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Convexity Conditions and the Legendre-Fenchel Transform for the Product of Finitely Many Positive Definite Quadratic Forms

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Abstract While the product of finitely many convex functions has been investigated in the field of global optimization, some fundamental issues such as the convexity condition and the Legendre-Fenchel transform for the product function remain unresolved. Focusing on quadratic forms, this paper is aimed at addressing the question: *When is the product of finitely many positive definite quadratic forms convex, and what is the Legendre-Fenchel transform for it*? First, we show that the convexity of the product is determined intrinsically by the condition number of so-called 'scaled matrices' associated with quadratic forms involved. The main result claims that if the condition number of these scaled matrices are bounded above by an explicit constant (which depends only on the number of quadratic forms involved), then the product function is convex. Second, we prove that the Legendre-Fenchel transform for the product of positive definite quadratic forms can be expressed, and the computation of the transform amounts to finding the solution to a system of equations (or equally, finding a Brouwer's fixed point of a mapping) with a special structure. Thus, a broader question than the open "Question 11" in Hiriart-Urruty (SIAM Rev. 49, 225–273, 2007) is addressed in this paper.

Keywords Matrix analysis · Convex analysis · Legendre-Fenchel transform · Quadratic forms · Positive definite matrices · Condition numbers

1 Introduction

Optimization problems having a product of convex functions as an objective or a constraint are called 'multiplicative programming' problems which have been extensively investigated in the field of global optimization (see e.g. [3, 4, 19–21, 32, 34,

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35]). The multiplicative programming problem may find applications in such areas as microeconomics, geometric design, finance, VLSI chip design, and system reliability [4, 11, 20]. The product function is not only used in optimization, but in other areas as well. For instance, the product of finitely many quadratic forms in random variables has been widely studied in probability and statistics [9, 10, 16, 18, 27–29].

Matsui [30] showed that the linear multiplicative programming problem is NP-hard. Thus the multiplicative programming is not an 'easy' class of optimization problems. Part of the reason can be understood from the fact that the product of convex functions is not convex in general. For instance, the product $(y^TAy)(y^TA^{-1}y)$, where A is an $n \times n$ positive definite matrix, is not convex in general. While the general multiplicative programming problem is NP-hard, for a given problem it is not always so negative if we can prove that the problem is convex. Thus a natural and fundamental question is: *when is the product of finitely many convex functions convex*? It is interesting to address this question since answering it may identify a subclass of multiplicative optimization problems that can be computationally tractable. However, developing a convexity condition for the product function is not straightforward, and very limited progresses on this issue were made so far: The product of univariate convex functions and the product of two positive definite quadratic forms in R^n were studied in [12] and [36], respectively.

On the other hand, the Legendre-Fenchel transform (LF-transform for short) plays a vital role in developing optimization theory and algorithms (see e.g. [1, 5, 6, 14, 33]), and it has wide applications also in other areas of applied mathematics [7, 25]. Recall that for a given function $h : R^n \to R$, the LF-transform of h is defined by

$$h^*(x) = \sup_{y \in R^n} x^T y - h(y).$$

From a practical application point of view, it is important to obtain an explicit expression of the LF-transform. Unfortunately, for the product of convex functions, the question of whether its LF-transform can be explicitly expressed remains open in many situations even for the product of quadratic forms. So another fundamental issue associated with the product function is: *what is the LF-transform of the product of finitely many convex functions*? It is worth mentioning that some recent efforts on effective computation and expression of the LF-transform, stimulated by different needs, can be found in [2, 7, 12, 15, 23–26, 37].

As in the situation of the convexity condition, there is very little knowledge about the LF-transform of the product of convex functions so far. The initial discussion on the product of univariate convex functions was given in [12], and the LF-transform of the product of two positive definite quadratic forms was posted as an open question in the field of nonlinear analysis and optimization (see 'Question 11' in [13]). Recently, this open question has been addressed in [36]. Let q_A denote the quadratic form $q_A(y) = (1/2)y^TAy$, where A is an $n \times n$ symmetric positive definite matrix. The following result was established in [36]: (i) If A, B are positive definite and $f = q_A q_B$ is convex, then f^* can be expressed explicitly as a function which is homogeneous of degree $\frac{4}{3}$, and the computation of f^* can be implemented via finding a root of a univariate polynomial equation; (ii) there exists a positive constant $\gamma > 0$

(which can be given explicitly) such that if the condition number of the scaled matrix $B^{-1/2}AB^{-1/2}$ is less than or equal to the constant γ, then the product $f = q_A q_B$ is convex.

However, it is quite challenging to provide a general answer to the aforementioned question concerning the convexity and LF-transform for the product of general convex functions. The aim of this paper is to address the question in the case of finitely many positive definite quadratic forms: *when is the product of finitely many positive definite quadratic forms convex, and what is the LF-transform for it*? The contribution of this paper is twofold: a general sufficient convexity condition for the product of quadratic forms is established and an explicit expression of its LF-transform is derived in this paper. First, the convexity result claims that if the condition number of 'scaled matrices' are not too large (bounded above by a constant which depends on the number of quadratic forms), then the product function is convex. To our knowledge, this is the first general convexity condition for the product of finitely many quadratic forms. Secondly, we prove that if the product function is convex then its LF-transform can be explicitly expressed as a nonnegative function which is positively homogeneous of degree $\frac{2m}{2m-1}$, where m is the number of quadratic forms (see Theorem 3.6 and Remark 3.7). Thus, a broader question than the open "Question 11" in [13] is addressed. The analysis in this paper shows that the computation of the LF-transform can be implemented via solving a system of smooth equations (or equally, finding a fixed-point of a smooth mapping) with a special structure. It should be mentioned that many discussions and the proof for the case of only two quadratic forms in [36] cannot be directly generalized to the case of more than two quadratic forms.

This paper is organized as follows. In Sect. 2, we establish a general sufficient convexity condition for the product of finitely many quadratic forms. In Sect. 3, a series of useful technical results are proved, based on which an explicit formula for the LF-transform of the product function is derived. Conclusions are given in the last section.

2 When Is the Product Function Convex?

Throughout this paper, R^n_{++} is used to denote the positive orthant of the n-dimensional Euclidean space R^n, i.e., the set of all vectors with positive components, and I is used to denote the identity matrix with an appropriate dimension. If M is a matrix, $M \succ 0$ means a symmetric, positive definite matrix, and $\kappa(M)$ denotes the condition number of M, i.e., $\kappa(M) = \lambda_{\max}(M)/\lambda_{\min}(M)$, the ratio of its largest and smallest eigenvalues.

Let $f : R^n \to R$ denote the product of finitely many quadratic forms, i.e.,

$$f(y) = \prod_{i=1}^{m} \left(\frac{1}{2} y^T A_i y\right) = \prod_{i=1}^{m} q_{A_i}(y)$$

where $m \geq 2$ and $A_i, i = 1, \dots, m$, are $n \times n$ symmetric matrices ($n \geq 1$). Clearly, the gradient and the Hessian matrix of f are given by

$$\nabla f(y) = \sum_{i=1}^{m} \left(\prod_{j=1, j\neq i}^{m} q_{A_j}(y) \right) A_i y, \tag{1}$$

$$\begin{aligned} \nabla^2 f(y) = & \sum_{i=1}^{m} \left(\prod_{j=1, j\neq i}^{m} q_{A_j}(y) \right) A_i \\ & + \sum_{i=1}^{m} \sum_{j=1, j\neq i}^{m} \left(\prod_{k=1, k\neq i, j}^{m} q_{A_k}(y) \right) A_i y y^T A_j. \end{aligned} \tag{2}$$

When $m = 2$, we see that (2) is reduced to

$$\nabla^2 f(y) = q_{A_1}(y) A_2 + q_{A_2}(y) A_1 + A_1 y y^T A_2 + A_2 y y^T A_1,$$

and when $m = 3$, (2) is reduced to

$$\begin{aligned} \nabla^2 f(y) = & q_{A_2}(y) q_{A_3}(y) A_1 + q_{A_1}(y) q_{A_3}(y) A_2 + q_{A_1}(y) q_{A_2}(y) A_3 \\ & + q_{A_1}(y) \left(A_2 y y^T A_3 + A_3 y y^T A_2 \right) + q_{A_2}(y) \left(A_1 y y^T A_3 + A_3 y y^T A_1 \right) \\ & + q_{A_3}(y) \left(A_1 y y^T A_2 + A_2 y y^T A_1 \right). \end{aligned}$$

Given the two positive definite matrices A_i and A_j, the term $A_i y y^T A_j$ is not necessarily positive semi-definite, and hence the product function $f(y)$ may lose its convexity. Since f is twice continuously differentiable in R^n, to develop a convexity condition for f, it is sufficient to identify the condition under which its Hessian matrix is positive semi-definite at any point $y \in R^n$. By (2), for any $x \in R^n$, we have

$$\begin{aligned} & x^T \nabla^2 f(y) x \\ & = \sum_{i=1}^{m} \left\{ \left[\frac{1}{2^{m-1}} \prod_{j=1, j\neq i}^{m} y^T A_j y \right] x^T A_i x \right. \\ & \qquad \left. + \sum_{j=1, j\neq i}^{m} \left[\frac{1}{2^{m-2}} \prod_{k=1, k\neq i, j}^{m} y^T A_k y \right] x^T A_i y y^T A_j x \right\} \\ & = \frac{1}{2^{m-1}} \Phi(x, y) \end{aligned} \tag{3}$$

where

$$\Phi(x, y) := \sum_{i=1}^{m} \left[\prod_{j=1, j\neq i}^{m} y^T A_j y \right] x^T A_i x$$

$$+2\sum_{i=1}^{m}\sum_{j=1,j\neq i}^{m}\left[\prod_{k=1,k\neq i,j}^{m} y^T A_k y\right](x^T A_i y)(y^T A_j x). \tag{4}$$

Thus, to prove that (2) is positive semi-definite for any $y \in R^n$, it suffices to show that $\Phi(x, y) \geq 0$ for any $x, y \in R^n$. We will make use of the result below.

