) = x + \theta\Delta x$, $s(\theta) = s + \theta\Delta s$ and

$$\mu_\infty(\theta) = \frac{\max(x(\theta)s(\theta)) + \min(x(\theta)s(\theta))}{2}.$$

Denote by

$$\theta_{\max} = \max\bigl\{t \in (0,1]\colon\ x(\theta) > 0,\ s(\theta) > 0,\ \forall\theta \in (0,t]\bigr\}. \tag{17}$$

Clearly, $\theta_* \leqslant \theta_{\max}$. We now specify the algorithm as follows.

Algorithm 3.1. Given a constant $\bar{\tau}$ such that $0 < \bar{\tau} \leqslant \frac{1}{2}$, and given τ such that $\bar{\tau} - \frac{1}{n}(1-\bar{\tau}) = \tau < \bar{\tau}$, and given an initial feasible point $(x^0, s^0) > 0$ such that $\delta_\infty(x^0, s^0, \mu_\infty^0) \leqslant \tau$ where $\mu_\infty^0 = \frac{\max(x^0s^0)+\min(x^0s^0)}{2}$. Set $k := 0$, and do the following steps:

Step 1 (Predictor). Solve system (2) with $(x,s) := (x^k, s^k)$ for the search direction $(\Delta x^k, \Delta s^k)$. Compute the damping parameter θ_*^k according to (16), and set

$$\bigl(\bar{x}^k, \bar{s}^k\bigr) = \bigl(x^k + \theta_*^k\Delta x^k, s^k + \theta_*^k\Delta s^k\bigr), \qquad \bar{\mu}_\infty^k = \frac{\max(\bar{x}^k\bar{s}^k) + \min(\bar{x}^k\bar{s}^k)}{2}.$$

Step 2 (Corrector). Solve the following system for the corrector direction $(\Delta\bar{x}^k, \Delta\bar{s}^k)$:

$$\begin{aligned} &A\Delta\bar{x}^k = 0,\\ &A^{\mathrm{T}}\Delta\bar{y}^k + \Delta\bar{s}^k = 0,\\ &\bar{s}^k\Delta\bar{x}^k + \bar{x}^k\Delta\bar{s}^k = \bar{\mu}_\infty^k e - \bar{x}^k\bar{s}^k. \end{aligned}$$

Set $(x^{k+1}, s^{k+1}) = (\bar{x}^k + \alpha^k\Delta\bar{x}^k, \bar{s}^k + \alpha^k\Delta\bar{s}^k)$, where

$$\alpha^k = \frac{2(\bar{\tau} - \tau)}{n(1 + \sqrt{1 - (\bar{\tau} - \tau)n\bar{\mu}_\infty^k / \min(\bar{x}^k\bar{s}^k)})}.$$

Set $k := k+1$. Repeat the above steps until certain stopping criterion, for instance $\mu_{\mathrm{gap}}^k \leqslant \varepsilon$, is satisfied.

Later, we will see that the steplength α^k is well-defined. We now start to analyze the algorithm and prove that the algorithm has an $\mathrm{O}(n\log\frac{(x^0)^{\mathrm{T}}s^0}{\varepsilon})$-iteration complexity. For simplicity, we suppress the iteration index k when there is no confusion arising.

Lemma 3.2. *Let* $(\Delta x, \Delta s)$ *be the predictor search direction. Let* $\theta_{\max}$ *be defined as* (17) *and*

$$\hat{\eta} = \min_{\Delta x_i\Delta s_i < 0} \frac{x_is_i}{|\Delta x_i\Delta s_i|} = \min_{w_i < 0} \frac{p_i}{|w_i|},$$

where $p := \frac{xs}{\mu_\infty}$ *and* $w := \frac{\Delta x\Delta s}{\mu_\infty}$. *If* $\theta_{\max} < 1$, *then* $\hat{\eta} \leqslant \eta_{\max} := (\theta_{\max})^2/(1-\theta_{\max})$.

