NORTH-HOLLAND

Exceptional Families and Finite-Dimensional Variational Inequalities over Polyhedral Convex Sets

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ABSTRACT

A new concept of exceptional family for nonlinear variational inequalities over polyhedral sets is introduced in this paper. It generalizes the concepts for complementarity problem introduced by Smith and Isac. We applied the new analytical tool to the study of existence problem for variational inequalities. It is shown that our existence condition is weaker than most of the sufficient conditions which have been known. © Elsevier Science Inc., 1997

1. INTRODUCTION

The variational inequality problems have been successfully applied in PIES model, traffic equilibria, spatial price equilibria, the prediction of interregional commodity flows, the solution of Nash equilibria, and the Walrasian or general equilibrium model of economic activities during the last three decades. It is well-known that both complementarity problems and convex nonlinear programs can be considered as special cases of the variational inequality problems (see, [1]).

The development of solution conditions for the problem has played a very important role in both theory and practical applications. So far, a large body of literature has developed on the existence (and uniqueness) of solutions to the problem, including the works by Cottle [2], Eaves [3] and [4], Karamardian [5], [6] and [7], Moré [8] and [9], Habetler and Price [10], Pang [11] and [12], Smith [13], Isac [14] and an article by Isac, Bulavski and Kalashnikov (to appear).

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In order to study the existence of economic equilibria, Smith [13] introduced the *exceptional sequence* for complementarity problem and applied his result to spatial price equilibrium. Independently, G. Isac discovered the exceptional sequence under the name of *opposite radial sequence.* As another application different from [13], Harker [15] presented an alternative proof of the existence of a solution to the network equilibrium problem by using Smith's result [13]. Recently, Isac, et al. (to be published) introduced the concept of (*regular*) *exceptional family of elements*, which generalized the concept in [13], and applied it to explicit/implicit complementarity and order complementarity problems. Specializing the results in [13] and Isac, et al. to linear complementarity problem (LCP) is discussed in [16]. However, the concept of exceptional family used by [13] and Isac, et al. can only be applied to complementarity problem, i.e., the set in problem is a cone.

In this paper, we develop a new concept of exceptional family for variational inequality over polyhedral set. We discover that the concepts in [12] and Isac, et al. are a special case of our concept of exceptional family. Moreover, we will establish a sufficient existence condition for the solution to variational problem and show that our sufficient condition is weaker than most of the well-known existence conditions. Indeed, we have also presented the alternative proof for many existence theorems.

2. EXCEPTIONAL FAMILY

We consider the following finite-dimensional variational inequality problem, denoted by VI(S, F), which is defined to find a vector x^* such that

$$(x - x^*)^T F(x^*) \geqslant 0 \quad \text{for all } x \in S \tag{2.1}$$

where S is polyhedral convex set, i.e.,

$$S = \{x \in R^n \colon Ax \leq b\}. \tag{2.2}$$

F: $S \to R^n$ is a function, A is an $m \times n$ matrix and $b \in R^m$. When $b \neq 0$, the set S is not convex cone. But if $b = 0$ and $A = -I$ (identity matrix), then the set S reduces to the nonnegative orthant, namely, $R^n_+ = \{x \in R^n$: $x \geqslant 0\}$. In this case (2.1) reduces to the well-known nonlinear complementarity problem NCP(F)

$$x \geqslant 0, \quad f(x) \geqslant 0, \quad x^T F(x) = 0.$$

If S is a bounded set, then by theorem 3.1 in [1], the problem (2.1) has at least one solution for any continuous function on S, and there is nothing to discuss. Throughout the paper, we assume that the polyhedral set S is unbounded and the interior $\text{int}(S) \neq \emptyset$. Our main goal is to develop some sufficient existence conditions for the solution to the problem (2.1) by a new analytical tool (exceptional family, see definition 2.1).

Let $\|\cdot\|$ denote the Euclidean norm function and the set $B_\alpha = \{x \in R^n: \|x\| \leq \alpha\}$ be the Euclidean ball with radius $\alpha > 0$, let S_α be defined by

$$S_\alpha = S \cap B_\alpha = \{x \in R^n: \ Ax \leq b\} \cap \{x \in R^n: \|x\| \leq \alpha\}.$$

Obviously, S_α is convex compact set provided that $S_\alpha \neq \emptyset$. Since S is unbounded, there exists some α_0 such that $\text{int}(S_{\alpha_0}) \neq \emptyset$ for $\alpha \geqslant \alpha_0$.

