Applied Mathematics and Computation 106 (1999) 221–235
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.nl/locate/amc

D-orientation sequences for continuous functions and nonlinear complementarity problems

Yun-Bin Zhao

Institute of Applied Mathematics, Chinese Academy of Sciences, P.O. Box 2734, Beijing 100080, People's Republic of China

Abstract

We introduce the new concept of d-orientation sequence for continuous functions. It is shown that if there does not exist a d-orientation sequence for a continuous function, then the corresponding complementarity problem (CP) has a solution. We believe that such a result characterizes an intrinsic property of CPs. As the concept of "exceptional family of elements", the notion of "d-orientation sequence of a function" is also a powerful tool for investigating the existence theorems of CPs. We use this new tool to establish, among other things, a new existence result for a class of P_*-mapping CPs. © 1999 Published by Elsevier Science Inc. All rights reserved.

Keywords: Complementarity problems; d-orientation sequence; p-order generalized coercivity; P_*-mapping

1. Introduction

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function. The well-known complementarity problem (CP) is to find a $x \in \mathbb{R}^n$ such that

$$x \geqslant 0, \quad f(x) \geqslant 0, \quad x^{\mathrm{T}} f(x) = 0. \tag{1}$$

Such a problem has been studied extensively for several decades due to its many successful applications in engineering, operations researches and even in economics [1,5,7]. Given a CP, the existence of a solution is not always assured.

PII: S0096-3003(98)10125-X

On the other hand, most algorithms developed to solve CPs often assume the existence of a solution. Because of these aspects, many authors investigated different classes of the functions and proposed a variety of existence theorems [1–10,13–16,21–23,28–30]. Seminally, Isac et al. [8] introduced the notion of *exceptional family of elements* associated with a continuous function, this concept includes as a special case of *exceptional sequence* due to Smith [23]. Moreover, Isac and Obuchowska [9] proved that many well-known existence conditions for CPs imply that the functions have no exceptional family of elements. Recently, Zhao et al. [28–30] generalized the Isac's concept to nonlinear variational inequality (VI) problems and introduced the notion of *exceptional family for VI*. This concept evinces a deep property of VI/CPs, actually, it becomes a powerful analysis method for the solvability of the problems.

In this paper, we will develop a new concept, namely, *d-orientation sequence of a function*. This concept, quite different from exceptional family, seems very important for it provides a new argument method for the solvability of a CP. By this concept, we will establish several new existence theorems for the problem (1). One of the existence results is related to P_*-mapping CP, which is very broad and encompasses a large number of interesting special cases, for instance, the monotone CPs. The concept of linear P_*-CP was first defined by Kojima et al. [18]. Recently, the P_*-CPs (linear and nonlinear) has been studied by several authors [18,12,20,25–27,11,31].

In Section 2, we introduce the concept of d-orientation sequence for a continuous function, and show the main result, that is, for any continuous function, there exists either a d-orientation sequence for the function or a solution to the corresponding CP. Therefore, "function is without d-orientation sequence" is a sufficient condition for a CP to posses a solution. In Section 3, we show a new existence condition which includes several well-known conditions as the special cases. An existence theorem for P_*-mapping CPs is given in Section 4.

2. General existence theorems

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function. Recall that a $\mathrm{VI}(K, f)$, is to find a solution x^* such that

$$(x - x^*)^{\mathrm{T}} f(x^*) \geqslant 0, \qquad \text{for all } x \in K,$$

where K is a closed convex set in $\mathbb{R}^n$. In particular, if the set $K = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n \colon\ x \geqslant 0\}$ then $\mathrm{VI}(K, f)$ reduces to CP.

Throughout the paper, let $d \in \mathbb{R}^n$ be a fixed vector in positive orthant, i.e., $d > 0$. For such d, let

$$K_r = \mathbb{R}^n_+ \cap \{x \in \mathbb{R}^n\colon\ x^{\mathrm{T}} d \leqslant r\}.$$

Clearly, K_r is a bounded convex set, so that $\mathrm{VI}(K_r, f)$ has at least one solution [5]. The following results are similar to that in Ref. [17].

Lemma 2.1. *Let f be a continuous function from* $\mathbb{R}^n$ *into* $\mathbb{R}^n$, *then the CP* (1) *has a solution if and only if there exists a scalar* $r > 0$ *such that the* $\mathrm{VI}(K_r, f)$ *has a solution* x^r *with* $(x^r)^{\mathrm{T}} d < r$.

Proof. If x^* solves the problem (1), then

$$(x - x^*)^{\mathrm{T}} f(x^*) \geqslant 0 \quad \text{for all } x \in \mathbb{R}^n_+.$$

Let $r > (x^*)^{\mathrm{T}} d$, from the above we have

$$(x - x^*)^{\mathrm{T}} f(x^*) \geqslant 0, \quad \text{for all } x \in K_r$$

which implies that x^* solves the problem $\mathrm{VI}(K_r, f)$.