Proof. By definition of $\theta_{\max}$, it follows that

$$\theta_{\max} = \min\left\{\min_{\Delta x_i < 0} \frac{x_i}{|\Delta x_i|}, \min_{\Delta s_i < 0} \frac{s_i}{|\Delta s_i|}\right\}.$$

Thus, there exists some i_0 such that either $x_{i_0} + \theta_{\max}\Delta x_{i_0} = 0$ or $s_{i_0} + \theta_{\max}\Delta s_{i_0} = 0$. Hence,

$$0 = (x_{i_0} + \theta_{\max}\Delta x_{i_0})(s_{i_0} + \theta_{\max}\Delta s_{i_0}) = (1-\theta_{\max})x_{i_0}s_{i_0} + (\theta_{\max})^2\Delta x_{i_0}\Delta s_{i_0}. \tag{18}$$

Since $\theta_{\max} < 1$ and $x_{i_0}s_{i_0} > 0$, equality (18) implies that $\Delta x_{i_0}\Delta s_{i_0} < 0$. Dividing both sides of (18) by $1-\theta_{\max}$, we have

$$x_{i_0}s_{i_0} + \eta_{\max}\Delta x_{i_0}\Delta s_{i_0} = 0,$$

where $\eta_{\max} = (\theta_{\max})^2/(1-\theta_{\max})$. By definition of $\hat{\eta}$, we conclude that $\eta_{\max} \geqslant \hat{\eta}$. $\square$

We now prove that $\eta_* = (\theta_*)^2/(1-\theta_*)$ has a lower bound.

Lemma 3.3. *Let $\eta_* = (\theta_*)^2/(1-\theta_*)$, where θ_* is the steplength in the predictor step. Then*

$$\eta_* \geqslant \min\left\{\frac{4(1-\tau)0.9}{1+\tau}, \frac{4(\bar{\tau}-\tau)}{1+\tau}\right\}\frac{1}{n}.$$

Proof. Noting that the predictor step starts from the point (x,s) that satisfies $\delta_\infty(x,s,\mu_\infty) \leqslant \tau$. There are only two cases.

Case 1. $\delta_\infty(x(\theta), s(\theta), \mu_\infty(\theta)) - \delta_\infty(x,s,\mu_\infty) < \bar{\tau} - \tau$ for all $0 < \theta < \theta_{\max}$. In this case, by definition of θ_*, we have that $\theta_* = \theta_{\max}$. There are only two sub-cases.

Sub-case 1. $\theta_{\max} = 1$. For this case, $\eta = \theta^2/(1-\theta) \to \infty$ as $\theta \to \theta_{\max} = 1$. Thus,

$$\eta^* > 0.9\min(p)/\|w\|_\infty \tag{19}$$

holds trivially, since $\eta_* = \infty$ in this case.

Sub-case 2. $\theta_{\max} < 1$. Since $\theta_* = \theta_{\max} < 1$, it follows that $\eta^* = \eta_{\max} := (\theta_{\max})^2/(1-\theta_{\max})$. By Lemma 3.2, we have $\eta_{\max} \geqslant \hat{\eta} \geqslant 0.9\min(p)/\|w\|_\infty$. Therefore, for this sub-case, inequality (19) remains valid.

Case 2. There exists a point $\tilde{\theta} \in (0, \theta_{\max})$ such that

$$\delta_\infty\big(x(\tilde{\theta}), s(\tilde{\theta}), \mu_\infty(\tilde{\theta})\big) - \delta(x,s,\mu_\infty) \geqslant \bar{\tau} - \tau > 0.$$

Since $\delta_\infty(x(0), s(0), \mu_\infty(0)) - \delta(x,s,\mu_\infty) = 0$, by continuity, there must exist a point $\theta' \in (0,\theta_{\max})$ such that

$$\delta_\infty\big(x(\theta'), s(\theta'), \mu_\infty(\theta')\big) - \delta(x,s,\mu_\infty) = \bar{\tau} - \tau.$$