Let $P_{S_\alpha}(\cdot)$ denote the projection operator on the set S_α, i.e., for any $z \in R^n$, $P_{S_\alpha}(z)$ in the unique solution to the following problem

$$\min\{\|x - z\|: \ x \in S_\alpha\}. \tag{2.3}$$

The lemma given below characterizes some property of the projection operator.

LEMMA 2.1. *Let* $\text{int}(S_\alpha) \neq \emptyset$, $x \in S_\alpha$ *and* $z \in R^n$, *then* $x = P_{S_\alpha}(z)$ *if and only if there exist some nonnegative vector* $\lambda = (\lambda_1, \dots, \lambda_m)^T \in R^m_+$ *and some scalar* $\mu \geqslant 0$ *such that*

$$2[(1+\mu)x - z] = -A^T\lambda \tag{2.4}$$

$$\lambda_i > 0 \Rightarrow (Ax - b)_i = 0, \quad i = 1, \dots, m \tag{2.5}$$

$$\mu > 0 \Rightarrow \|x\|^2 - \alpha^2 = 0. \tag{2.6}$$

PROOF. By the definition of $P_{S_\alpha}(z)$, $x = P_{S_\alpha}(z)$ if and only if x is the unique solution to the problem (2.3) which is equivalent to the following convex programming

$$\min\{\|x - z\|^2: \ Ax \leq b, \ x^T x \leq \alpha^2, \ x \in R^n\}. \tag{2.7}$$

Since $\mathrm{int}(S_\alpha) \neq \emptyset$, the Slater optimality condition holds, therefore $x = P_{S_\alpha}(z)$ if and only if x is the Kukn-Tucher point of the convex programming (2.7). Namely, there exist some scalar $\mu \geqslant 0$ and nonnegative vector $\lambda \in R_+^m$ such that

$$-2(z-x) + A^T\lambda + 2\mu x = 0 \qquad (a_1)$$

$$Ax \leq b, \quad x^T x - \alpha^2 \leq 0 \qquad (a_2)$$

$$\lambda_i (Ax - b)_i = 0 \quad \text{for all } i = 1, \dots, m \qquad (a_3)$$

$$\mu(x^T x - \alpha^2) = 0. \qquad (a_4)$$

Since $x \in S_\alpha$, (a_1) through (a_4) are easily seen to be equivalent to (2.4) through (2.6). The proof is completed.

By the same argument as lemma 2.1, it is easy to show the next result.

LEMMA 2.2. *Let $x \in S$ and $z \in R^n$, S is given by (2.2), then $x = P_S(z)$ if and only if there exists some vector $\lambda \in R_+^m$ such that x and z satisfy the following*

$$2(z-x) = A^T\lambda \qquad (2.8)$$

$$\lambda_i > 0 \Rightarrow (Ax - b)_i = 0. \qquad (2.9)$$

It is well-known that x^* solves the problem $\mathrm{VI}(S, F)$ if and only if x^* is the fixed point to the mapping $H(x) = P_S(x - F(x))$, i.e.,

$$x^* = P_S(x^* - F(x^*)) \qquad (2.10)$$

(see, [1], Proposition 2.3). Setting $z = x^* - F(x^*)$ and $x = x^*$, then by lemma 2.2, x^* is the solution to variational inequality problem $\mathrm{VI}(S, F)$ if and only if there exists some vector $\lambda^* \in R_+^m$ such that

$$F(x^*) = -\frac{1}{2} A^T \lambda^* \qquad (2.11)$$

$$\lambda_i^* > 0 \Rightarrow (Ax^* - b)_i = 0. \qquad (2.12)$$

Similarly, x^* solves the variational problem $\mathrm{VI}(S_\alpha, F)$, namely

$$(x - x^\alpha)^T (F(x^\alpha) \geqslant 0 \quad \text{for all } x \in S_\alpha$$