Conversely, suppose that there exists some scalar $r > 0$ such that $\mathrm{VI}(K_r, f)$ has a solution x^r with $(x^r)^{\mathrm{T}} d < r$. Then

$$(x - x^r)^{\mathrm{T}} f(x^r) \geqslant 0 \quad \text{for all } x^r \in K_r. \tag{2}$$

It suffices to show that

$$(x - x^r)^{\mathrm{T}} f(x^r) \geqslant 0 \quad \text{for all } x \in \mathbb{R}^n_+ \setminus K_r. \tag{3}$$

Actually, let x be an arbitrary vector in $\mathbb{R}^n_+ \setminus K_r$. Since

$$p(\lambda) = \lambda x + (1 - \lambda) x^r \in \mathbb{R}^n_+ \quad \text{for all } \lambda \in [0, 1].$$

Notice that $(x^r)^{\mathrm{T}} d < r$, there exists a sufficiently small scalar $\lambda^* > 0$ such that $p(\lambda^*)^{\mathrm{T}} d < r$, hence $p(\lambda^*) \in K_r$ and by Eq. (2), we have

$$\begin{aligned} 0 \leqslant (p(\lambda^*) - x^r)^{\mathrm{T}} f(x^r) &= (\lambda^* x + (1 - \lambda^*) x^r - x^r)^{\mathrm{T}} f(x^r) \\ &= \lambda^* (x - x^r)^{\mathrm{T}} f(x^r) \end{aligned}$$

which implies Eq. (3) holding, hence x^r is a solution to the problem (1). □

The following definition makes the concept of d-orientation sequence of a function precise.

Definition 2.1. Given $d > 0$, we say that $\{x^r\} \subset \mathbb{R}^n_+$ is a d-orientation sequence of the function f if $\|x^r\| \to \infty$ as $r \to \infty$, and for each x^r there exists a positive scalar, denoted by μ^r, such that

$$f_i(x^r) = -\mu^r d_i \quad \text{if } x_i^r > 0, \tag{4a}$$

$$f_i(x^r) \geqslant -\mu^r d_i \quad \text{if } x_i^r = 0. \tag{4b}$$

Remark 2.1. We use the term "d-orientation sequence" is the reason that all the vectors $f(x^r)$ corresponding to $x^r > 0$ have the same direction $-d$, i.e., for each $x^r > 0, f(x^r) = -\mu^r d$ for some scalar $\mu^r > 0$. This is quite different from the concept of exceptional family of elements for a function [8], which is defined as follows

$$f_i(x^r) = -\mu^r x_i^r \quad \text{for all } x_i^r > 0,$$

$$f_i(x^r) \geqslant 0 \quad \text{for all } x_i^r = 0,$$

where $\mu^r > 0$ is some positive scalar. It is evident that $f(x^r) = -\mu^r x^r$ for $x^r > 0$.

Remark 2.2. At a glance, the d-orientation sequence seems to be analogous to the *d-regularity of a mapping* introduced by Karamardian [15]. However, this two concepts are quite different. The d-regularity is defined as follows: "A mapping $f(x) - f(0)$ is said to be d-regular if the following equation has no solution in $\{(x,t) \in \mathbb{R}^{n+1}\colon\ x \in \mathbb{R}^n_+, t \geqslant 0\}$ with $x \neq 0$:

$$f_i(x) = -td_i + f_i(0) \quad \text{if } x_i^r > 0,$$

$$f_i(x) \geqslant -td_i + f_i(0) \quad \text{if } x_i^r = 0."$$

The main results of this section is as follows.

Theorem 2.1. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function, then there exists either a solution for the CP* (1) *or a d-orientation sequence of the function f.*

Proof. Assume that there exists no solution for the CP, we show that there exists a d-orientation sequence of f. Indeed, under this assumption, it follows from Lemma 2.1 that, for each $r > 0$, there exists no solution x^r of $\mathrm{VI}(K_r, f)$ such that $(x^r)^{\mathrm{T}} d < r$. Since K_r is bounded, the solution set of $\mathrm{VI}(K_r, f)$ is always nonempty [5]. Therefore, for any $r > 0$ the solution x^r of $\mathrm{VI}(K_r, f)$ must satisfy $(x^r)^{\mathrm{T}} d = r$. We now show that such a sequence $\{x^r\}$ is a d-orientation sequence of f.

Since x^r is a solution to $\mathrm{VI}(K_r, f)$, we have

$$x^r = P_{K_r}(x^r - f(x^r)).$$

That is, x^r is the unique solution to the following convex program which satisfies Slater's constrained qualification.

$$\begin{array}{ll} \min & \dfrac{1}{2}\|y - [x^r - f(x^r)]\|^2 \\ \text{s.t.} & y \in \{x \in \mathbb{R}^n\colon\ x \geqslant 0, \quad x^{\mathrm{T}} d \leqslant r\}. \end{array}$$

Hence x^r must satisfy the following Karush–Kuhn–Tucker condition. That is, there exist a vector $\lambda^r \in \mathbb{R}^n_+$ and a scalar $\mu^r \geqslant 0$ such that

$$[x^r - (x^r - f(x^r))] - \lambda^r + \mu^r d = 0, \tag{5a}$$

$$(\lambda_i^r)^{\mathrm{T}} x_i^r = 0 \quad \text{for all } i = 1, \dots, n, \tag{5b}$$

$$\mu^r((x^r)^{\mathrm{T}} d - r) = 0, \tag{5c}$$

$$x^r \geqslant 0, \quad (x^r)^{\mathrm{T}} d \leqslant r. \tag{5d}$$

Notice that x^r is a solution to $\mathrm{VI}(K_r, f)$ and that $(x^r)^{\mathrm{T}} d = r$, the relations (5c) and (5d) hold trivially. Hence the above conditions reduces to the following

$$f(x^r) = \lambda^r - \mu^r d, \quad (\lambda^r)^{\mathrm{T}} x^r = 0,$$

which can be written as

$$f_i(x^r) = -\mu^r d_i \quad \text{ if } x_i^r > 0,$$

$$f_i(x^r) \geqslant -\mu^r d_i \quad \text{if } x_i^r = 0.$$

We now show $\mu^r > 0$. Indeed, if $\mu^r = 0$, then the above relations reduce to

$$f(x^r) \geqslant 0, \quad x^r \geqslant 0, \quad (x^r)^{\mathrm{T}} f(x^r) = 0,$$

so that x^r is a solution to CP. This is in contradiction with the assumption at the beginning of the proof. Since $d > 0$, $\{x^r\} \subset \mathbb{R}^n_+$ and $(x^r)^{\mathrm{T}} d = r$, the sequence $\|x^r\|$ must tend to $+\infty$ as $r \to +\infty$. By the Definition 2.1, $\{x^r\}$ is a d-orientation sequence of f. □