Lemma 2.1 ([17], Theorem 7.4.34). *Let M be a given $n \times n$ matrix and $M \succ 0$. Then*

$$\left(x^T M y\right)^2 \leq \left(\frac{\lambda_{\max}(M) - \lambda_{\min}(M)}{\lambda_{\max}(M) + \lambda_{\min}(M)}\right)^2 (x^T M x)(y^T M y)$$

for every pair of orthogonal vectors $x, y \in R^n$, i.e., $x^T y = 0$.

It should be stressed that the vectors x, y in the lemma above are required to be orthogonal.

For any $M \succ 0$, in the remainder of this paper we denote by

$$\chi(M) = \frac{\lambda_{\max}(M) - \lambda_{\min}(M)}{\lambda_{\max}(M) + \lambda_{\min}(M)} = \frac{\kappa(M) - 1}{\kappa(M) + 1}.$$

For any pair of matrices $A, B \succ 0$, it is easy to verify that $\kappa(B^{-1/2}AB^{-1/2}) = \kappa(A^{-1/2}BA^{-1/2})$, and thus $\chi(B^{-1/2}AB^{-1/2}) = \chi(A^{-1/2}BA^{-1/2})$. Hence, when we consider the condition number of these matrices, we do not distinguish between $B^{-1/2}AB^{-1/2}$ and $A^{-1/2}BA^{-1/2}$.

The next result plays a key role in developing our main convexity condition for the product function.

Lemma 2.2 *Let $\eta > 0$ be any given positive number. For any $n \times n$ matrices $A \succ 0$ and $B \succ 0$, if $\chi(B^{-1/2}AB^{-1/2}) \leq \sqrt{\frac{2\eta}{\eta+1}}$, then*

$$\Gamma_{(A,B,\eta)}(x, y) := \eta\left(x^T A x y^T B y + x^T B x y^T A y\right) + 2(x^T A y)(x^T B y) \geq 0$$

for any vectors $x, y \in R^n$.

Proof Denote by $P = B^{-1/2}AB^{-1/2}$. By the nonsingular linear transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} B^{-1/2} & 0 \\ 0 & B^{-1/2} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

we may reformulate Γ as follows:

$$\begin{aligned}
&\Gamma_{(A,B,\eta)}(x, y)\\
&\quad = \eta\left[u^T(B^{-1/2}AB^{-1/2})u v^T v + u^T u v^T(B^{-1/2}AB^{-1/2})v\right]\\
&\qquad + 2u^T(B^{-1/2}AB^{-1/2})v u^T v
\end{aligned}$$

$$= \eta \left[(u^T P u) v^T v + u^T u (v^T P v) \right] + 2(u^T P v) u^T v$$
$$=: \theta_{(P,\eta)}(u, v).$$

Thus, to prove $\Gamma_{(A,B,\eta)}(x, y) \geq 0$ for any $x, y \in R^n$, it is sufficient to show that

$$\theta_{(P,\eta)}(u, v) \geq 0 \tag{5}$$

for any $u, v \in R^n$. In fact, if $u^T v = 0$, it is evident from (5) that $\theta_{(P,\eta)}(u, v) \geq 0$. Thus, in what follows we assume that $u^T v \neq 0$. Let L_u denote the subspace generated by u and $L_u^\perp$ be the orthogonal subspace of L_u, i.e.,

$$L_u = \{tu : t \in R\}, \qquad L_u^\perp = \{w : u^T w = 0, w \in R^n\}.$$

Since $u^T v \neq 0$ (i.e., $v \notin L_u^\perp$), the vector v can be represented as $v = \widehat{u} + \widehat{v}$ for some $\widehat{u} \in L_u$ and $\widehat{v} \in L_u^\perp$. By the structure of L_u, the vector $\widehat{u} = tu$ for some $t \in R$ where $t \neq 0$ (since otherwise $v = \widehat{v} \in L_u^\perp$). From (5), we see that $\theta_{(P,\eta)}$ is homogeneous of degree 2 in v. Thus,

$$\theta_{(P,\eta)}(u, v) = \theta_{P,\eta}(u, tu + \widehat{v}) = \theta_{P,\eta}(u, t(u + \widehat{v}/t)) = t^2 \theta_{(P,\eta)}(u, u + \widehat{v}/t).$$

Notice that $\widehat{v}/t \in L_u^\perp$. Thus, to prove $\theta_{P,\eta}(u, v) \geq 0$ it is sufficient to prove that

$$\theta_{(P,\eta)}(u, u + z) \geq 0 \quad \text{for any } z \text{ such that } u^T z = 0.$$

First, for any z such that $u^T z = 0$, we note that

$$\begin{aligned}
&\theta_{(P,\eta)}(u, u + z) \\
&\quad = \eta \left[u^T P u (u + z)^T (u + z) + u^T u (u + z)^T P (u + z) \right] + 2u^T P (u + z) u^T (u + z) \\
&\quad = \eta \left[u^T P u (u^T u + z^T z) + u^T u (u^T P u + 2u^T P z + z^T P z) \right] \\
&\qquad + 2u^T P u u^T u + 2u^T P z u^T u \\
&\quad = (u^T P u) u^T u \left\{ 2(\eta + 1) + \eta \frac{z^T z}{u^T u} + 2(\eta + 1) \frac{u^T P z}{u^T P u} + \eta \frac{z^T P z}{u^T P u} \right\}.
\end{aligned} \tag{6}$$

Since $u^T z = 0$, by Lemma 2.1 we see that $|u^T P z| \leq \chi(P) \sqrt{u^T P u z^T P z}$ which implies that $u^T P z \geq -\chi(P) \sqrt{u^T P u z^T P z}$. Therefore, from (6) we have

$$\begin{aligned}
&\theta_{(P,\eta)}(u, u + z) \\
&\quad \geq (u^T P u) u^T u \left(2(\eta + 1) + \eta \frac{z^T z}{u^T u} - 2(\eta + 1) \chi(P) \frac{\sqrt{u^T P u z^T P z}}{u^T P u} + \eta \frac{z^T P z}{u^T P u} \right) \\
&\quad = (u^T P u) u^T u \left\{ \eta \frac{z^T z}{u^T u} + \left(2(\eta + 1) - 2(\eta + 1) \chi(P) \sqrt{\frac{z^T P z}{u^T P u}} + \eta \frac{z^T P z}{u^T P u} \right) \right\} \\
&\quad \geq 0.
\end{aligned}$$

The last inequality follows from the fact that when $\chi(P) \le \sqrt{\frac{2\eta}{\eta+1}}$, the quadratic function $2(\eta+1) - 2(\eta+1)\chi(P)t + \eta t^2 \ge 0$ for any $t \in R$. □

It should be mentioned that Lemma 2.2 is also true for $\eta = 0$, in which case A and B are collinear. We now prove the main result of this section.

Theorem 2.3 *Let $A_i \succ 0$, $i = 1, \ldots, m$, be $n \times n$ matrices. If*

$$\kappa(A_j^{-1/2} A_i A_j^{-1/2}) \le \frac{(2m+1) + 2\sqrt{4m-2}}{2m-3} \quad \text{for all } i, j = 1, \ldots, m, \ i \ne j \tag{7}$$

(which is equivalent to $\chi(A_j^{-1/2} A_i A_j^{-1/2}) \le \sqrt{\frac{2}{2m-1}}$ for all $i, j = 1, \ldots, m, i \ne j$), then the product of m quadratic forms $f = \prod_{i=1}^m q_{A_i}$ is convex.