if and only if

$$x^{\alpha} = P_{S_{\alpha}}(x^{\alpha} - F(x^{\alpha})) \tag{2.13}$$

Replacing z and x in lemma 2.1 by $x^{\alpha} - F(x^{\alpha})$ and x^{a}, respectively. From (2.4) through (2.6), it follows that (2.13) can be equivalently formulated as follows: there exists some vector $\lambda \in R_{+}^{m}$ and some scalar $\mu \geqslant 0$ such that

$$F(x^{\alpha}) = -\tfrac{1}{2} A^{T}\lambda - \mu x^{\alpha} \tag{2.14}$$

$$\lambda_{i} > 0 \Rightarrow (Ax^{\alpha} - b)_{i} = 0 \tag{2.15}$$

$$\mu > 0 \Rightarrow \| x^{\alpha}\|^{2} = \alpha^{2}. \tag{2.16}$$

Motivated by the above observation, now we develop the concept of exceptional family for VI(S, F).

DEFINITION 2.1. *For any F: $S \to R^{n}$, S is a polyhedral convex set defined by* (2.2), *we say the sequence* $\{x^{\alpha}\}_{\alpha > 0} \subset S$ *is an exceptional family for the variational inequality problem* (2.1) *if for each* $\alpha > 0$, $\| x^{\alpha}\| = \alpha$, *and there exist some positive scalar* $\mu_{\alpha} > 0$ *and some vector* $\lambda^{\alpha} \in R_{+}^{m}$ *such that*

$$F(x^{\alpha}) = -\mu_{\alpha} x^{\alpha} - \tfrac{1}{2} A^{T}\lambda^{\alpha} \tag{2.17}$$

$$\lambda_{i}^{\alpha} > 0 \Rightarrow (Ax^{\alpha} - b)_{i} = 0. \tag{2.18}$$

Clearly, our concept of exceptional family is the generalization of the corresponding concept introduced in [13] and Isac for nonlinear complementarity problem NCP(F). The latter is a special case of the above definition. In fact, when $b = 0$ and $A = -I$, VI(S, F) reduces to NCP(F) and (2.17) and (2.18) reduce to

$$F_{i}(x^{\alpha}) = -\mu_{\alpha} x^{\alpha}, \quad \text{if } x_{i}^{\alpha} > 0 \tag{2.19}$$

$$F_{i}(x^{\alpha}) \geqslant 0 \quad \text{if } x_{i}^{\alpha} = 0. \tag{2.20}$$

It is the exceptional family introduced by Smith [13] and Isac. The next lemma establishes the relations between variational inequality VI(S, F) and VI(S_{α}, F).

LEMMA 2.3[17]. *Let F be a continuous mapping from S into R^n, then* VI(S, F) *has at least one solution if and only if there exists an $\alpha > 0$ such that $x^\alpha \in S_\alpha$ is the solution to* VI(S_α, F) *with $\|x^\alpha\| < \alpha$.*

Now we prove our main result on the existence of the solution to variational inequality problem (2.1). In fact the theorem below generalized the results concerning NCP(F) established in [13] and Isac et al.

THEOREM 2.1. *Let F be a continuous mapping from S into R^n, then either the variational inequality* VI(S, F) *has a solution or there exists an exceptional family.*

PROOF. Suppose that the problem (2.1) has no solution. Then by lemma 2.3, there exists no $\alpha > 0$ such that $x^\alpha \in S_\alpha$ is the solution to VI(S_α, F) with property $\|x^\alpha\| < \alpha$. But F is continuous and for each $\alpha > 0$ the set S_α is compact convex set. By theorem 3.1 in [1], VI(S_α, F) has a solution for each $\alpha > 0$. Therefore, there exists sequence $\{x^\alpha\}_{\alpha>0} \subset S$ with property $\|x^\alpha\| = \alpha$ and each x^α is the solution to VI(S_α, F). Now we prove that $\{x^\alpha\}_{\alpha>0}$ is an exceptional family for VI(S, F). Actually, for each $\alpha > 0$, by (2.14)–(2.16), there exist $\lambda^\alpha \in R^m_+$ and $\mu_\alpha \geqslant 0$ such that

$$F(x^\alpha) = -\tfrac{1}{2}A^T\lambda^\alpha - \mu_\alpha x^\alpha \tag{2.21}$$

$$\lambda_i^\alpha > 0 \Rightarrow (Ax^\alpha - b)_i = 0. \tag{2.22}$$

It suffices to show $\mu_\alpha > 0$ for all $\alpha > 0$. But if $\mu_\alpha = 0$ for some $\alpha > 0$, then (2.21) and (2.22) reduce to (2.11) and (2.12), respectively. Hence x^α is a solution to VI(S, F). It is a contradiction. Therefore $\{x^\alpha\}_{\alpha>0}$ is an exceptional family for VI(S, F).