Let θ' be the smallest solution to the above equation. Let $\eta' = (\theta')^2/(1-\theta')$. We now prove that η' is bounded from below. Since either $\eta' \geqslant 0.9\min(p)/\|w\|_\infty$ or $\eta' \leqslant 0.9\min(p)/\|w\|_\infty$, it is sufficient to prove that η' has a lower bound if $\eta' \leqslant 0.9\min(p)/\|w\|_\infty$. We first note that

$$\max(xs + \eta\Delta x\Delta s) \leqslant \max(xs) + \eta\|\Delta x\Delta s\|_\infty, \tag{20}$$

and that for all $\eta \leqslant 0.9\min(p)/\|w\|_\infty$ we have

$$\min(xs + \eta\Delta x\Delta s) \geqslant \min(xs) - \eta\|\Delta x\Delta s\|_\infty > 0. \tag{21}$$

It is easy to see that for any given number $c > 0$, the function $\varphi(t) = \frac{t-c}{t+c}$ is increasing with respect to t, and the function $\chi(t) = \frac{c-t}{c+t}$ is decreasing with respect to t. Thus, by (20), (21) and Lemma 3.1, for any $\eta \leqslant 0.9\min(p)/\|w\|_\infty$ we have

$$\begin{aligned}
\delta_\infty\big(x(\theta), s(\theta), \mu_\infty(\theta)\big) &= \frac{\max(x(\theta)s(\theta)) - \min(x(\theta)s(\theta))}{\max(x(\theta)s(\theta)) + \min(x(\theta)s(\theta))} \\
&= \frac{\max((1-\theta)xs + \theta^2\Delta x\Delta s) - \min((1-\theta)xs + \theta^2\Delta x\Delta s)}{\max((1-\theta)xs + \theta^2\Delta x\Delta s) + \min((1-\theta)xs + \theta^2\Delta x\Delta s)} \\
&= \frac{\max(xs + \eta\Delta x\Delta s) - \min(xs + \eta\Delta x\Delta s)}{\max(xs + \eta\Delta x\Delta s) + \min(xs + \eta\Delta x\Delta s)} \\
&\leqslant \frac{\max(xs) + \eta\|\Delta x\Delta s\|_\infty - (\min(xs + \eta\Delta x\Delta s)}{\max(xs) + \eta\|\Delta x\Delta s\|_\infty + (\min(xs + \eta\Delta x\Delta s)} \\
&\leqslant \frac{\max(xs) + \eta\|\Delta x\Delta s\|_\infty - (\min(xs) - \eta\|\Delta x\Delta s\|_\infty)}{\max(xs) + \eta\|\Delta x\Delta s\|_\infty + (\min(xs) - \eta\|\Delta x\Delta s\|_\infty)} \\
&= \frac{\max(xs) - \min(xs)}{\max(xs) + \min(xs)} + \frac{2\eta\|\Delta x\Delta s\|_\infty}{\max(xs) + \min(xs)} \\
&= \delta_\infty(x,s,\mu_\infty) + \eta\|w\|_\infty
\end{aligned} \tag{22}$$

where $w = \Delta x \Delta s/\mu_\infty$. The first inequality above follows from (20) and the increasing property of $\varphi(t)$, and the second inequality follows from (21) and decreasing property of $\chi(t)$. The last equality above follows from the fact

$$\frac{\max(xs) + \min(xs)}{2} = \mu_\infty.$$

Since $\delta_\infty(x, s, \mu_\infty) \leqslant \tau$, we have $1 - \tau \leqslant p_i = x_i s_i/\mu_\infty \leqslant 1 + \tau$. Thus, by Lemma 2.1, we have

$$\|w\|_\infty \leqslant \frac{1}{4\mu_\infty} \left\| (xs)^{-1/2} xs \right\|_2^2 = \frac{x^{\mathrm{T}} s}{4\mu_\infty} \leqslant \frac{n(1+\tau)}{4}. \tag{23}$$