Corollary 2.1. *If there exists no d-orientation sequence for the mapping* $f: \mathbb{R}^n \to \mathbb{R}^n$, *then the corresponding CP has a solution.*

We believe that Theorem 2.1 elicits an intrinsic property of any CPs. The concept of d-orientation sequence of a function, as exceptional family of elements introduced in [8], seems to be important, because it allows to prove a number of alternative theorems, and consequently sufficient conditions for the existence of a solution of CP. The sufficient condition function without d-orientation sequence is a very weak conditions. To demonstrate this, we show that the well-known Karamardian's condition (most of known existence theorems were elicited from this condition), and Isac and Gowda's condition imply the above sufficient condition.

Definition 2.2. We say that $f: \mathbb{R}^n \to \mathbb{R}^n$ satisfies the Karamardian's condition on $\mathbb{R}^n_+$ if there exists $D \subset \mathbb{R}^n_+$ convex compact set such that for all $x \in \mathbb{R}^n_+ \setminus D$, there exists $y \in D$ such that $(x - y)^{\mathrm{T}} f(x) \geqslant 0$.

Theorem 2.2. *If the function* $f: \mathbb{R}^n \to \mathbb{R}^n$ *satisfies the Karamardian's condition, then there exists no d-orientation sequence for f.*

Proof. Suppose that opposite is true, that is, f has a d-orientation sequence for f. Denote this sequence by $\{x^r\}$, we will show that the function f does not satisfy the Karamardian's condition on $\mathbb{R}^n_+$. To this end we will prove that for every compact convex set $D \subset \mathbb{R}^n_+$ there exists $r > 0$ such that $x^r \in \mathbb{R}^n_+ \setminus D$ and

$$(x^r - y)^{\mathrm{T}} f(x^r) < 0$$

for all $y \in D$. Let D be such a set. For any x^r and $y \in D$, if $x_i^r > 0$, by Eq. (4a) we have

$$(x_i^r - y_i) f_i(x^r) = (x_i^r - y_i)(-\mu^r d_i).$$

If $x_i^r = 0$, by Eq. (4b) and noting that $y_i \geqslant 0$, we have

$$(x_i^r - y_i) f_i(x^r) = (0 - y_i) f_i(x^r) \leqslant (0 - y_i)(-\mu^r d_i).$$

Hence

$$(x^r - y)^{\mathrm{T}} f(x^r) \leqslant (x^r - y)^{\mathrm{T}}(-\mu^r d) = -\mu^r[(x^r)^{\mathrm{T}} d - y^{\mathrm{T}} d]. \tag{6}$$

Since D is a compact set, there exists some $c > 0$ such that $y^{\mathrm{T}} d \leqslant c$ for all $y \in D$. since $\|x^r\| \to +\infty$, hence

$$(x^r - y)^{\mathrm{T}} f(x^r) \leqslant -\mu^r[(x^r)^{\mathrm{T}} d - y^{\mathrm{T}} d] < 0 \text{ as } r \to +\infty,$$

which shows that the Karamardian's condition does not hold. $\square$

Definition 2.3 ([10]). We say that $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is monotone decreasing on rays with respect to $\mathbb{R}^n_+$ if there exists $t_0 > 0$ such that for every $x \in \mathbb{R}^n_+$ and every s, t with $s \geqslant t \geqslant t_0$, we have

$$x^{\mathrm{T}}(\phi(tx) - \phi(sx)) \geqslant 0.$$

Lemma 2.2 ([9]). *ϕ is monotone decreasing on rays with respect to $\mathbb{R}^n_+$, if and only if for every $\alpha \geqslant 1$ and every $x \in \mathbb{R}^n_+$, we have*

$$x^{\mathrm{T}}(\phi(x) - \phi(\alpha x)) \geqslant 0.$$

Definition 2.4. We say that $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies Isac and Gowda's condition if there exists a positive scalar $p > 0$ such that $\phi(x) = \|x\|^{p-1} x - f(x)$ is a monotone decreasing on rays with respect to $\mathbb{R}^n_+$.

Remark 2.3. When $p = 1$, then $\phi(x) = x - f(x)$, which was studied in Ref. [9] by using the concept of exceptional family of elements.