COROLLARY 2.1. *Let F be a continuous function from S into R^n, if the variational inequality* VI(S, F) *has no exceptional family, then* VI(S, F) *has at least one solution.*

Because of the above results, it is of interest to decide when the problem VI(S, F) has no exceptional family and hence guarantee the existence of solution to VI(S, F). In the next section, we point out that this sufficient condition, namely, without exceptional family, is weaker than most of the well-known existence conditions discussed extensively in literature. Isac et al. gave an NCP(F) example to demonstrate the condition "without exceptional family" is strictly weaker than the well-known coercivity condition.

3. NONEXISTENCE OF EXCEPTIONAL FAMILY

PROPOSITION 3.1. *Let F: $S \to R^n$ be a continuous function, if for every sequence $\{(x^\alpha, \lambda^\alpha)\}_{\alpha>0} \subset S \times R_+^m$ with property $\|x^\alpha\| \to \infty$, the following condition holds*

$$F(x^\alpha)^T x^\alpha + \tfrac{1}{2} b^T \lambda^\alpha \geqslant 0 \quad \textit{for some } \alpha > 0,$$

then VI(S, F) *has no exceptional family.*

PROOF. Suppose that there exists an exceptional family $\{x^\alpha\}_{\alpha>0} \subset S$, then by definition for each $\alpha > 0$ there exists a scalar $\mu_\alpha > 0$ and a vector $\lambda^\alpha \in R_+^m$ such that (2.17) and (2.18) hold. Therefore

$$F(x^\alpha)^T x^\alpha = -\mu_\alpha \|x^\alpha\|^2 - \tfrac{1}{2}(x^\alpha)^T A^T \lambda^\alpha. \tag{3.1}$$

Furthermore, since (2.18) implies that $(x^a)^T A^T \lambda^\alpha = b^T \lambda^\alpha$, we have

$$F(x^\alpha)^T x^\alpha + \tfrac{1}{2} b^T \lambda^\alpha = -\mu_\alpha \|x^\alpha\|^2 < 0 \quad \text{for all } \alpha > 0. \tag{3.2}$$

This is contradiction.

COROLLARY 3.1. *If the vector $b \in R_+^m$ and the function satisfies the following condition: for every sequence $\{x^\alpha\}_{\alpha>0} \subset S$ with property $\|x^\alpha\| \to \infty$ and $F(x^\alpha)^T x^\alpha \geqslant 0$ and some $\alpha > 0$, then the problem* (2.1) *has no exceptional family.*

The above result can be viewed as an extension of corollary 4.5 in [13]. Specializing it to linear complementarity problem, we have the following.

COROLLARY 3.2. *Let $F(x) = Mx + q$, where M is an $n \times n$ matrix, $q \in R_+^n$ and $b \in R_+^m$, then* VI$(S, Mx + q)$ *has no exceptional family.*

PROPOSITION 3.2. *Let F be a mapping from S into R^n (not necessarily continuous). If there exists some $x^0 \in S$ such that one of the following two conditions hold*

(c_1) $S(x^0) = \{x \in S: (x - x^0)^T F(x) < 0\}$ *is nonempty and bounded;*
(c_2) *for each infinite sequence $\|x^\alpha\| \to \infty$, where $\{x^\alpha\} \subset S$, F satisfies*

$$(x^\alpha - x^0)^T F(x^\alpha) \geqslant 0 \quad \text{for some } \alpha \text{ such that } \|x^\alpha\| > \|x^0\|,$$

then the variational inequality VI(S, F) *has no exceptional family.*

PROOF. Conversely, we assume that the problem VI(S, F) has an exceptional family $\{x^\alpha\}_{\alpha > 0} \subset S$, by definition 2.1, we have $\|x^\alpha\| = \alpha$ for each $\alpha > 0$ and there exist scalar $\mu_\alpha > 0$ and nonnegative vector $\lambda^\alpha \in R^m$ such that (2.17) and (2.18) hold. By (3.2), noting that $Ax^0 - b \leq 0$, we have