Noting that (22) holds for any $\eta \leqslant 0.9 \min(p)/\|w\|_\infty$. Thus, if $\eta' \leqslant 0.9 \min(p)/\|w\|_\infty$, from (22) and (23), it follows that

$$\eta' \geqslant \frac{(\delta_\infty(x(\theta'), s(\theta'), \mu_\infty(\theta') - \delta_\infty(x, s, \mu_\infty))}{\|w\|_\infty} \geqslant \frac{4(\bar\tau - \tau)}{n(1+\tau)}.$$

Therefore, we have

$$\eta' \geqslant \min\left\{ \frac{0.9 \min(p)}{\|w\|_\infty}, \frac{4(\bar\tau - \tau)}{n(1+\tau)} \right\}.$$

Notice that

$$\delta_\infty\big(x(\theta'), s(\theta'), \mu_\infty(\theta')\big) \leqslant \bar\tau + \delta(x, s, \mu_\infty) - \tau \leqslant \bar\tau,$$

which also implies that $(x(\theta'), s(\theta')) > 0$. By definition of θ_*, we conclude that $\theta' \leqslant \theta_*$. This in turn implies that $\eta_* \geqslant \eta'$. Therefore, both cases 1 and 2 imply that

$$\eta_* \geqslant \min\left\{ \frac{0.9 \min(p)}{\|w\|_\infty}, \frac{4(\bar\tau - \tau)}{n(1+\tau)} \right\}$$
$$\geqslant \min\left\{ \frac{4(1-\tau)0.9}{1+\tau}, \frac{4(\bar\tau - \tau)}{1+\tau} \right\} \frac{1}{n}.$$

The last inequality follows from (23) and $1 + \tau \geqslant \min(p) \geqslant 1 - \tau$. □

In what follows, we consider only the case $n \geqslant 2$.

Lemma 3.4. *Suppose that $n \geqslant 2$. Let $\bar\tau$ be given as in Algorithm* 3.1. *There exists a constant* $0.25 \leqslant \gamma \leqslant 1 - \frac{(1+\bar\tau)\bar\tau}{2(1-\bar\tau)}$ *independent of n such that*

$$(1 - \theta_*)\left(1 - \alpha + \alpha \frac{\bar\mu_\infty}{\bar\mu_{\mathrm{gap}}}\right) \leqslant 1 - \gamma\theta_*,$$

where θ_* and α are the steplength in predictor and corrector steps in Algorithm 3.1, *respectively.*

Proof. By choice of $\bar\tau$ and τ, we have

$$\bar\tau - \tau < \frac{1 - \bar\tau}{n} \leqslant \frac{\min(\bar x \bar s)}{n \bar\mu_\infty},$$

i.e.,

$$1 \geqslant \frac{(\bar\tau - \tau) n \bar\mu_\infty}{\min(\bar x \bar s)}.$$

This implies that the steplength α used in the corrector step is well-defined. First we note that

$$\alpha = \frac{2(\bar\tau - \tau)}{n(1 + \sqrt{1 - (\bar\tau - \tau) n \bar\mu_\infty / \min(\bar x^k \bar s^k)})} \leqslant \frac{2(\bar\tau - \tau)}{n}.$$

On the other hand, since $n \geqslant 2$ and $\bar\tau - \frac{1-\bar\tau}{n} = \tau < \bar\tau < 1$, it is easy to see that $0.9(1 - \tau) \geqslant \bar\tau - \tau$. As a result, by Lemma 3.3 we have

$$\eta_* \geqslant \frac{4(\bar\tau - \tau)}{n(1+\tau)}.$$

Since $0 < \bar\tau \leqslant \frac{1}{2}$, we have

$$\frac{4}{3} \leqslant \frac{2(1-\bar\tau)}{(1+\bar\tau)\bar\tau} < \infty.$$

Therefore, there exists a constant $0.25 \leqslant \gamma \leqslant 1 - \frac{(1+\bar\tau)\bar\tau}{2(1-\bar\tau)}$ such that