Theorem 2.3. *If the continuous function $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies the Isac and Gowda's condition, then there exists no d-orientation sequence for f (hence the problem* (1) *has a solution).*

Proof. Assume the contrary, that is, there exists a d-orientation sequence for f. We denote this d-orientation sequence by $\{x^r\}$. Since $\phi(x) = \|x\|^{p-1}x - f(x)$ is monotone decreasing on rays with respect to $\mathbb{R}^n_+$, by Lemma 2.2, we have

$$x^{\mathrm{T}}(\phi(x) - \phi(\alpha x)) \geqslant 0, \quad \text{for all } x \in \mathbb{R}^n_+ \quad \text{and} \quad \alpha \geqslant 1. \tag{7}$$

Notice that $\|x^r\| \to +\infty$ as $r \to \infty$, there exists a scalar $r_0 > 0$ such that for all $r \geqslant r_0$, we have $\|x^r\| \geqslant 1$. Setting $\alpha = \|x^r\|$ and $x = x^r/\|x^r\|$ in Eq. (7), we have

$$\left(\frac{x^r}{\|x^r\|}\right)^{\mathrm{T}}[\phi(x^r/\|x^r\|) - \phi(x^r)] \geqslant 0$$

which is equivalent to

$$(x^r)^{\mathrm{T}}[\phi(x^r/\|x^r\|) - \|x^r\|^{p-1}x^r + f(x^r)] \geqslant 0. \tag{8}$$

Since $\{x^r\}$ is a d-orientation sequence for f, i.e., there exists $\{\mu^r\} > 0$ such that Eqs. (4a) and (4b) holds, hence

$$(x^r)^{\mathrm{T}}f(x^r) = -\mu^r\sum_{i\in I_+}x^r_i d_i = -\mu^r(x^r)^{\mathrm{T}}d$$

where $I_+ = \{i\colon\ x^r_i > 0\}$. Substituting this into Eq. (8), we have

$$(x^r)^{\mathrm{T}}[\phi(x^r/\|x^r\|)] - \|x^r\|^{p+1} - \mu^r(x^r)^{\mathrm{T}}d \geqslant 0.$$

Moreover

$$(x^r)^{\mathrm{T}}[\phi(x^r/\|x^r\|)] \geqslant \|x^r\|^{p+1}.$$

Since $T(\cdot)$ is a continuous function, it is bounded on the set $B(0,1) = \{x\colon \|x\| \leqslant 1\}$, therefore, there exists some $M > 0$ such that

$$M\|x^r\| \geqslant (x^r)^{\mathrm{T}}[\phi(x^r/\|x^r\|)] \geqslant \|x^r\|^{p+1}$$

which implies that

$$M \geqslant \|x^r\|^p \to +\infty \text{ as } r \to \infty.$$

A contradiction. □

It is possible that there exists a d-orientation sequence for f and a solution for the corresponding CP. See the following example.

Example 2.1. Let $f : \mathbb{R}^2_+ \to \mathbb{R}^2$ be defined by

$$f(x) = (1 - x_2, 2 - x_1)^{\mathrm{T}}, \quad x = (x_1, x_2)^{\mathrm{T}}.$$

It may readily be verified that $(0,0)$ is a solution to the problem (1). Let $d = (1,1)^{\mathrm{T}} > 0$, then it is evident that the sequence $\{x^r = (2+r, 1+r)^{\mathrm{T}}\}$ is a d-orientation sequence for this function, actually,

$$f_1(x^r) = (1 - x_2^r) = -r, \quad f_2(x^r) = -r.$$

Since $x_i^r > 0 (i = 1, 2)$, we have

$$f_i(x^r) = -\mu^r d_i \quad \text{for } i = 1, 2,$$

where $\mu^r = r, d_i = 1$.

This example also shows that while "f without d-orientation sequence" is a sufficient condition for the existence of a solution to CPs, however, it is not necessary in general. An interesting problem is when this sufficient condition is also necessary, the following result makes an affirmative answer to this question.

Theorem 2.4. *If $f : \mathbb{R}^n \to \mathbb{R}^n$ is a pseudo-monotone mapping, i.e., for each distinct pair $x, y \in \mathbb{R}^n$.*

$$f(x)^{\mathrm{T}}(y - x) \geqslant 0 \text{ implies } f(y)^{\mathrm{T}}(y - x) \geqslant 0,$$

then the CP (1) *has a solution if and only if there exists no d-orientation sequence for f.*

Proof. It suffices to show that if the CP has a solution, then there exists no d-orientation sequence for f. Actually, suppose x^* is such a solution for the problem, i.e.,

$$x^* \geqslant 0, \quad f(x^*) \geqslant 0, \quad (x^*)^{\mathrm{T}} f(x^*) \geqslant 0.$$

Equivalently

$$(x - x^*)^{\mathrm{T}} f(x^*) \geqslant 0 \quad \text{for all } x \in \mathbb{R}^n_+.$$

Since f is pseudo-monotone mapping, from the above we have

$$(x - x^*) f(x) \geqslant 0 \quad \text{for all } x \in \mathbb{R}^n_+. \tag{9}$$

Suppose that f has a d-orientation sequence, denoted by $\{x^r\}$, then by the same argument as Eq. (6), we have

$$(x^r - x^*) f(x^r) \leqslant -\mu^r((x^r)^{\mathrm{T}} d - (x^*)^{\mathrm{T}} d).$$

Since $\|x^r\| \to \infty$ as $r \to \infty$, the above inequality implies that

$$(x^r - x^*)^{\mathrm{T}} f(x^r) < 0 \quad \text{for all sufficiently large } r$$

which is in contradiction with Eq. (9). □

The similar result concerning exceptional family for VI problems has been proved by Zhao et al. [28–30].

3. *p*-Order generalized coercivity

The above results indicate that defining the conditions under which a continuous function does not posses a d-orientation sequence would provide new practical existence results of CPs. Along this idea, in the remainder of the paper, we will show two new existence theorems, one is related to so-called p-order generalized coercivity, the other is related to so-called P_*-mapping which is defined in Section 4. The two new existence theorems include several previous existence results as the special cases.