Proof Denote by

$$\Omega(x, y) := \sum_{i=1}^m \left(\prod_{k=1, k\ne i}^m y^T A_k y \right) (x^T A_i x).$$

Note that for any vectors x, y we have

$$\begin{aligned} &\sum_{i=1}^m \sum_{j=1, j\ne i}^m \left(\prod_{k=1, k\ne i, j}^m y^T A_k y \right) x^T A_i x y^T A_j y \\ &\quad = \sum_{i=1}^m \sum_{j=1, j\ne i}^m \left(\prod_{k=1, k\ne i}^m y^T A_k y \right) x^T A_i x \\ &\quad = (m-1) \sum_{i=1}^m \left(\prod_{k=1, k\ne i}^m y^T A_k y \right) x^T A_i x \\ &\quad = (m-1)\Omega(x, y). \end{aligned} \tag{8}$$

On the other hand, we have

$$\begin{aligned} &\sum_{i=1}^m \sum_{j=1, j\ne i}^m \left(\prod_{k=1, k\ne i, j}^m y^T A_k y \right) (y^T A_i y)(x^T A_j x) \\ &= \sum_{i=1}^m \sum_{j=1, j\ne i}^m x^T A_j x \left(\prod_{k=1, k\ne j}^m y^T A_k y \right) \\ &= \sum_{i=1}^m \left\{ \sum_{j=1}^m x^T A_j x \left(\prod_{k=1, k\ne j}^m y^T A_k y \right) - x^T A_i x \left(\prod_{k=1, k\ne i}^m y^T A_k y \right) \right\} \end{aligned}$$

$$= \sum_{i=1}^{m} \left\{ \Omega(x,y) - x^T A_i x \left(\prod_{k=1,k\neq i}^{m} y^T A_k y \right) \right\}$$
$$= (m-1)\Omega(x,y). \tag{9}$$

Thus, (8) and (9) imply that

$$\sum_{i=1}^{m} \sum_{j=1,j\neq i}^{m} \left(\prod_{k=1,k\neq i,j}^{m} y^T A_k y \right) x^T A_i x y^T A_j y$$
$$= \sum_{i=1}^{m} \sum_{j=1,j\neq i}^{m} \left(\prod_{k=1,k\neq i,j}^{m} y^T A_k y \right) y^T A_i y x^T A_j x,$$

and hence

$$\sum_{i=1}^{m} \sum_{j=1,j\neq i}^{m} \left(\prod_{k=1,k\neq i,j}^{m} y^T A_k y \right) x^T A_i x y^T A_j y$$
$$= \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1,j\neq i}^{m} \left(\prod_{k=1,k\neq i,j}^{m} y^T A_i y \right) \left\{ (x^T A_i x) y^T A_j y + (y^T A_i y) x^T A_j x \right\}. \tag{10}$$

By (4), (8) and (10), we have

$$\Phi(x,y)$$
$$= \Omega(x,y) + 2 \sum_{i=1}^{m} \sum_{j=1,j\neq i}^{m} \left[\prod_{k=1,k\neq i,j}^{m} y^T A_k y \right] \left(x^T A_i y \right) x^T A_j y$$
$$= \frac{1}{(m-1)} \sum_{i=1}^{m} \sum_{j=1,j\neq i}^{m} \left(\prod_{k=1,k\neq i,j}^{m} y^T A_k y \right) (x^T A_i x) y^T A_j y$$
$$+ 2 \sum_{i=1}^{m} \sum_{j=1,j\neq i}^{m} \left(\prod_{k=1,k\neq i,j}^{m} y^T A_k y \right) \left(x^T A_i y \right) \left(x^T A_j y \right)$$
$$= \frac{1}{2(m-1)} \sum_{i=1}^{m} \sum_{j=1,j\neq i}^{m} \left[\prod_{k=1,k\neq i,j}^{m} y^T A_k y \right]$$
$$\times \left\{ (x^T A_i x) y^T A_j y + y^T A_i y (x^T A_j x) \right\}$$
$$+ 2 \sum_{i=1}^{m} \sum_{j=1,j\neq i}^{m} \left[\prod_{k=1,k\neq i,j}^{m} y^T A_k y \right] \left(x^T A_i y \right) \left(x^T A_j y \right)$$

$$= \sum_{i=1}^{m} \sum_{j=1, j\neq i}^{m} \prod_{k=1,k\neq i,j}^{m} y^T A_k y$$

$$\times \left\{ \frac{1}{2(m-1)} \left[x^T A_i x y^T A_j y + y^T A_i y x^T A_j x \right] + 2x^T A_i y x^T A_j y \right\}$$

$$= \sum_{i=1}^{m} \sum_{j=1, j\neq i}^{m} \left(\prod_{k=1,k\neq i,j}^{m} y^T A_k y \right) \Gamma_{(A_i, A_j, \frac{1}{2(m-1)})}(x, y), \tag{11}$$

where $\Gamma_{(A_i, A_j, \frac{1}{2(m-1)})}(x, y)$ is defined as in Lemma 2.2 by setting $A = A_i$, $B = A_j$ and $\eta = \frac{1}{2(m-1)}$. Since $\eta = \frac{1}{2(m-1)}$, we have that $\sqrt{\frac{2\eta}{\eta+1}} = \sqrt{\frac{2}{2m-1}}$. If

$$\chi(A_j^{-1/2} A_i A_j^{-1/2}) \leq \sqrt{\frac{2}{2m-1}} = \sqrt{\frac{2\eta}{\eta+1}}$$

for all $i, j = 1, \ldots, m$ and $i \neq j$, then by applying Lemma 2.2 to the matrix pair (A_i, A_j) and $\eta = \frac{1}{2(m-1)}$ we deduce that

$$\Gamma_{(A_i, A_j, \frac{1}{2(m-1)})}(x, y) \geq 0 \quad \text{for any } x, y \in R^n.$$

Thus, it follows from (11) that $\Phi(x, y) \geq 0$ for any vectors $x, y \in R^n$. Notice that

$$\chi\left(A_j^{-1/2} A_i A_j^{-1/2}\right) = \frac{\kappa(A_j^{-1/2} A_i A_j^{-1/2}) - 1}{\kappa(A_j^{-1/2} A_i A_j^{-1/2}) + 1},$$

which implies that $\chi(A_j^{-1/2} A_i A_j^{-1/2}) \leq \sqrt{\frac{2}{2m-1}}$ if and only if $\kappa(A_j^{-1/2} A_i A_j^{-1/2})$ satisfies (7). By (3), we conclude that the Hessian matrix of the product function f is positive semi-definite, and thus f is convex. □

It is worth noting that the upper bound (7) of condition numbers depends on the number of quadratic forms involved. Intuitively, the more functions are involved, the more likely the product function loses its convexity. Note that the upper bound (7) decreases as m increases. So (7) does indicate that the more quadratic forms are involved, the more restrictive conditions need to be imposed on the condition number of scaled matrices in order to retain the convexity of the product function.

When $m = 3$ (the product of three quadratic forms), we see that

$$\frac{(2m+1) + 2\sqrt{4m-2}}{2m-3} = \frac{7 + 2\sqrt{10}}{3} \approx 4.4415.$$

Corollary 2.4 *Let A, B, C be $n \times n$ matrices. If $A, B, C \succ 0$ and*

$$\kappa(B^{-1/2} A B^{-1/2}),\ \kappa(C^{-1/2} B C^{-1/2}),\ \kappa(A^{-1/2} C A^{-1/2}) \leq \frac{7 + 2\sqrt{10}}{3},$$

then the product $f = q_A q_B q_C$ *is convex.*

When $m = 2$, we have

$$\frac{(2m+1)+2\sqrt{4m-2}}{2m-3} = 5 + 2\sqrt{6} \approx 9.899.$$

In this case, Theorem 2.3 is reduced to the next result, which was first proved in [36].

Corollary 2.5 (Zhao [36]) *For any* $n \times n$ *matrices* A *and* B, *if* $A, B \succ 0$ *and* $\kappa(B^{-1/2}AB^{-1/2}) \le 5 + 2\sqrt{6}$, *then the product* $f = q_A q_B$ *is convex.*

Theorem 2.3 provides a sufficient convexity condition for the product of finitely many quadratic forms ($2 \le m < \infty$). This is the first general sufficient convexity for the product function. At present, we do not know whether the condition (7) can be further improved in general cases. Even for the case $m = 2$, the question about whether or not the threshold $5 + 2\sqrt{6}$ in Corollary 2.5 can be improved is not clear. However, if the matrix with a special structure is considered, the threshold can be improved, as indicated by the following result.

Proposition 2.6 *Let* $A, B \succ 0$ *be* 2×2 *matrices. Suppose that* A, B *can be simultaneously diagonalizable, i.e., there exists an orthogonal matrix* U *such that*

$$A = U^T \begin{bmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{bmatrix} U, \qquad B = U^T \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix} U,$$

and the diagonal entries satisfy $\beta_1\gamma_1 = \beta_2\gamma_2$. *Then* $f = q_A q_B$ *is convex if and only if* $\kappa(B^{-1/2}AB^{-1/2}) \le 17 + 12\sqrt{2}$.