$$\begin{aligned}(x^\alpha - x^0)^T F(x^\alpha) &= F(x^\alpha)^T x^\alpha - F(x^\alpha)^T x^0 \\ &= -\mu_\alpha \|x^\alpha\|^2 - \tfrac{1}{2} b^T \lambda^\alpha + \mu_\alpha (x^\alpha)^T x^0 + \tfrac{1}{2}(Ax^0)^T \lambda \alpha \\ &= -\mu_\alpha \left(\|x^\alpha\|^2 - (x^\alpha)^T x^0\right) + \tfrac{1}{2}(Ax^0 - b)^T \lambda^\alpha \\ &\leq -\mu_\alpha \left(\|x^\alpha\|^2 - (x^\alpha)^T x^0\right) \\ &\leq -\mu_\alpha \|x^\alpha\| (\|x^\alpha\| - \|x^0\|).\end{aligned}$$

For sufficiently large α, i.e., there exists an a_0 such that for all $\alpha > \alpha_0$, we have $\|x^\alpha\| > \|x^0\|$. From the above, it is easy to see that the condition (c_1) and (c_2) can not be satisfied. This completes the proof.

The condition (c_1) in the proposition can be replaced by

(c_3) $\overline{S(x^0)} = \{x \in S: (x - x^0)^T F(x) \leq 0\}$ is bounded.

In this case, it is evident that all solutions to the problem VI(S, F) are contained in $\overline{S(x^0)}$, i.e., the solution set is a compact set. By theorem 2.1 and the above proposition 3.2, we reobtain the following results (see, [1], proposition 3.3). But our proof, which bases on exceptional family, is very simple.

COROLLARY 3.3[1]. *Let F: $S \to R^n$ be continuous. If there is s vector $x^0 \in S$ such that the condition (c_1) holds, then there exists at least one solution to* VI(S, F). *Furthermore if the condition (c_3) holds, then the solution set of* VI(S, F) *is compact.*

It should be noted that while the condition "nonexistence of exceptional family" is sufficient for the existence of solution to VI(S, F), it is in general not necessary. Smith [13] gave an example of complementarity problem which has both a solution and an exceptional family. However, under some assumption on F, the condition is also necessary. The next result refines theorem 2.1 in the pseudo-monotonicity case.

COROLLARY 3.4. *Let F: $S \to R^n$ be a continuous pseudo-monotone mapping, then one and only one of the following alternative holds.*

(a_1) *the variational problem* VI(S, F) *has at least a solution;*
(a_2) *the variational problem* VI(S, F) *has an exceptional family.*

PROOF. If (a_1) doesn't hold, then (a_2) hold by theorem 2.1. Now assume that (a_1) holds. Let x^* be a solution to the problem VI(S, F), then

$$(x - x^*)^T F(x^*) \geq 0 \quad \text{for all } x \in S.$$

Since F is pseudo-monotone, the above inequality implies that

$$(x - x^*)^T F(x) \geqslant 0 \quad \text{for all } x \in S.$$

Let $x^0 = x^*$. It follows obviously that the condition (c_2) in proposition 3.2 holds. Hence (a_2) doesn't hold. The proof is completed.

Denote the dual cone of the set S by S^*, i.e.,

$$S^* = \{y \in R^n\colon\ y^T x \geqslant 0 \qquad \text{for all } x \in S\}.$$

COROLLARY 3.5. *Suppose of F is pseudo-monotone with respect to S and there exists $x^0 \in S$ such that $F(x^0) \in \operatorname{int}(S^*)$, then the problem* VI($S$, F) *has no exceptional family.*

By proposition 3.4 in [1], the above assumption on F implies the solution existence for VI(S, F), therefore corollary 3.5 is the immediate consequence of corollary 3.4.