Definition 3.1. $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be a p-order generalized coercive function, if there exists some point $\hat{x} \in \mathbb{R}^n_+$ and a scalar $p \in (-\infty, 1]$ such that for each sequence $\{x^\alpha\} \subset \mathbb{R}^n_+$ with $\|x^\alpha\| \to +\infty$

$$\limsup_{x^\alpha \in \mathbb{R}^n_+, \|x^\alpha\| \to \infty} \frac{f(x^\alpha)^{\mathrm{T}}(x^\alpha - \hat{x})}{\|x^\alpha\|^p} > 0. \tag{10}$$

Theorem 3.1. *If f is a p-order generalized coercive function, then there exists no d-orientation sequence for the function f.*

Proof. We show this results by contradiction, suppose that $\{x^r\}$ be a d-orientation sequence for f. For any $p \in (-\infty, 1]$, replacing y by $\hat{x}$ in Eq. (6) and by using $\|x^r\| \to \infty$, we have

$$\frac{f(x^r)(x^r - \hat{x})}{\|x^r\|^p} \leqslant \frac{-\mu^r((x^r)^{\mathrm{T}}d - \hat{x}^{\mathrm{T}}d)}{\|x^r\|^p} \leqslant 0$$

which implies that f cannot be a p-order generalized coercive function. □

The two corollaries below are immediate consequences of Theorem 3.1.

Corollary 3.1. *If f is a coercive function* ([22,17,5]), *i.e., there exists a $\hat{x} \in \mathbb{R}^n_+$ such that*

$$\lim_{x \in \mathbb{R}^n_+, \|x\| \to \infty} \frac{f(x)^{\mathrm{T}}(x - \hat{x})}{\|x\|} = +\infty, \tag{11}$$

then there exists no d-orientation sequence for f.

Corollary 3.2. *If f satisfies the following condition* (*Guo and Yao* [4])

$$\liminf_{\|x\| \to +\infty, x \in \mathbb{R}^n_+} x^{\mathrm{T}} f(x) > 0, \tag{12}$$

then there exists no d-orientation sequence for f.

It is easy to see that the p-order generalized coercivity Eq. (10) is strictly weaker than the well-known coercive condition (11) and Guo and Yao's condition (12). See the following example.

Example 3.1. Given $\beta > 0$, we consider the function $f : \mathbb{R}^n \to \mathbb{R}^n$, where

$$f = \frac{x}{1 + \|x\|^{2+\beta}}.$$

We show this function is a $(-\beta)$-order generalized coercive mapping, but it is not a coercive function, and this function does not satisfy Guo and Yao's condition. Indeed, for any $\hat{x} \in \mathbb{R}^n_+$, since

$$\lim_{x\in\mathbb{R}^n_+, \|x\|\to+\infty} \frac{f(x)^{\mathrm{T}}(x-\hat{x})}{\|x\|} = \lim_{x\in\mathbb{R}^n_+, \|x\|\to+\infty} \frac{x^{\mathrm{T}}(x-\hat{x})}{\|x\|(1+\|x\|^{\beta+2})} = 0.$$

Thus f is not a coercive function. Noting that

$$\liminf_{x\in\mathbb{R}^n_+, \|x\|\to\infty} x^{\mathrm{T}} f(x) = \lim_{x\in\mathbb{R}^n_+, \|x\|\to+\infty} \frac{\|x\|^2}{1+\|x\|^{\beta+2}} = 0.$$

so that f does not satisfy Guo and Yao's condition. However, for any $\beta > 0$ this function is a $(-\beta)$-order generalized coercive mapping, since for any $\hat{x} \in \mathbb{R}^n_+$

$$\begin{aligned}\lim_{x\in\mathbb{R}^n_+, \|x\|\to+\infty} \frac{f(x)^{\mathrm{T}}(x-\hat{x})}{\|x\|^{-\beta}} &= \lim_{x\in\mathbb{R}^n_+, \|x\|\to+\infty} \frac{\|x\|^{\beta}(\|x\|^2 - x^{\mathrm{T}}\hat{x})}{1+\|x\|^{\beta+2}} \\ &= \lim_{x\in\mathbb{R}^n_+, \|x\|\to+\infty} \frac{\|x\|^{\beta+2}[1 - x^{\mathrm{T}}\hat{x}/\|x\|^2]}{1+\|x\|^{\beta+2}} = 1.\end{aligned}$$

which implies that for any sequence $\{x^\alpha\} \in \mathbb{R}^n_+$ with $\|x^\alpha\| \to \infty$

$$\limsup_{\|x^\alpha\|\to\infty} \frac{f(x^\alpha)^{\mathrm{T}}(x^\alpha - \hat{x})}{\|x^\alpha\|^{-\beta}} = 1 > 0.$$

It should be noted that the p-order generalized coercivity is not a necessary condition to assure the CP having a solution.

Example 3.2. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$, where

$$f(x) = (x_1 \sin(x_1 + x_2), x_2 \sin(x_1 + x_2))^{\mathrm{T}}, \quad x = (x_1, x_2)^{\mathrm{T}}.$$

Obviously, (0,0) is a solution to the corresponding CP. However, for any $p \in (-\infty, 1]$, this function is not a p-order generalized coercive mapping. Indeed for any $\hat{x} = (\hat{x}_1, \hat{x}_2)^{\mathrm{T}} \in \mathbb{R}^n_+$, let $\{x^k = (k\pi, k\pi)^{\mathrm{T}}\}$, then

$$\limsup_{\|x^k\|\to+\infty} \frac{f(x^k)(x^k - \hat{x})}{\|x^k\|^p} = 0.$$

It is evident that Guo and Yao's condition implies the condition below:

$$\text{There exits } \beta > 0 \text{ such that } x^{\mathrm{T}} f(x) \geqslant 0 \\ \text{for all } x \in \mathbb{R}^n_+ \text{ with } \|x\| \geqslant \beta. \tag{13}$$

It is easy to see that such a condition is strictly weaker than Guo and Yao's condition. An implication of the following result is that the condition (13) also implies that there exists no d-orientation sequence for f.