Proof Without loss of generality, we assume that $\beta_1 \ge \beta_2$ which together with $\beta_1\gamma_1 = \beta_2\gamma_2$ implies that $\frac{\beta_1}{\gamma_1} \ge \frac{\beta_2}{\gamma_2}$. Notice that

$$B^{-1/2}AB^{-1/2} = U^T \begin{bmatrix} \frac{\beta_1}{\gamma_1} & 0 \\ 0 & \frac{\beta_2}{\gamma_2} \end{bmatrix} U.$$

Thus, $\kappa(B^{-1/2}AB^{-1/2}) = (\frac{\beta_1}{\gamma_1})/(\frac{\beta_2}{\gamma_2}) = \frac{\beta_1\gamma_2}{\beta_2\gamma_1}$. By setting $z = Uy$, f can be written as

$$f(y_1, y_2) = \frac{1}{2}(\beta_1 z_1^2 + \beta_2 z_2^2) \cdot \frac{1}{2}\left(\gamma_1 z_1^2 + \gamma_2 z_2^2\right) =: g(z_1, z_2).$$

Clearly, f is convex if and only if g is convex. Consider the Hessian matrix of g, which is given by

$$\nabla^2 g(z_1, z_2) = \frac{1}{2} \begin{bmatrix} 6\beta_1\gamma_1 z_1^2 + (\beta_1\gamma_2 + \beta_2\gamma_1)z_2^2 & 2(\beta_1\gamma_2 + \beta_2\gamma_1)z_1 z_2 \\ 2(\beta_1\gamma_2 + \beta_2\gamma_1)z_1 z_2 & 6\beta_2\gamma_2 z_2^2 + (\beta_1\gamma_2 + \beta_2\gamma_1)z_1^2 \end{bmatrix}. \tag{12}$$

If g is convex in R^n, then (12) must be positive semi-definite at any point in R^n. In particular, it must be positive semi-definite at $(z_1, z_2) = (1, 1)$, thus

$$\nabla^2 g(1,1) = \frac{1}{2}\begin{bmatrix} 6\beta_1\gamma_1 + (\beta_1\gamma_2 + \beta_2\gamma_1) & 2(\beta_1\gamma_2 + \beta_2\gamma_1) \\ 2(\beta_1\gamma_2 + \beta_2\gamma_1) & 6\beta_2\gamma_2 + (\beta_1\gamma_2 + \beta_2\gamma_1) \end{bmatrix} \succeq 0$$

which implies that

$$\begin{aligned}
0 &\le \det\begin{bmatrix} 6\beta_1\gamma_1 + (\beta_1\gamma_2 + \beta_2\gamma_1) & 2(\beta_1\gamma_2 + \beta_2\gamma_1) \\ 2(\beta_1\gamma_2 + \beta_2\gamma_1) & 6\beta_2\gamma_2 + (\beta_1\gamma_2 + \beta_2\gamma_1) \end{bmatrix} \\
&= (\beta_1\gamma_2 + \beta_2\gamma_1)^2 \det\begin{bmatrix} \frac{6\beta_1\gamma_1}{\beta_1\gamma_2+\beta_2\gamma_1} + 1 & 2 \\ 2 & \frac{6\beta_2\gamma_2}{\beta_1\gamma_2+\beta_2\gamma_1} + 1 \end{bmatrix} \\
&= 3(\beta_1\gamma_2 + \beta_2\gamma_1)^2 \left(12\frac{\beta_1\gamma_1\beta_2\gamma_2}{(\beta_1\gamma_2 + \beta_2\gamma_1)^2} + 2\frac{\beta_1\gamma_1 + \beta_2\gamma_2}{\beta_1\gamma_2 + \beta_2\gamma_1} - 1\right).
\end{aligned}$$

Since $\beta_1\gamma_1 = \beta_2\gamma_2$, we have that $\beta_1\gamma_1 + \beta_2\gamma_2 = 2\sqrt{\beta_1\gamma_1\beta_2\gamma_2}$. Substituting this into the inequality above, we have

$$12\left(\frac{\sqrt{\beta_1\gamma_1\beta_2\gamma_2}}{\beta_1\gamma_2 + \beta_2\gamma_1}\right)^2 + 4\left(\frac{\sqrt{\beta_1\gamma_1\beta_2\gamma_2}}{\beta_1\gamma_2 + \beta_2\gamma_1}\right) - 1 \ge 0. \tag{13}$$

Conversely, if (13) holds, we can prove that g is convex. Indeed, since the diagonal entries of (12) is nonnegative, it is sufficient to prove that $\det(\nabla^2 g) \ge 0$. In fact, noting that $\beta_1\gamma_1 = \beta_2\gamma_2$, we have

$$\begin{aligned}
&\det(\nabla^2 g(z_1, z_2)) \\
&\quad = \frac{1}{4}\left\{\left[6\beta_1\gamma_1 z_1^2 + (\beta_1\gamma_2 + \beta_2\gamma_1)\, z_2^2\right]\left[6\beta_2\gamma_2 z_2^2 + (\beta_1\gamma_2 + \beta_2\gamma_1)\, z_1^2\right]\right. \\
&\qquad \left. - 4\,(\beta_1\gamma_2 + \beta_2\gamma_1)^2\, z_1^2 z_2^2\right\} \\
&\quad = \frac{1}{4}\left[6\beta_1\gamma_1\,(\beta_1\gamma_2 + \beta_2\gamma_1)\,(z_1^4 + z_2^4) + 36\beta_1\gamma_1\beta_2\gamma_2 z_1^2 z_2^2\right. \\
&\qquad \left. - 3\,(\beta_1\gamma_2 + \beta_2\gamma_1)^2\, z_1^2 z_2^2\right] \\
&\quad \ge \frac{1}{4}\left[12\beta_1\gamma_1\,(\beta_1\gamma_2 + \beta_2\gamma_1)\, z_1^2 z_2^2 + 36\beta_1\gamma_1\beta_2\gamma_2 z_1^2 z_2^2 - 3\,(\beta_1\gamma_2 + \beta_2\gamma_1)^2\, z_1^2 z_2^2\right] \\
&\quad = \frac{1}{4}\left[12\sqrt{\beta_1\gamma_1\beta_2\gamma_2}\,(\beta_1\gamma_2 + \beta_2\gamma_1) + 36\beta_1\gamma_1\beta_2\gamma_2 - 3\,(\beta_1\gamma_2 + \beta_2\gamma_1)^2\right] z_1^2 z_2^2 \\
&\quad = \frac{3}{4}\,(\beta_1\gamma_2 + \beta_2\gamma_1)^2\left[4\frac{\sqrt{\beta_1\gamma_1\beta_2\gamma_2}}{\beta_1\gamma_2 + \beta_2\gamma_1} + 12\frac{\beta_1\gamma_1\beta_2\gamma_2}{(\beta_1\gamma_2 + \beta_2\gamma_1)^2} - 1\right] z_1^2 z_2^2 \\
&\quad \ge 0.
\end{aligned}$$

The first inequality above follows from the fact $z_1^4 + z_2^4 \geq 2z_1^2 z_2^2$. Therefore, g is convex if and only if the positive numbers $\beta_1, \beta_2, \gamma_1, \gamma_2$ satisfy the inequality (13). Notice that the quadratic function $12t^2 + 4t - 1 \geq 0$ if and only if either $t \leq -\frac{1}{2}$ or $t \geq \frac{1}{6}$. Thus, (13) holds if and only if

$$\frac{1}{6} \leq \frac{\sqrt{\beta_1\gamma_1\beta_2\gamma_2}}{\beta_1\gamma_2 + \beta_2\gamma_1} = \frac{\sqrt{(\beta_1\gamma_2)/(\beta_2\gamma_1)}}{(\beta_1\gamma_2)/(\beta_2\gamma_1) + 1} = \frac{\sqrt{\kappa(B^{-1/2}AB^{-1/2})}}{\kappa(B^{-1/2}AB^{-1/2}) + 1} \tag{14}$$

which is equivalent to $\kappa(B^{-1/2}AB^{-1/2}) \leq 17 + 12\sqrt{2}$. □

Remark 2.7 The proposition above shows that if $\kappa(B^{-1/2}AB^{-1/2}) > 17 + 12\sqrt{2}$, the product of two quadratic forms considered in Proposition 2.6 is not convex. As we mentioned earlier, we do not know at present whether the bound '$5 + 2\sqrt{6}$' in Corollary 2.5 can be improved without affecting the result of the corollary. If it can be improved to a certain level $\gamma^* > 5 + 2\sqrt{6}$ without damaging the result of Corollary 2.5, then Proposition 2.6 indicates that γ^* must not exceed $17 + 12\sqrt{2}$.