DEFINITION 3.1[8],[1]. A mapping F: $S \to R^n$ is said to be a

(1) P-function on S if

$$\max_{1\leq i\leq n} [F_i(x) - F_i(y)](x_i - y_i) > 0 \quad \text{for all } x,\ y \in S,\ x \neq y;$$

(2) Uniform P-function on S if there exists a scalar $c > 0$ such that

$$\max_{1\leq i\leq n} [F_i(x) - F_i(y)](x_i - y_i) \geqslant c\|x - y\|_2^2 \quad \text{for all } x,\ y \in S,\ x \neq y. \tag{3.3}$$

Let X be a convex, closed subset in R^n. For nonlinear P-function, the variational inequality VI(X, F) has at most one solution, but may not necessarily have a solution. Moré [8] gave an example to show the case. So in order to guarantee the nonexistence of exceptional family for P-function, some other restrictive assumption will be imposed. However, for uniform P-function the problem VI(S, F) always has a unique solution. In the following, among other things, we will present a new proof for the results under some assumption on S by using the concept of exceptional family (see, proposition 3.3 in detail).

Now we introduce the concept of uniform diagonal dominance function. This class of functions seems to have similar property to uniform P-function. Denote the max-norm by $\|\cdot\|_\infty$, i.e., $\|x\|_\infty = \max_{1\leq i\leq n}|x_i|$. Let $f_i(x)(i = 1,\ldots, n)$ be the component of the map F.

DEFINITION 3.2. A mapping F: $S \to R^n$ is said to be a uniform diagonal dominance function with respect to S if for any x, $y \in S$, $x \neq y$ and any index k with $|x_k - y_k| = \|x - y\|_\infty$, there exists a positive scalar c such that

$$(x_k - y_k)(f_k(x) - f_k(y)) \geqslant c\|x - y\|_\infty^2. \tag{3.4}$$

These maps can be viewed as the nonlinear generalization of the strictly diagonally dominant matrix with positive diagonal entrices, i.e., $M = (a_{ij})_{n\times n}$ with property

$$a_{ii} > \sum_{j\neq i} |a_{ij}| \quad \text{for all } i = 1,\ldots, n. \tag{3.5}$$

In fact, if M satisfies (3.5), let

$$c = \sum_{1 \le i \le n} \left\{ a_{ii} - \sum_{j \ne i} |a_{ij}| \right\} > 0.$$

Then for any $x \neq y$ and index k with $|x_k - y_k| = \|x - y\|_\infty$, we have

$$\begin{aligned} (x_k - y_k)(A(x - y))_k &= \sum_{j=1}^{n} a_{kj}(x_k - y_k)(x_j - y_j) \\ &\geqslant a_{kk}|x_k - y_k|^2 - \sum_{j \ne k} |a_{kj}||x_k - y_k||x_j - y_j| \\ &\geqslant \left(a_{kk} - \sum_{j \ne k} |a_{kj}| \right) |x_k - y_k|^2 \\ &\geqslant c\|x - y\|_\infty^2. \end{aligned}$$

Conversely, suppose that $M = (a_{ij})_{n \times n}$ satisfies (3.4), namely for any $x \neq y$ in R^n and any index k with $|x_k - y_k| = \|x - y\|_\infty$, it follows that

$$(x_k - y_k)(A(x - y))_k \geqslant c\|x - y\|_\infty^2. \tag{3.6}$$

Specially, set $y = 0$ and $x_j = -\operatorname{sgn} a_{kj}$ for all $j \neq k$ and $x_k = 1$, then $x \neq 0$ and $|x_k| = \|x\|_\infty$. Hence from (3.6)

$$c = c\|x\|_\infty^2 \le x_k \sum_{j=1}^{n} a_{kj} x_j = a_{kk} - \sum_{j \ne k} |a_{kj}|.$$

Note that the above condition holds for all $k \in \{1, 2, \dots, n\}$, therefore M satisfies (3.5).