Theorem 3.2. *If there exists a constant $\beta > 0$ such that for all $x \in \mathbb{R}^n_+$ with $\|x\| \geqslant \beta$, there exists a $y \in \mathbb{R}^n_+$ such that $\|y\| < \beta$ and $(x-y)^{\mathrm{T}} f(x) \geqslant 0$, then the function f is without d-orientation sequence.*

Proof. We suppose that f has an d-orientation sequence $\{x^r\}$, by Eq. (6)

$$(x^r - y)^{\mathrm{T}} f(x^r) \leqslant -\mu^r((x^r)^{\mathrm{T}} d - y^{\mathrm{T}} d).$$

Since $\|x^r\| \to +\infty$ and $\|y\| < \beta$, the above inequality implies $(x^r - y)^{\mathrm{T}} f(x^r) < 0$ for all sufficiently large r, this is in contradiction with the assumption of the theorem.

Remark 3.1. When $y = 0$, the condition of Theorem 3.2 reduces to the condition (13). Theorem 3.2 also shows that when reduced to CP, the condition "quasimonotone" of (Theorem 3.1, [6]) due to Hadjisavvas and Schaible is not necessary.

Definition 3.2. We say the mapping $f: \mathbb{R}^n \to \mathbb{R}^n$ is nonnegative at infinity if for any $y \in \mathbb{R}^n_+$, there exists a positive real number $\beta(y)$ such that $(x-y)^{\mathrm{T}} f(x) \geqslant 0$ for every $x \in \mathbb{R}^n_+$ such that $\|y\| \geqslant \beta(y)$.

Remark 3.2. If we substitute the term "$(x-y)^{\mathrm{T}} f(x) \geqslant 0$" by "$(x-y)^{\mathrm{T}} f(x) > 0$" in the above definition, then the above concept reduces to concept *positive at infinity* introduced by Schaible and Yao [24].

Theorem 3.3. *If $f: \mathbb{R}^n \to \mathbb{R}^n$ is nonnegative at infinity then there exists no d-orientation sequence for f.*

Proof. Along the same idea of proof of Theorem 5 in Ref. [9] and by using Eq. (6), it is easy to verify the result.

4. Generalized P_*-complementarity problems

In this section, we will define a new class of nonlinear mappings, which is broad enough to encompass the monotone mappings as the special cases. We show the corresponding CP has a solution under strictly feasibility condition (i.e., there exists some $u \geqslant 0$ such that $f(u) > 0$).

Definition 4.1. A mapping $f: \mathbb{R}^n \to \mathbb{R}^n$ is said to be a P_*-mapping, if there exists a nonnegative constant γ such that the following inequality holds for any distinct point $x, y \in \mathbb{R}^n$.

$$(1+\gamma) \sum_{j \in I_+(x,y,f)} (x_j - y_j)(f_j(x) - f_j(y)) + \min_{1 \leqslant j \leqslant n} (x_j - y_j)(f_j(x) - f_j(y)) \geqslant 0, \tag{14}$$

where

$$I_+(x,y,f) = \{j: (x_j - y_j)(f_j(x) - f_j(y)) \geqslant 0\}.$$

Let j_0 denotes a min index, i.e.,

$$(x_{j_0} - y_{j_0})(f_{j_0}(x) - f_{j_0}(x)) = \min_{1 \leqslant j \leqslant n} (x_j - y_j)(f_j(x) - f_j(y)) \tag{15}$$

then Eq. (14) can be written as

$$\begin{aligned}(x-y)^{\mathrm{T}}(f(x) - f(y)) \geqslant & -\gamma \sum_{j \in I_+(x,y,f)} (x_j - y_j)(f_j(x) - f_j(y)) \\ & + \sum_{j \in I_-(x,y,f) \setminus \{j_0\}} (x_j - y_j)(f_j(x) - f_j(y)),\end{aligned}$$

where

$$I_-(x,y,f) = \{j: (x_j - y_j)(f_j(x) - f_j(y)) < 0\}.$$

It is easy to see that a monotone mapping must be a P_*-mapping. The linear P_*-mapping (i.e., $Mx + q$, where M is P_*-matrix) was first defined by Kojima et al. [18], then Väliaho [26,27] pointed out that the class of P_*-matrices are just the class of sufficient matrices. Recently the CP with P_*-mapping has been studied in the field of interior-point algorithm, see for example, [12,25, 20,11,31], etc.

The following is our main result in this section.