$$\frac{2(1-\bar\tau)(1-\gamma)}{(1+\bar\tau)\bar\tau} \geqslant 1.$$

Since $\tau < \bar\tau$, it follows that

$$\frac{2(1-\bar\tau)(1-\gamma)}{(1+\tau)\bar\tau} > 1.$$

Thus, we have

$$\begin{aligned}\alpha &< \frac{2(\bar\tau-\tau)}{n}\frac{2(1-\bar\tau)(1-\gamma)}{(1+\tau)\bar\tau} \leqslant \frac{\eta_*(1-\bar\tau)(1-\gamma)}{\bar\tau} \\ &= \frac{\theta_*^2(1-\bar\tau)(1-\gamma)}{(1-\theta_*)\bar\tau} \leqslant \frac{\theta_*(1-\bar\tau)(1-\gamma)}{(1-\theta_*)\bar\tau},\end{aligned}$$

i.e.,

$$(1-\theta_*)\frac{\alpha\bar\tau}{1-\bar\tau} \leqslant \theta_*(1-\gamma).$$

Adding both sides of the above inequality by $1-\theta_*$ yields

$$(1-\theta_*)\left(1+\frac{\alpha\bar\tau}{1-\bar\tau}\right) \leqslant 1-\gamma\theta_*.$$

Since $\|\bar x\bar s/\bar\mu_\infty - e\|_\infty \leqslant \bar\tau$, which implies that $1-\bar\tau \leqslant \bar x_i \bar s_i/\bar\mu_\infty \leqslant 1+\bar\tau$, we have

$$\frac{\bar\mu_{\mathrm{gap}}}{\bar\mu_\infty} = \frac{\bar x^{\mathrm T}\bar s}{n\bar\mu_\infty} \geqslant \frac{\min(\bar x\bar s)}{\bar\mu_\infty} \geqslant 1-\bar\tau,$$

i.e., $\bar\mu_\infty/\bar\mu_{\mathrm{gap}} \leqslant 1/(1-\bar\tau)$. Therefore, we have

$$\begin{aligned}(1-\theta_*)\left(1-\alpha+\alpha\frac{\bar\mu_\infty}{\bar\mu_{\mathrm{gap}}}\right) &\leqslant (1-\theta_*)\left(1-\alpha+\frac{\alpha}{1-\bar\tau}\right) \\ &= (1-\theta_*)\left(1+\frac{\alpha\bar\tau}{1-\bar\tau}\right) \leqslant 1-\gamma\theta_*.\end{aligned}$$

The proof is complete. □

We now prove the main result in this section.

Theorem 3.1. *Suppose that $n \geqslant 2$. Let $\bar\tau$ and τ be given as in Algorithm* 3.1*, and let (x^k, s^k, μ_∞^k) and $(\bar x^k, \bar s_k, \bar\mu_\infty^k)$ be generated by Algorithm* 3.1*. Then the following properties hold.*

(i) *$\delta_\infty(x^k, s^k, \mu_\infty^k) \leqslant \tau$ and $\delta_\infty(\bar x^k, \bar s^k, \bar\mu_\infty^k) \leqslant \bar\tau$ for all k.*
(ii) *$\mu_{\mathrm{gap}}^{k+1} \leqslant (1-\gamma\theta_*^k)\mu_{\mathrm{gap}}^k$ for all k, where γ is given as in Lemma* 3.4.
(iii) *There exists a constant $\sigma \in (0,1)$ independent of n such that $\theta_*^k \geqslant \frac{\sigma}{n}$, and hence the algorithm has an $\mathrm{O}(n\log\frac{(x^0)^{\mathrm T}s^0}{\varepsilon})$-iteration complexity.*