Remark 2.8 By setting $y = B^{1/2}x$, the product function can be written as

$$(x^T Ax)(x^T Bx) = y^T(B^{-1/2}AB^{-1/2})y(y^T y),$$

which implies that the convexity of the product function is completely determined by such a scaled matrix as $B^{-1/2}AB^{-1/2}$. Thus, from an algebraic point of view, it is natural to impose a condition on the scaled matrix in order to obtain the convexity of the product function, as shown by Theorems 2.3 and its corollaries. The condition (7) that is equivalent to $\chi(A_j^{-1/2}A_iA_j^{-1/2}) \leq \sqrt{\frac{2}{2m-1}}$, $i, j = 1, \ldots, m$, $i \neq j$ can be understood from a geometric point of view. In fact, denote the angle between A and B as $\theta(A, B) = \arccos\langle A, B\rangle/(\|A\|_F\|B\|_F)$ where $\langle A, B\rangle = \mathrm{tr}(AB)$ and $\|\cdot\|_F$ is Frobenius norm. Then it is easy to see that $\chi(B^{-1/2}AB^{-1/2}) = 0$ if and only if $\theta(A, B) = 0$, in which case A and B are collinear. Thus, the condition (7) basically means the angle between each pair of matrices does not exceed a certain threshold. For the case $m = 2$, Proposition 2.6 indicates that the result of Theorem 2.3 does not hold if the threshold is higher than $17 + 12\sqrt{2}$. In other words, when the angle between the matrices exceeds a certain threshold (the worst scenario occurs when $\theta(A, B)$ is close to $\pi/2$ in which case $\xi(B^{-1/2}AB^{-1/2}) \approx \infty$), then the product function will lose its convexity.

3 Expression of Legendre-Fenchel Transform

In this section, we address a more challenging question than the one (Question 11) in [13]: *What is the LF-transform for the product of finitely many positive-definite quadratic forms*? To this end, let us first prove a series of useful technical results concerning the existence and/or uniqueness of the solution to certain nonlinear equations.

Lemma 3.1 *Let $A_i \succ 0$, $i = 1, \dots, m$, be $n \times n$ matrices, and let $0 \neq x \in R^n$ be an arbitrarily given vector. Then for each i $(i = 1, \dots, m)$ the nonlinear equation*

$$\left(\prod_{j=1, j\neq i}^{m} q_{A_j}(y)\right) A_i y = x \tag{15}$$

has a unique solution which is given by

$$y^{(i)} = \left(\frac{2^{m-1}}{\prod_{j=1, j\neq i}^{m} x^T A_i^{-1} A_j A_i^{-1} x}\right)^{\frac{1}{2m-1}} A_i^{-1} x, \tag{16}$$

where $i = 1, \dots, m$.

Proof It is easy to verify that (16) is a solution to (15). Thus, it suffices to prove that (16) is the only solution to (15). Indeed, let y be an arbitrary solution to (15). Then, we have $A_i[(\prod_{j=1, j\neq i}^{m} q_{A_j}(y))y] = x$. Let u be the unique solution to $A_i u = x$, i.e., $u = A_i^{-1} x$. Thus,

$$\left(\prod_{j=1, j\neq i}^{m} q_{A_j}(y)\right) y = A_i^{-1} x = u,$$

from which we see that $y \neq 0$ since $x \neq 0$. Notice that $\prod_{j=1, j\neq i}^{m} q_{A_j}(y) > 0$ for all $i = 1, \dots, m$. Denote by $\beta = 1/(\prod_{j=1, j\neq i}^{m} q_{A_j}(y))$. Then the equality above can be written as $y = \beta u$. Substituting it back into (15), we have

$$\left(\prod_{j=1, j\neq i}^{m} q_{A_j}(\beta u)\right) A_i(\beta u) = x,$$

i.e.,

$$\beta^{2(m-1)+1} \left(\prod_{j=1, j\neq i}^{m} q_{A_j}(u)\right) A_i u = x.$$

Since $A_i u = x \neq 0$, the inequality above implies that $\beta^{2(m-1)+1}(\prod_{j=1, j\neq i}^{m} q_{A_j}(u)) = 1$. Hence

$$\beta = \left(\frac{1}{\prod_{j=1, j\neq i}^{m} q_{A_j}(u)}\right)^{\frac{1}{2(m-1)+1}} = \left(\frac{2^{m-1}}{\prod_{j=1, j\neq i}^{m} x^T A_i^{-1} A_j A_i^{-1} x}\right)^{\frac{1}{2m-1}},$$

which implies that

$$y = \beta u = \left(\frac{2^{m-1}}{\prod_{j=1, j\neq i}^{m} x^T A_i^{-1} A_j A_i^{-1} x}\right)^{\frac{1}{2m-1}} A_i^{-1} x.$$

Thus the solution to (15) is unique and is given by (16). □

An immediate result from Lemma 3.1 is the following lemma.

Lemma 3.2 *Let $A_i \succ 0, i = 1, \ldots, m$, be $n \times n$ matrices, and let $x^{(i)} \neq 0$, $i = 1, \ldots, m$, be given vectors. Then the following system with respect to y*

$$\begin{cases} (\prod_{j=1, j\neq 1}^{m} q_{A_j}(y))A_1 y = x^{(1)}, \\ (\prod_{j=1, j\neq 2}^{m} q_{A_j}(y))A_2 y = x^{(2)}, \\ \vdots \\ (\prod_{j=1, j\neq m}^{m} q_{A_j}(y))A_m y = x^{(m)} \end{cases} \tag{17}$$

has a solution if and only if $y^{(1)} = y^{(2)} = \cdots = y^{(m)}$, where

$$y^{(i)} = \left(\frac{2^{m-1}}{\prod_{j=1, j\neq i}^{m} (x^{(i)})^T A_i^{-1} A_j A_i^{-1} x^{(i)}}\right)^{\frac{1}{2m-1}} A_i^{-1} x^{(i)}, \quad i = 1, \ldots, m.$$

Moreover, if the system (17) *has a solution, its solution must be unique.*

Proof Given a set of vectors $x^{(i)} \neq 0$, $i = 1, \ldots, m$, by Lemma 3.1 each individual equation of (17) always has a unique solution. Thus, if the system (17) has a solution, such a solution must be unique. However, the whole system of equations may not have a common solution unless $x^{(i)} \neq 0$, $i = 1, \ldots, m$, are chosen such that all the vectors $y^{(i)}$, $i = 1, \ldots, m$, are equal. That is, $x^{(i)}$ $(i = 1, \ldots, m)$ must satisfy the following condition:

$$\begin{cases} (\frac{2^{m-1}}{\prod_{j=1, j\neq 2}^{m} (x^{(2)})^T A_2^{-1} A_j A_2^{-1} x^{(2)}})^{\frac{1}{2m-1}} A_2^{-1} x^{(2)} = y^{(1)}, \\ (\frac{2^{m-1}}{\prod_{j=1, j\neq 3}^{m} (x^{(3)})^T A_3^{-1} A_j A_3^{-1} x^{(3)}})^{\frac{1}{2m-1}} A_3^{-1} x^{(3)} = y^{(1)}, \\ \vdots \\ (\frac{2^{m-1}}{\prod_{j=1, j\neq m}^{m} (x^{(m)})^T A_m^{-1} A_j A_m^{-1} x^{(m)}})^{\frac{1}{2m-1}} A_m^{-1} x^{(m)} = y^{(1)}, \end{cases} \tag{18}$$

where $y^{(1)} = (\frac{2^{m-1}}{\prod_{j=1, j\neq 1}^{m} (x^{(1)})^T A_1^{-1} A_j A_1^{-1} x^{(1)}})^{\frac{1}{2m-1}} A_1^{-1} x^{(1)}$. □

Before we prove the next result, let us first define a useful mapping. Given a vector $0 \neq x \in R^n$, let $\mathcal{F}^{(x)} = (\mathcal{F}_2^{(x)}, \mathcal{F}_3^{(x)}, \ldots, \mathcal{F}_m^{(x)})^T$ be a mapping from R_{++}^{m-1} to R_{++}^{m-1}. Its components are defined as

$$F_j^{(x)}(\alpha_2, \ldots, \alpha_m) = \frac{x^T D^{-T} A_1^{-1} A_j A_1^{-1} D^{-1} x}{x^T D^{-T} A_1^{-1} D^{-1} x}, \quad j = 2, \ldots, m \tag{19}$$

where $A_i \succ 0$ for $i = 1, \dots, m$ and

$$D = I + \frac{1}{\alpha_2} A_2 A_1^{-1} + \cdots + \frac{1}{\alpha_m} A_m A_1^{-1} = \left(A_1 + \frac{1}{\alpha_2} A_2 + \cdots + \frac{1}{\alpha_m} A_m \right) A_1^{-1}. \quad (20)$$

The mapping $\mathcal{F}^{(x)}$ plays a key role in the proof of the next result.