Theorem 3.1. *For any P_*-mapping $f: \mathbb{R}^n \to \mathbb{R}^n$, if there exists a point $u \in \mathbb{R}^n_+$ such that $f(u) > 0$, then the function f has no d-orientation sequence* (*hence the problem* (1) *has at least a solution.*)

Proof. Assume that contrary, let $\{x^r\} \subset \mathbb{R}^n_+$ be the d-orientation sequence for f, then by the Definition 2.1, there exists a positive sequence $\{\mu^r\} > 0$ such that Eqs. (4a) and (4b) hold, hence

$$(f_i(x^r) - f_i(u))(x_i^r - u_i) \begin{cases} = -(\mu^r d_i + f_i(u))(x_i^r - u_i) & \text{if } x_i^r > 0, \\ \leqslant -(\mu^r d_i + f_i(u))(x_i^r - u_i) & \text{if } x_i^r = 0. \end{cases}$$

That is, for all $i = 1, \ldots, n$, we have

$$(f_i(x^r) - f_i(u))(x_i^r - u_i) \leqslant -(\mu^r d_i + f_i(u))(x_i^r - u_i). \tag{16}$$

Since $\|x^r\| \to \infty$, there exists at least one component index i_0 such that $x_{i_0}^r \to \infty$ as $r \to +\infty$. Notice that $\mu^r > 0$ and $d_i > 0$, it is evident that

$$\begin{aligned}(f_{i_0}(x^r) - f_{i_0}(u))(x_{i_0}^r - u_{i_0}) &\leqslant -(\mu^r d_{i_0} + f_{i_0}(u))(x_{i_0}^r - u_{i_0}) \\ &\leqslant -f_{i_0}(u)(x_{i_0}^r - u_{i_0}) \to -\infty\end{aligned} \tag{17}$$

which implies the index set $I_-(x^r, u, f) \neq \emptyset$, thus by the definition of P_*-mapping, the set $I_+(x^r, u, f) \neq \emptyset$. There exists a subsequence of $\{x^r\}$, denoted by $\{x^{r_j}\}$ $(j = 1, 2, \ldots)$ such that for some fixed index p and q

$$(f_p(x^{r_j}) - f_p(u))(x_p^{r_j} - u_p) = \min_{1 \leqslant j \leqslant n} (x_i^{r_j} - u_i)(f_i(x^{r_j}) - f_i(u)) \tag{18}$$

and

$$(f_q(x^{r_j}) - f_q(u))(x_q^{r_j} - u_q) = \max_{1 \leqslant j \leqslant n} (x_i^{r_j} - u_i)(f_i(x^{r_j}) - f_i(u)) > 0, \tag{19}$$

hold for all the sequence $\{x^{r_j}\}$. Since f is a P_*-mapping, and by using Eqs. (18), (14) and (19), we have

$$\begin{aligned}(f_{i_0}(x^r) - f_{i_0}(u))(x_{i_0}^r - u_{i_0}) &\geqslant (f_p(x^{r_j}) - f_p(u))(x_p^{r_j} - u_p) \\ &\geqslant -(1+\gamma) \sum_{I_+(x,u,f)} (x_i^{r_j} - u_i)(f_i(x^{r_j}) - f_i(u)) \\ &\geqslant -(1+\gamma)(n-1) \max_{1 \leqslant j \leqslant n} (x_i^{r_j} - u_i)(f_i(x^{r_j}) - f_i(u)) \\ &= -(1+\gamma)(n-1)(f_q(x^{r_j}) - f_q(u))(x_q^{r_j} - u_q).\end{aligned} \tag{20}$$

When $x_i^{r_j} > u_i$, then

$$(x_i^{r_j} - u_i)(f_i(x^{r_j}) - f_i(u)) = -(\mu^r d_i + f_i(u))(x_i^{r_j} - u_i) < 0.$$

Hence from Eq. (19), we conclude that $0 \leqslant x_q^{r_j} \leqslant u_q$. If $x_q^{r_j} = 0$, then

$$(f_q(x^{r_j}) - f_q(u))(x_q^{r_j} - u_q) \leqslant (\mu^{r_j} d_q + f_q(u))u_q.$$

If $0 < x_q^{r_j} \leqslant u_q$, then

$$\begin{aligned}(f_q(x^{r_j}) - f_q(u))(x_q^{r_j} - u_q) &= (\mu^{r_j} d_q + f_q(u))(u_q - x_q^{r_j}) \\ &\leqslant (\mu^{r_j} d_q + f_q(u))u_q.\end{aligned}$$

Therefore, Eq. (20) can be written as follows

$$(f_{i_0}(x^r) - f_{i_0}(u))(x_{i_0}^r - u_{i_0}) \geqslant -(1+\gamma)(n-1)(\mu^{r_j} d_q + f_q(u))u_q.$$

That is

$$-(\mu^{r_j} d_{i_0} + f_{i_0}(u))(x_{i_0}^{r_j} - u_{i_0}) \geqslant -(1+\gamma)(n-1)(\mu^{r_j} d_q + f_q(u))u_q.$$

Multiplying both sides by $1/(x_{i_0}^{r_j} - u_{i_0})$, and rearranging the terms, we have

$$-\mu^{r_j}\left[d_{i_0}-\frac{(1+\gamma)(n-1)d_q}{x_{i_0}^{r_j}-u_{i_0}}\right]\geqslant f_{i_0}(u)-\frac{(1+\gamma)(n-1)f_q(u)u_q}{x_{i_0}^{r_j}-u_{i_0}}.$$

Notice that $x_{i_0}^{r_j}\to+\infty$, the above inequality is impossible to hold for sufficiently large r_j, since in the case, the left-hand side of the above inequality is negative, however, the right-hand side tends to $f_{i_0}(u)$ which is a positive number. $\square$.

Remark 4.1. For nonlinear monotone mapping f, the feasible condition, i.e., "there exists a point $u\in\mathbb{R}^n_+$ such that $f(u)\geqslant 0$" cannot assure the existence of a solution to the corresponding nonlinear CP. Megiddo [19] gave an example to show the case. Notice that a monotone mapping must be a P_*-mapping, therefore, we conclude that the strictly feasible condition, i.e. "there exists a point $u\in\mathbb{R}^n_+$ such that $f(u)>0$" of Theorem 4.1 cannot be replaced by *feasibility condition* to assure the same result.