Proof. (i) can be proved by deduction. It is sufficient to show that if $\delta_\infty(x^k, s^k, \mu_\infty^k) \leqslant \tau$, then $\delta_\infty(\bar x^k, \bar s^k, \bar\mu_\infty^k) \leqslant \bar\tau$ and $\delta_\infty(x^{k+1}, s^{k+1}, \mu_\infty^{k+1}) \leqslant \tau$. We now assume that $\delta_\infty(x^k, s^k, \mu_\infty^k) \leqslant \tau$. The fact $\delta_\infty(\bar x^k, \bar s^k, \bar\mu_\infty^k) \leqslant \bar\tau$ follows immediately from the choice of the steplength θ_*^k. Since $\bar\tau < 1$, this also implies that $(\bar x^k, \bar s^k)$ is positive. We now prove that the next iterate (x^{k+1}, s^{k+1}) is back inside the original neighborhood, i.e., $\delta_\infty(x^{k+1}, s^{k+1}, \mu_\infty^{k+1}) \leqslant \tau$. Note that

$$\left\|\frac{\Delta\bar{x}^k\Delta\bar{s}^k}{\bar{\mu}_\infty^k}\right\|_\infty \leqslant \frac{1}{4\bar{\mu}_\infty^k}\left\|(\bar{x}\bar{s})^{-1/2}\left(\bar{\mu}_\infty^k e-\bar{x}^k\bar{s}^k\right)\right\|_2^2 \leqslant \frac{\bar{\mu}_\infty^k}{4}\left\|(\bar{x}\bar{s})^{-1/2}\right\|_2^2\left\|e-\frac{\bar{x}^k\bar{s}^k}{\bar{\mu}_\infty^k}\right\|_\infty^2 \leqslant \frac{\bar{\mu}_\infty^k}{4}\frac{n}{\min(\bar{x}^k\bar{s}^k)}\left\|e-\frac{\bar{x}^k\bar{s}^k}{\bar{\mu}_\infty^k}\right\|_\infty^2.$$

Therefore,

$$\begin{aligned}
\left\|\frac{x^{k+1}s^{k+1}}{\bar{\mu}_\infty^k}-e\right\|_\infty &= \left\|\frac{\bar{x}^k\bar{s}^k+\alpha^k(\bar{\mu}_\infty^k e-\bar{x}^k\bar{s}^k)+(\alpha^k)^2\Delta\bar{x}^k\Delta\bar{s}^k}{\bar{\mu}_\infty^k}-e\right\|_\infty \\
&= \left\|\left(1-\alpha^k\right)\left(\frac{\bar{x}^k\bar{s}^k}{\bar{\mu}_\infty^k}-e\right)+\left(\alpha^k\right)^2\frac{\Delta\bar{x}^k\Delta\bar{s}^k}{\bar{\mu}_\infty^k}\right\|_\infty \\
&\leqslant \left(1-\alpha^k\right)\left\|\frac{\bar{x}^k\bar{s}^k}{\bar{\mu}_\infty^k}-e\right\|_\infty+\left(\alpha^k\right)^2\left\|\frac{\Delta\bar{x}^k\Delta\bar{s}^k}{\bar{\mu}_\infty^k}\right\|_\infty \\
&\leqslant \left(1-\alpha^k\right)\left\|\frac{\bar{x}^k\bar{s}^k}{\bar{\mu}_\infty^k}-e\right\|_\infty+\frac{(\alpha^k)^2\bar{\mu}_\infty^k}{4}\frac{n}{\min(\bar{x}^k\bar{s}^k)}\left\|e-\frac{\bar{x}^k\bar{s}^k}{\bar{\mu}_\infty^k}\right\|_\infty^2 \\
&= \left(1-\alpha^k+\left(\alpha^k\right)^2\frac{n\bar{\mu}_\infty^k}{4\min(\bar{x}^k\bar{s}^k)}\left\|\frac{\bar{x}^k\bar{s}^k}{\bar{\mu}_\infty^k}-e\right\|_\infty\right)\left\|\frac{\bar{x}^k\bar{s}^k}{\bar{\mu}_\infty^k}-e\right\|_\infty \\
&\leqslant \left(1-\alpha^k+\left(\alpha^k\right)^2\frac{n\bar{\tau}\bar{\mu}_\infty^k}{4\min(\bar{x}^k\bar{s}^k)}\right)\bar{\tau}.
\end{aligned} \tag{24}$$