Lemma 3.3 *Let $A_i \succ 0, i = 1, \dots, m$ and let $0 \neq x \in R^n$ be an arbitrarily given vector in R^n. Then the following system of equations in variables $\alpha_2, \dots, \alpha_m$ has a solution in R^{m-1}_{++}:*

$$\begin{cases} \alpha_2 = \frac{x^T (A_1 + \sum_{k=2}^m \frac{1}{\alpha_k} A_k)^{-1} A_2 (A_1 + \sum_{k=2}^m \frac{1}{\alpha_k} A_k)^{-1} x}{x^T (A_1 + \sum_{k=2}^m \frac{1}{\alpha_k} A_k)^{-1} A_1 (A_1 + \sum_{k=2}^m \frac{1}{\alpha_k} A_k)^{-1} x}, \\ \alpha_3 = \frac{x^T (A_1 + \sum_{k=2}^m \frac{1}{\alpha_k} A_k)^{-1} A_3 (A_1 + \sum_{k=2}^m \frac{1}{\alpha_k} A_k)^{-1} x}{x^T (A_1 + \sum_{k=2}^m \frac{1}{\alpha_k} A_k)^{-1} A_1 (A_1 + \sum_{k=2}^m \frac{1}{\alpha_k} A_k)^{-1} x}, \\ \vdots \\ \alpha_m = \frac{x^T (A_1 + \sum_{k=2}^m \frac{1}{\alpha_k} A_k)^{-1} A_m (A_1 + \sum_{k=2}^m \frac{1}{\alpha_k} A_k)^{-1} x}{x^T (A_1 + \sum_{k=2}^m \frac{1}{\alpha_k} A_k)^{-1} A_1 (A_1 + \sum_{k=2}^m \frac{1}{\alpha_k} A_k)^{-1} x} \end{cases} \quad (21)$$

and any solution $(\alpha_2, \dots, \alpha_m) \in R^{m-1}_{++}$ of the system above satisfies that

$$\alpha_i \in [\lambda_{\min}(P_i), \lambda_{\max}(P_i)], \quad i = 2, \dots, m \quad (22)$$

where $P_i = A_1^{-1/2} A_i A_1^{-1/2}$, $i = 2, \dots, m$.

Proof Given $x \neq 0$, let the mapping $\mathcal{F}^{(x)} : R^{m-1}_{++} \to R^{m-1}_{++}$ be defined by (19) where D is given by (20). Consider the following compact and convex set

$$S = [\lambda_{\min}(P_2), \lambda_{\max}(P_2)] \times \cdots \times [\lambda_{\min}(P_m), \lambda_{\max}(P_m)] \quad (23)$$

which is the Cartesian product of $m-1$ intervals. Notice that $\mathcal{F}^{(x)}(\alpha_2, \dots, \alpha_m)$ is continuous on S, and that for any $(\alpha_2, \dots, \alpha_m) \in S$, it follows from (19) that

$$\mathcal{F}_j^{(x)}(\alpha_2, \dots, \alpha_m) = \frac{x^T D^{-T} A_1^{-1} A_j A_1^{-1} D^{-1} x}{x^T D^{-T} A_1^{-1} D^{-1} x} = \frac{z^T (A_1^{-1/2} A_j A_1^{-1/2}) z}{z^T z} = \frac{z^T P_j z}{z^T z},$$

where $z = A_1^{-1/2} D^{-1} x$, and $P_j = A_1^{-1/2} A_j A_1^{-1/2}$. By Rayleigh-Ritz Theorem,

$$\lambda_{\min}(P_j) \le \mathcal{F}_j^{(x)}(\alpha_2, \dots, \alpha_m) \le \lambda_{\max}(P_j), \quad j = 2, \dots, m. \quad (24)$$

Therefore, we conclude that $\mathcal{F}^{(x)}(S) \subseteq S$. By Brouwer's fixed-point theorem, the mapping $\mathcal{F}^{(x)}$ has a fixed point in S, i.e., there is a vector $(\alpha_2, \dots, \alpha_m)$ in S such that

$$(\alpha_2, \dots, \alpha_m)^T = \mathcal{F}^{(x)}(\alpha_2, \dots, \alpha_m),$$

namely

$$\begin{cases} \alpha_2 = \dfrac{x^T D^{-T} A_1^{-1} A_2 A_1^{-1} D^{-1} x}{x^T D^{-T} A_1^{-1} D^{-1} x}, \\ \alpha_3 = \dfrac{x^T D^{-T} A_1^{-1} A_3 A_1^{-1} D^{-1} x}{x^T D^{-T} A_1^{-1} D^{-1} x}, \\ \vdots \\ \alpha_m = \dfrac{x^T D^{-T} A_1^{-1} A_m A_1^{-1} D^{-1} x}{x^T D^{-T} A_1^{-1} D^{-1} x}, \end{cases}$$

which, by (20), is nothing but (21). Thus, the solution of (21) coincides with the fixed point of the mapping $\mathcal{F}^{(x)}$. Notice that (22) follows directly from the fact $\alpha_j = F_j^{(x)}(\alpha_2, \ldots, \alpha_m)$ and (24). □

Lemma 3.4 *Let $A_i \succ 0$, $i = 1, \ldots, m$ and let $0 \neq x \in R^n$ be an arbitrarily given vector in R^n. For any given positive vector $(\alpha_2, \ldots, \alpha_m)^T \in R^{m-1}_{++}$, the following system of equations (in variables $x^{(1)}, \ldots, x^{(m)} \in R^n$)*

$$\begin{cases} x^{(1)} + x^{(2)} + \cdots + x^{(m)} = x, \\ A_2 A_1^{-1} x^{(1)} - \alpha_2 x^{(2)} = 0, \\ A_3 A_1^{-1} x^{(1)} - \alpha_3 x^{(3)} = 0, \\ \vdots \\ A_m A_1^{-1} x^{(1)} - \alpha_m x^{(m)} = 0 \end{cases} \tag{25}$$

has a unique solution which is given by

$$\begin{bmatrix} x^{(1)} \\ x^{(2)} \\ \vdots \\ x^{(m)} \end{bmatrix} = \begin{bmatrix} A_1 \big(A_1 + \sum_{k=2}^m \frac{1}{\alpha_k} A_k\big)^{-1} x \\ \frac{1}{\alpha_2} A_2 \big(A_1 + \sum_{k=2}^m \frac{1}{\alpha_k} A_k\big)^{-1} x \\ \vdots \\ \frac{1}{\alpha_m} A_m \big(A_1 + \sum_{k=2}^m \frac{1}{\alpha_k} A_k\big)^{-1} x \end{bmatrix}. \tag{26}$$

Proof The system (25) can be written as

$$\begin{bmatrix} I & I & I & \cdots & I \\ A_2 A_1^{-1} & -\alpha_2 I & 0 & \cdots & 0 \\ A_3 A_1^{-1} & 0 & -\alpha_3 I & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_m A_1^{-1} & 0 & 0 & \cdots & -\alpha_m I \end{bmatrix} \begin{bmatrix} x^{(1)} \\ x^{(2)} \\ x^{(3)} \\ \vdots \\ x^{(m)} \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{27}$$

For any given $(\alpha_2, \ldots, \alpha_m) > 0$, it is easy to check that the coefficient matrix above is nonsingular, and its inverse is given by

$$\begin{bmatrix} D^{-1} & \frac{1}{\alpha_2} D^{-1} & \cdots & \frac{1}{\alpha_m} D^{-1} \\ \frac{1}{\alpha_2} A_2 A_1^{-1} D^{-1} & \frac{1}{\alpha_2}\left(\frac{1}{\alpha_2} A_2 A_1^{-1} D^{-1} - I\right) & \cdots & \frac{1}{\alpha_2}\left(\frac{1}{\alpha_m} A_2 A_1^{-1} D^{-1}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\alpha_m} A_m A_1^{-1} D^{-1} & \frac{1}{\alpha_m}\left(\frac{1}{\alpha_2} A_m A_1^{-1} D^{-1}\right) & \cdots & \frac{1}{\alpha_m}\left(\frac{1}{\alpha_m} A_m A_1^{-1} D^{-1} - I\right) \end{bmatrix}$$

where D is given by (20). Thus, the solution to the system (27) is unique and given by

$$\begin{bmatrix} x^{(1)} \\ x^{(2)} \\ \vdots \\ x^{(m)} \end{bmatrix} = \begin{bmatrix} D^{-1} x \\ \frac{1}{\alpha_2} A_2 A_1^{-1} D^{-1} x \\ \vdots \\ \frac{1}{\alpha_m} A_m A_1^{-1} D^{-1} x \end{bmatrix}.$$

Substituting (20) into the above leads to (26). □

As we have mentioned earlier, to ensure that the system (17) has a solution the vectors $x^{(i)}$ $(i = 1, \ldots, m)$ should satisfy certain conditions. The next result shows how to construct such vectors.