References

[1] R.W. Cottle, J.S. Pang, R.W. Stone, The Linear Complementarity Problem, Academic Press, Boston, 1992.

[2] R.W. Cottle, J.C. Yao, Pseudomonotone complementarity problem in Hilbert space J. Optim. Theory Appl. 75 (1992) 281–295.

[3] B.C. Eaves, On the basic theorem of complementarity problem, Math. Programming 1 (1971) 68–75.

[4] J.S. Guo, J.C. Yao, The variational inequalities, nonlinear complementarity problems, SIAM J. Control Optim. 16 (1978) 504–511.

[5] P.T. Harker, J.S. Pang, Finite-dimensional variational inequality and nonlinear complementarity problems: a survey of theory, algorithms and applications, Math. Programming 48B (1990) 161–220.

[6] N. Hadjisavvas, S. Schaible, Quasimonotone variational inequalities in Banach space, J. Optim. Theory Appl. 90 (1996) 95–111.

[7] G. Isac, Complementarity Problems, Lecture Notes in Mathematics, Springer, Nr. 1528, 1992.

[8] G. Isac, V. Bulavski, V. Kalashnikov, Exceptional families, topological degree and complementarity problems, J. Global Optim. 10 (1997) 1–15.

[9] G. Isac, W.T. Obuchowska, Functions without exceptional family of elements and complementarity problems, J. Optim. Theory Appl. 99 (1998) 147–163.

[10] G. Isac, M.S. Gowda, Operators of class $(S)^1_+$, Altman's condition and the complementarity problem, J. Fac. Sci. Univ. Tokyo, Sect. IA. 40 (1993) 1–16.

[11] B. Jansen, K. Roos, T. Terlaky, A. Yoshise, Polynomiality of primal dual affine scaling algorithm for nonlinear complementarity problems, TR 95-83, faculty of Technical Mathematics and Computer Science, Delf University of Technology, Delft, the Netherlands, 1995.

[12] J. Ji, F.A. Potra, R. Sheng, A predictor-corrector method for solving the P_*-matrix LCP from infeasible starting points, Optim. Meth. Software 6 (1995) 109–126.

[13] S. Karamardian, An existence theorem for the complementarity problem, J. Optim. Theory Appl. 19 (1976) 227–232.

[14] S. Karamardian, Generalized complementarity problem, J. Optim. Theory Appl. 8 (1971) 161–167.

[15] S. Karamardian, The complementarity problem, Math. Programming 2 (1972) 107–129.
[16] S. Karamardian, Complementarity problems over cones with monotone and pseudomonotone maps, J. Optim. Theory Appl. 18 (1976) 445–454.
[17] D. Kinderlehrer, G. Stampacchia, An Introduction to Variational Inequalities and Their Applications, Academic Press, New York, 1980.
[18] M. Kojima, N. Megiddo, T. Noma, A. Yoshise, A Unified Approach to Interior Point Algorithm for Linear Complementarity Problems, Lecture Notes in Computer Sciences, Vol. 538, Springer, New York, 1991.
[19] N. Megiddo, A monotone complementarity problem with feasible solutions but no complementarity solutions, Math. Programming 12 (1977) 131–132.
[20] J.M. Miao, A quadratically convergent $O((\kappa+1)\sqrt{n}L$-iteration algorithm for the P_*-mapping matrix linear complementarity problem, Math. Programming 69 (1995) 355–374.
[21] J.J. Moré, Classes of function and feasibility condition in nonlinear complementarity problems, Math. Programming 6 (1974) 327–338.
[22] J.J. Moré, Coercivity conditions in nonlinear complementarity problem, SIAM Review 16 (1974) 1–16.
[23] T.E. Smith, A solution condition for complementarity problems, with an application to spatial price equilibrium, Appl. Math. Comput. 15 (1984) 61–69.
[24] S. Schaible, J.C. Yao, On the equivalence of nonlinear complementarity problems and least element problems, Math. Programming 70 (1995) 191–200.
[25] F.A. Potra, R. Sheng, Predictor-corrector algorithm for solving $P_*(\kappa)$-matrix LCP from arbitrary positive starting point, Math. Programming 76 (1997) 223–244.
[26] H. Väliaho, P_* matrices are just sufficient, Linear Algebra Appl. 239 (1996) 103–108.
[27] H. Väliaho, Determining the handicap of a sufficient matrix, Linear Algebra Appl. 253 (1997) 279–298.
[28] Y.B. Zhao, J.Y. Han, Exceptional family of elements for a variational inequality and its applications, Technical Report, Institute of Applied Mathematics, Chinese Academy of Sciences, Beijing, 1996. Journal of Global Optimization, forthcoming.
[29] Y.B. Zhao, Exceptional family and finite-dimensional variational inequality over polyhedral sets, Appl. Math. Comput. 87 (1997) 111–126.
[30] Y.B. Zhao, J.Y. Han, H.D. Qi, Exceptional family and existence theorems for variational inequalities, Technical Report, Institute of Applied Mathematics, Chinese Academy of Science, Beijing, China, 1997. Journal of Optimization Theory and Applications, 101 (1999), forthcoming.
[31] Y.B. Zhao, J.Y. Han, On the polynomiality of two interior-point methods for non-monotone nonlinear complementarity problems, Technical Report, Institute of Applied Mathematics, Chinese Academy of Science, Beijing, China, 1997. Journal of Optimization Theory and Applications, 103 (1999), forthcoming.