From the beginning of the proof of Lemma 3.4, we have

$$1 \geqslant (\bar{\tau}-\tau)n\bar{\mu}_\infty^k/\min\left(\bar{x}^k\bar{s}^k\right).$$

Thus, the following quadratic equation in t has at least one solution

$$1-t+t^2\frac{n\bar{\tau}\bar{\mu}_\infty^k}{4\min(\bar{x}^k\bar{s}^k)}=\frac{\tau}{\bar{\tau}}. \tag{25}$$

It is easy to see that

$$\alpha^k=\frac{2(\bar{\tau}-\tau)}{n(1+\sqrt{1-(\bar{\tau}-\tau)n\bar{\mu}_\infty^k/\min(\bar{x}^k\bar{s}^k)}}$$

is the least solution to Eq. (25). Thus, it follows from (24) that

$$\left\|\frac{x^{k+1}s^{k+1}}{\bar{\mu}_\infty^k}-e\right\|_\infty \leqslant \left(\frac{\tau}{\bar{\tau}}\right)\bar{\tau}=\tau.$$

This also implies that $(x^{k+1}, s^{k+1}) > 0$. Since

$$\mu_\infty^{k+1}=\left(\max\left(x^{k+1}s^{k+1}\right)+\min\left(x^{k+1}s^{k+1}\right)\right)/2$$

is the global minimizer of the proximity measure function $\delta_\infty(x^{k+1}, s^{k+1}, \cdot)$, we conclude that

$$\delta_\infty\left(x^{k+1}, s^{k+1}, \mu_\infty^{k+1}\right)=\left\|\frac{x^{k+1}s^{k+1}}{\mu_\infty^{k+1}}-e\right\|_\infty \leqslant \tau,$$

as desired.

We now prove (ii). Notice that $(\Delta\bar{x}^k)^{\mathrm{T}}\Delta\bar{s}^k=0$. By using that fact that $\bar{\mu}_{\mathrm{gap}}^k=(1-\theta_*^k)\mu_{\mathrm{gap}}^k$ and by Lemma 3.4, we have

$$\begin{aligned}
\mu_{\text{gap}}^{k+1} &= \frac{(x^{k+1})^{\mathrm{T}} s^{k+1}}{n} = \frac{(\bar{x}^k + \alpha^k \Delta\bar{x}^k)^{\mathrm{T}}(\bar{s}^k + \alpha^k \Delta\bar{s}^k)}{n} \\
&= \frac{(\bar{x}^k)^{\mathrm{T}}\bar{s}^k + \alpha^k((\bar{x}^k)^{\mathrm{T}}\Delta\bar{s}^k + (\bar{s}^k)^{\mathrm{T}}\Delta\bar{x}^k)}{n} \\
&= (1-\alpha^k)\bar{\mu}_{\text{gap}}^k + \alpha^k \bar{\mu}_\infty^k \\
&= \bar{\mu}_{\text{gap}}^k \left(1 - \alpha^k + \alpha^k \frac{\bar{\mu}_\infty^k}{\bar{\mu}_{\text{gap}}}\right) \\
&\leqslant (1-\theta_*^k)\left(1 - \alpha^k + \alpha^k \frac{\bar{\mu}_\infty^k}{\bar{\mu}_{\text{gap}}}\right)\mu_{\text{gap}}^k \\
&\leqslant (1-\gamma\theta_*^k)\mu_{\text{gap}}^k.
\end{aligned}$$