Lemma 3.5 *Let $A_i \succ 0, i = 1, \ldots, m$ and let $0 \neq x \in R^n$ be an arbitrarily given vector in R^n. If the vectors $x^{(1)}, x^{(2)}, \ldots, x^{(m)}$ are given by (26) where $(\alpha_2, \ldots, \alpha_m) \in R^{m-1}_{++}$ is a solution to the system (21), then the system (17) has a unique solution which can be represented as*

$$y^* = \left(\frac{2^{m-1}}{\prod_{j=2}^m x^T C^{-1} A_j C^{-1} x} \right)^{\frac{1}{2m-1}} C^{-1} x, \quad C = A_1 + \sum_{k=2}^m \frac{1}{\alpha_k} A_k. \tag{28}$$

Proof Since $(x^{(1)}, \ldots, x^{(m)})$ is determined by (26), we have

$$x^{(1)} = A_1 \left(A_1 + \sum_{k=2}^m \frac{1}{\alpha_k} A_k \right)^{-1} x = A_1 C^{-1} x, \tag{29}$$

$$x^{(i)} = \frac{1}{\alpha_i} A_i \left(A_1 + \sum_{k=2}^m \frac{1}{\alpha_k} A_k \right)^{-1} x = \frac{1}{\alpha_i} A_i C^{-1} x, \quad i = 2, \ldots, m, \tag{30}$$

where $(\alpha_2, \ldots, \alpha_m)$ is a solution to (21), which always exists by Lemma 3.3. Thus, for each $i = 2, \ldots, m$, by (29) and (30) and we have

$$\left(\frac{\prod_{j=2}^m (x^{(1)})^T A_1^{-1} A_j A_1^{-1} x^{(1)}}{\prod_{j=1, j\neq i}^m (x^{(i)})^T A_i^{-1} A_j A_i^{-1} x^{(i)}} \right)^{\frac{1}{2m-1}}$$

$$= \left(\frac{\prod_{j=2}^m x^T C^{-1} A_j C^{-1} x}{\left(\frac{1}{\alpha_i}\right)^{2(m-1)} \prod_{j=1, j\neq i}^m x^T C^{-1} A_j C^{-1} x} \right)^{\frac{1}{2m-1}}$$

$$= \left(\frac{x^T C^{-1} A_i C^{-1} x}{\left(\frac{1}{\alpha_i}\right)^{2(m-1)} x^T C^{-1} A_1 C^{-1} x} \right)^{\frac{1}{2m-1}} = \left(\frac{\alpha_i}{\left(\frac{1}{\alpha_i}\right)^{2(m-1)}} \right)^{\frac{1}{2m-1}} = \alpha_i. \quad (31)$$

The last second equality follows from the fact that $(\alpha_2, \dots, \alpha_m)$ is a solution to (21). Since $(x^{(1)}, \dots, x^{(m)})$ given by (26) is the solution to (25), it satisfies that

$$\begin{cases} A_2 A_1^{-1} x^{(1)} - \alpha_2 x^{(2)} = 0, \\ A_3 A_1^{-1} x^{(1)} - \alpha_3 x^{(3)} = 0, \\ \vdots \\ A_m A_1^{-1} x^{(1)} - \alpha_m x^{(m)} = 0, \end{cases}$$

which can be written as

$$\begin{cases} A_1^{-1} x^{(1)} = \alpha_2 A_2^{-1} x^{(2)}, \\ A_1^{-1} x^{(1)} = \alpha_3 A_3^{-1} x^{(3)}, \\ \vdots \\ A_1^{-1} x^{(1)} = \alpha_m A_m^{-1} x^{(m)}. \end{cases}$$

This together with (31) implies that $(x^{(1)}, \dots, x^{(m)})$ satisfies (18). Thus, we have $y^{(1)} = y^{(2)} = \cdots = y^{(m)}$ where $y^{(i)}, i = 1, \dots, m$, are given as in Lemma 3.2. By Lemma 3.2, the nonlinear system (17) has a unique solution which can be represented as

$$y^* = \left(\frac{2^{m-1}}{\prod_{j=2}^m (x^{(1)})^T A_1^{-1} A_j A_1^{-1} x^{(1)}} \right)^{1/(2m-1)} A_1^{-1} x^{(1)} = y^{(1)}.$$

Substituting (29) into the above yields (28). □

We have all ingredients to prove the main result of this section.

Theorem 3.6 *Let $A_i \succ 0$, $i = 1, \dots, m$, be $n \times n$ matrices, and assume that the product function $f = \prod_{i=1}^m q_{A_i}$ is convex. Then $f^*(0) = 0$ and for $x \neq 0$,*

$$f^*(x) = (2m-1) \left(\frac{1}{\prod_{k=2}^m \alpha_k} \right)^{\frac{1}{2m-1}} \left(\frac{x^T \left(A_1 + \sum_{k=2}^m \frac{1}{\alpha_k} A_k \right)^{-1} x}{2m} \right)^{\frac{m}{2m-1}} \quad (32)$$

where $(\alpha_2, \dots, \alpha_m) \in R_{++}^{m-1}$ is an arbitrary solution to the system (21) at x.

Proof When $x = 0$, it is evident that $f^*(0) = 0$. Thus, in the remainder of this proof, we assume that $x \neq 0$. First, let $(\alpha_2^*, \ldots, \alpha_m^*) \in R_{++}^{m-1}$ be a solution to the system (21). By Lemma 3.3, such a solution always exists. Second, let us consider the following system in variables $x^{(1)}, x^{(2)}, \ldots, x^{(m)} \in R^n$:

$$\begin{cases} x^{(1)} + x^{(2)} + \cdots + x^{(m)} = x, \\ A_2 A_1^{-1} x^{(1)} - \alpha_2^* x^{(2)} = 0, \\ A_3 A_1^{-1} x^{(1)} - \alpha_3^* x^{(3)} = 0, \\ \vdots \\ A_m A_1^{-1} x^{(1)} - \alpha_m^* x^{(m)} = 0. \end{cases} \tag{33}$$

By Lemma 3.4, the system (33) has a unique solution, denoted by $(x_*^{(1)}, x_*^{(2)}, \ldots, x_*^{(m)})$, which can be represented as (26) with $(\alpha_2, \ldots, \alpha_m) = (\alpha_2^*, \ldots, \alpha_m^*)$. Based on this fact, by Lemma 3.5, the following system

$$\left(\prod_{j=1, j\neq i}^{m} q_{A_j}(y) \right) A_i y = x_*^{(i)}, \quad i = 1, \ldots, m \tag{34}$$

has a unique solution which can be represented as

$$y^* = \left(\frac{2^{m-1}}{\prod_{j=2}^{m} x^T C^{-1} A_j C^{-1} x} \right)^{\frac{1}{2m-1}} C^{-1} x, \tag{35}$$

where

$$C = A_1 + \sum_{k=2}^{m} \frac{1}{\alpha_k^*} A_k.$$

Since $(x_*^{(1)}, x_*^{(2)}, \ldots, x_*^{(m)})$ is the solution to (33), from the first equation of (33), we have $x_*^{(1)} + x_*^{(2)} + \cdots + x_*^{(m)} = x$. Thus, substituting y^* into (34) and adding them up, we have

$$\sum_{i=1}^{m} \left(\prod_{j=1, j\neq i}^{m} q_{A_j}(y^*) \right) A_i y^* = \sum_{i=1}^{m} x_*^{(i)} = x,$$

which by (1) indicates that

$$x = \nabla f(y^*). \tag{36}$$

Since f is convex, (36) implies that for the given x the function $x^T y - f(y)$ attains its maximum value at y^*. Thus,

$$f^*(x) = \sup_{y \in R^n} x^T y - f(y) = x^T y^* - f(y^*). \tag{37}$$

Note that f is homogeneous of degree $2m$, by (36) again it is easy to verify that have

$$x^T y^* = \nabla f(y^*)^T y^* = 2m f(y^*). \tag{38}$$

Therefore, by (37), (38) and (35), we have

$$f^*(x) = x^T y^* - f(y^*) = \left(1 - \frac{1}{2m}\right) x^T y^*$$
$$= \left(\frac{2m-1}{2m}\right) \left(\frac{2^{m-1}}{\prod_{j=2}^m x^T C^{-1} A_j C^{-1} x}\right)^{\frac{1}{2m-1}} x^T C^{-1} x. \tag{39}$$

Since $(\alpha_2^*, \ldots, \alpha_m^*) \in R_{++}^{m-1}$ is a solution to (21), we have

$$x^T C^{-1} A_j C^{-1} x = \alpha_j^* x^T C^{-1} A_1 C^{-1} x, \quad j = 2, \ldots, m,$$

which implies that

$$\sum_{j=2}^m \frac{1}{\alpha_j^*} x^T C^{-1} A_j C^{-1} x = (m-1) x^T C^{-1} A_1 C^{-1} x, \tag{40}$$

and

$$\prod_{j=2}^m x^T C^{-1} A_j C^{-1} x = \left(x^T C^{-1} A_1 C^{-1} x\right)^{m-1} \prod_{j=2}^m \alpha_j^*. \tag{41}$$

By (40), we have

$$x^T C^{-1} A_1 C^{-1} x = x^T C^{-1} \left(A_1 + \sum_{k=2}^m \frac{1}{\alpha_k^*} A_k - \sum_{k=2}^m \frac{1}{\alpha_k^*} A_k\right) C^{-1} x$$
$$= x^T C^{-1} \left(C - \sum_{k=2}^m \frac{1}{\alpha_k^*} A_k\right) C^{-1} x$$
$$= x^T C^{-1} x - \sum_{k=2}^m \frac{1}{\alpha_k^*} x^T C^{-1} A_k C^{-1} x$$
$$= x^T C^{-1} x - (m-1) x^T C^{-1} A_1 C^{-1}.$$

Thus,

$$x^T C^{-1} A_1 C^{-1} x = \frac{1}{m} x^T C^{-1} x. \tag{42}$$

Combining (42) and (41) leads to

$$\prod_{j=2}^m x^T C^{-1} A_j C^{-1} x = \left(\frac{1}{m} x^T C^{-1} x\right)^{m-1} \prod_{j=2}^m \alpha_j^*.$$