Finally, we prove that $\theta_*^k \geqslant \frac{\sigma}{n}$ where $\sigma \in (0,1)$ is a constant independent of n. Actually, in the proof of Lemma 3.4, we have pointed out that $\eta_*^k \geqslant 4(\bar{\tau} - \tau)/n(1+\tau)$. Notice that $\bar{\tau} - \tau = \frac{1-\bar{\tau}}{n}$. We conclude that $\eta_*^k \geqslant \frac{\nu}{n^2}$ for some constant ν independent of n. Observe that the function $\xi(t) := \frac{2t}{t+\sqrt{t^2+4t}}$ is increasing function on $(0,\infty)$. Thus, we obtain

$$\theta_*^k = \frac{2\eta_*^k}{\eta_*^k + \sqrt{\eta_*^k + 4\eta_*^k}} \geqslant \frac{2(\nu/n^2)}{\nu/n^2 + \sqrt{(\nu/n^2)^2 + 4\nu/n^2}} \geqslant \frac{\sigma}{n}$$

where $0 < \sigma = \frac{2\sqrt{\nu}}{\sqrt{\nu}+\sqrt{\nu+4}} < 1$. Therefore, the algorithm has an $\mathrm{O}(n \log \frac{(x^0)^{\mathrm{T}} s^0}{\varepsilon})$-iteration complexity. □

Before closing this section, we point out that both Algorithms 2.1 and 3.1 are quadratically convergent in the sense that the duality gap sequences generated by these algorithms converge to zero quadratically. Actually, by a proof similar to that of Theorem 4.1 in [11], we can easily obtain the following result.

Theorem 3.2. *Let (x^k, s^k) be generated by Algorithm* 2.1 *or Algorithm* 3.1. *Then the algorithm is quadratically convergent in the sense that $\mu_{\text{gap}}^{k+1} = \mathrm{O}((\mu_{\text{gap}}^k)^2)$. Moreover, every accumulation point of the sequence (x^k, s^k) is a strictly complementary solution of the problem.*

4. Conclusions and future work

Most numerical experiments demonstrate that interior-point algorithms working in wider neighborhoods perform better than those counterparts using smaller neighborhoods. Inspired by this fact, we present in this paper a unified method to enlarge the neighborhoods of interior-point algorithms. As an example, we consider so-called predictor–corrector methods, and show how to use the least value of a proximity measure function to enlarge the neighborhoods of the original methods. We also prove that the proposed algorithms in this paper retain the best known iteration complexity and local superlinear convergence of the original algorithms. Our methods can be viewed as a new design for interior-point methods.

It is worth mentioning that the following proximity function is also widely used in interior-point algorithms:

$$\delta_\infty^-(x,s,\mu) = \left\| \left(\frac{xs}{\mu} - e\right)^- \right\|_\infty,$$

where the operation $(\cdot)^-$ performs componentwise, i.e., for any vector $y \in R^n$, the ith component of the vector $(y)^-$ is $\min(0, y_i)$, $i = 1, \dots, n$. In R_{++}^n, we note that $\delta_\infty^-(x,s,\mu) = 0$ for all $\mu \in (0, \min(xs)]$, and $\delta_\infty^-(x,s,\mu) > 0$ for all $\mu > \min(xs)$. This implies that the global minimum point of $\delta_\infty^-(x,s,\mu)$ with respect to μ is not unique. In fact, any $\mu \in (0, \min(xs)]$ is the global minimum point and the least value of the proximity function is zero. Therefore, if we take μ to be one of these minimum points, the neighborhood is enlarged to be the whole positive orthant, i.e., R_{++}^n. In this case, we can say that the interior-point algorithms do not need a particular neighborhood of central path, and thus the algorithms do not require any proximity measure function. The analysis for such an interior-point algorithm is a worthwhile and interesting future work.

It should be mentioned that in [5] the convergence of a predictor–corrector algorithm with ∞-norm-neighborhood with parameter α (corresponding to $\bar{\tau}$ under consideration in this paper) was chosen to be in the open interval $(0, 1)$. However, we prove only the convergence of Algorithm 3.1 in this paper with the parameter $\bar{\tau}$ in the interval $(0, \frac{1}{2}]$. At the present, we do not know whether Algorithm 3.1 with parameter $\bar{\tau}$ in $(\frac{1}{2}, 1)$ is convergent or not. We leave this important question as a future research topic.

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