Substituting this into (39), we have

$$f^*(x) = \left(\frac{2m-1}{2m}\right)\left(\frac{2^{m-1}}{\left(\frac{1}{m}x^TC^{-1}x\right)^{m-1}\prod_{j=2}^m \alpha_j^*}\right)^{\frac{1}{2m-1}} x^TC^{-1}x$$

$$= \left(\frac{2m-1}{2m}\right)\left(\frac{(2m)^{m-1}}{\prod_{j=2}^m \alpha_j^*}\right)^{\frac{1}{2m-1}} \left(x^TC^{-1}x\right)^{\frac{m}{2m-1}}$$

$$= (2m-1)\left(\frac{1}{\prod_{j=2}^m \alpha_j^*}\right)^{\frac{1}{2m-1}} \left(\frac{x^TC^{-1}x}{2m}\right)^{\frac{m}{2m-1}},$$

as desired. □

Remark 3.7 Let $\alpha(x) = (\alpha_2(x), \dots, \alpha_m(x))$ denote a solution to the system (21) at x. Then it is also a solution to the system (21) at λx for any $\lambda \in R$, i.e., $\alpha(\lambda x) = \alpha(x)$ for any $\lambda \in R$. Thus, it is easy to see that f^*, given by (32), is positively homogeneous of degree $\frac{2m}{2m-1}$. This is consistent with a general result concerning the LF-transform of a convex function that is homogeneous of degree $2m$. In fact, Lasserre [22] showed that if a function which is positively homogeneous of p degree (convexity of the function is not required), then its LF-transform is positively homogeneous of q degree, where $1/p + 1/q = 1$. Thus, if the product function f is not convex, its LF-transform remains homogeneous of degree $\frac{2m}{2m-1}$, in which case, however, the formula for f^* is not clear at present (Theorem 3.6 above provides the formula of f^* when f is convex). Moveover, if the product function f is strictly convex, then f^* given by (32) will be differentiable and strictly convex. While this property cannot be seen immediately from (32), it can follow from a well known result in [8] (see also, Corollary 4.1.3 in [14]).

Remark 3.8 We see from Theorem 3.6 that f^* is finite everywhere and $f^* > 0$ for all $x \in R^n \backslash \{0\}$. It should be noted that the convexity assumption on f is only needed in our analysis in order to derive formula (32). The finiteness and nonnegativeness of f^* do not rely on this assumption. The finiteness can follow directly from the coercivity of the product function f (see e.g., Proposition 1.3.8 in [14]). Noting that f^* is convex and homogeneous of degree $2m/(2m-1) > 1$, the nonnegativeness of f^* follows directly from Lemma 5.1 in [2] (which claims that any function that is convex and homogeneous of degree $p > 1$ must be nonnegative in its domain). Due to the special structure of the product function f, the finiteness and nonnegativeness of f^* can also be verified by the following estimate: From (22) and (32), it is easy to see that there exist two positive constants ξ_1, ξ_2 such that $\xi_1 \|x\|^{\frac{2m}{2m-1}} \le f^*(x) \le \xi_2 \|x\|^{\frac{2m}{2m-1}}$.

It is interesting to consider two special cases: $m = 2, 3$. First, by setting $m = 2$ in (32), we have

$$\begin{aligned} f^*(x) &= 3\left(\frac{1}{\alpha_2}\right)^{\frac{1}{3}} \left[x^T \left(A_1 + \frac{1}{\alpha_2} A_2\right)^{-1} x/4\right]^{\frac{2}{3}} \\ &= 3(\alpha_2)^{\frac{1}{3}} \left(x^T (\alpha_2 A_1 + A_2)^{-1} x/4\right)^{\frac{2}{3}}, \end{aligned}$$

and the system (18) collapses to

$$\begin{aligned} \alpha_2 &= \frac{x^T [A_1 + \frac{1}{\alpha_2} A_2]^{-1} A_2 [A_1 + \frac{1}{\alpha_2} A_2]^{-1} x}{x^T [A_1 + \frac{1}{\alpha_2} A_2]^{-1} A_1 [A_1 + \frac{1}{\alpha_2} A_2]^{-1} x} \\ &= \frac{x^T (\alpha_2 A_1 + A_2)^{-1} A_2 (\alpha_2 A_1 + A_2)^{-1} x}{x^T (\alpha_2 A_1 + A_2)^{-1} A_1 (\alpha_2 A_1 + A_2)^{-1} x}. \end{aligned}$$

Thus, an immediate result from Theorem 3.6 is as follows.

Corollary 3.9 [36] *Let $A \succ 0$ and $B \succ 0$ and the product $f = q_A q_B$ be convex. Then $f^*(0) = 0$ and for $x \neq 0$, $f^*(x) = 3\alpha^{\frac{1}{3}} (x^T (A + \alpha B)^{-1} x/4)^{\frac{2}{3}}$, where α is a root to the univariate equation at x: $\alpha = \frac{x^T (A+\alpha B)^{-1} A (A+\alpha B)^{-1} x}{x^T (A+\alpha B)^{-1} B (A+\alpha B)^{-1} x}$.*

Similarly, when $m = 3$, Theorem 3.6 is reduced to the next result.

Corollary 3.10 *Let $A_1 \succ 0$, $A_2 \succ 0$, $A_3 \succ 0$ be $n \times n$ matrices and let the product $f = q_{A_1} q_{A_2} q_{A_3}$ be convex. Then $f^*(0) = 0$ and for $x \neq 0$,*

$$f^*(x) = 5\left(\frac{1}{\alpha\gamma}\right)^{\frac{1}{5}} \left(\frac{x^T (A_1 + \frac{1}{\alpha} A_2 + \frac{1}{\gamma} A_3)^{-1} x}{6}\right)^{\frac{3}{5}},$$

where $(\alpha, \gamma) > 0$ is a solution to the following system of equations at x:

$$\begin{aligned} \alpha &= \frac{x^T (A_1 + \frac{1}{\alpha} A_2 + \frac{1}{\gamma} A_3)^{-1} A_2 (A_1 + \frac{1}{\alpha} A_2 + \frac{1}{\gamma} A_3)^{-1} x}{x^T (A_1 + \frac{1}{\alpha} A_2 + \frac{1}{\gamma} A_3)^{-1} A_1 (A_1 + \frac{1}{\alpha} A_2 + \frac{1}{\gamma} A_3)^{-1} x}, \\ \gamma &= \frac{x^T (A_1 + \frac{1}{\alpha} A_2 + \frac{1}{\gamma} A_3)^{-1} A_3 (A_1 + \frac{1}{\alpha} A_2 + \frac{1}{\gamma} A_3)^{-1} x}{x^T (A_1 + \frac{1}{\alpha} A_2 + \frac{1}{\gamma} A_3)^{-1} A_1 (A_1 + \frac{1}{\alpha} A_2 + \frac{1}{\gamma} A_3 1) x}. \end{aligned}$$

Roughly speaking, the computation of the LF-transform for the product of m quadratic forms amounts to finding a solution to the system (21). From the proof of Lemma 3.3, this also amounts to computing a fixed point of the mapping $\mathcal{F}^{(x)}(\alpha_2, \ldots, \alpha_m)$. As the size of the system (21) dependents proportionally on the number of quadratic forms involved, the computational complexity of f^* also depends directly on the number of quadratic forms. The more quadratic forms are involved, the more efforts are required for the evaluation of the LF-transform. It is not difficult to see that (21) is actually a polynomial system and hence it is sufficiently smooth. Newton's method can be employed to solve the system (21). Since the solution of the system lies in the box (23), the bisection method may be applied, and some fixed-point methods can be used as well.

4 Conclusions

A general sufficient convexity condition for the product of finitely many quadratic forms was developed in this paper. The main result claims that the product function is convex if the condition numbers of the so-called 'scaled matrices' are bounded above by a certain constant which can be explicitly given in terms of the number of quadratic forms. This result indicates that the more distinct quadratic forms are involved, the more restrictive condition should be imposed on these quadratic forms in order to retain the convexity of the product function (in another word, the more quadratic forms are involved, the more likely the product function loses its convexity). The convexity condition developed in this paper makes it possible to identify the computationally tractable multiplicative optimization problems, and makes it also possible to employ some efficient modern convex optimization methods [31] to solve some (quadratic) multiplicative programming problems instead of relying merely on global optimization methods. On the other hand, a more general question than the open 'Question 11' in [13] has been addressed in this paper. The main result (Theorem 3.6) shows that the Legendre-Fenchel transform of the product of finitely many quadratic forms can be explicitly expressed as a finite function with some parameters which can be obtained by solving a system of equations with a special structure (or equivalently, by computing a fixed point of a smooth mapping). This result makes it possible to compute efficiently the LF-transform for the product of finitely many quadratic forms. From a duality point of view, this result might also lead to an effective duality-type algorithm for some multiplicative optimization problems.

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