PROPOSITION 3.3. *If the continuous mapping F: $S \to R^n$ satisfies one of the following*

(a_1) *F is a uniform P-function with respect to S,*

(a_2) *F is a uniform diagonal dominance function with respect to S, and A is an $n \times n$ diagonal matrix, then the variational inequality* VI(S, F) *has no exceptional family.*

PROOF. (a_1) Let y be a fixed vector in S. Suppose that there exists an exceptional family $\{x^\alpha\}_{\alpha>0}$. Since it is an infinite sequence, there exists a subsequence $\{x^{\alpha_j}\}_{\alpha_j>0}$ such that for some fixed index i_0

$$[F_{i_0}(x^{\alpha_j}) - F_{i_0}(y)](x_{i_0}^{\alpha_j} - y_{i_0}) = \max_{1\le i\le n} [F_i(x^{\alpha_j}) - F_i(y)](x_i^{\alpha_j} - y_i)$$

holds for all α_j. Furthermore, since F is a uniform P-function, we have

$$[F_{i_0}(x^{\alpha_j}) - f_{i_0}(y)](x_{i_0}^{\alpha_j} - y_{i_0}) \geqslant c\|x^{\alpha_j} - y\|_2^2, \tag{3.7}$$

From (2.17) and (2.18) and notice that A is diagonal and $Ay \le b$. We have

$$\begin{aligned}
&[F_{i_0}(x^{\alpha_j}) - F_{i_0}(y)](x_{i_0}^{\alpha_j} - y_{i_0})\\
&\quad = F_{i_0}(x^{\alpha_j})(x_{i_0}^{\alpha_j} - y_{i_0}) - F_{i_0}(y)(x_{i_0}^{\alpha_j} - y_{i_0})\\
&\quad = \left[-\mu_{\alpha_j} x_{i_0}^{\alpha_j} - \tfrac{1}{2}(A^T\lambda^{\alpha_j})_{i_0}\right](x_{i_0}^{\alpha_j} - y_{i_0}) - F_{i_0}(y)(x_{i_0}^{\alpha_j} - y_{i_0})\\
&\quad = -\mu_{\alpha_j}\left[(x_{i_0}^{\alpha_j})^2 - x_{i_0}^{\alpha_j} y_{i_0}\right] + \tfrac{1}{2}\left[y_{i_0}(A^T\lambda^{\alpha_j})_{i_0} - (x_{i_0}^{\alpha_j})(A^T\lambda^{\alpha_j})_{i_0}\right]\\
&\qquad - F_{i_0}(y)(x_{i_0}^{\alpha_j} - y_{i_0})\\
&\quad = -\mu_{\alpha_j}\left[(x_{i_0}^{a_j})^2 - x_{i_0}^{\alpha_j} y_{i_0}\right] + \tfrac{1}{2}\left[\lambda_{i_0}^{\alpha_j}(Ay)_{i_0} - \lambda_{i_0}^{\alpha_j}(Ax^\alpha)_{i_0}\right]\\
&\qquad - F_{i_0}(y)(x_{i_0}^{\alpha_j} - y_{i_0})\\
&\quad = -\mu_{\alpha_j}\left[(x_{i_0}^{\alpha_j})^2 - x_{i_0}^{\alpha_j} y_{i_0}\right] + \tfrac{1}{2}\lambda_{i_0}^{\alpha_j}(Ay - b)_{i_0} - F_{i_0}(y)(x_{i_0}^{a_j} - y_{i_0})\\
&\quad \le -\mu_{\alpha_j}\left[(x_{i_0}^{a_j})^2 - x_{i_0}^{a_j} y_{i_0}\right] - F_{i_0}(y)(x_{i_0}^{a_j} - y_{i_0}).
\end{aligned}$$

Combining (3.7) and the above inequality, we have

$$\frac{-\mu_{\alpha_j}\left[(x_{i_0}^{a_j})^2 - x_{i_0}^{a_j} y_{i_0}\right] - F_{i_0}(y)(x_{i_0}^{a_j} - y_{i_0})}{\|x^{a_j} - y\|_2^2} \geqslant c > 0.$$

But $|x^{a_j}| \to \infty$ and $\mu_{\alpha_j} > 0$ for all $\alpha_j > 0$. This is a contradiction.

(a_2) Let F be a uniform diagonal dominance mapping and y be a fixed vector in S. If these exists an exceptional family $\{x^{\alpha}\}_{\alpha>0}$, then there is a subsequent $\{x^{\alpha_j}\}_{\alpha_j>0}$ such that for some index i_0, $|x_{i_0}^{a_j} - y_{i_0}| = \|x^{\alpha_j} - y\|_{\infty}$ for all $j = 1, \cdots$. by (3.4), we have

$$\left(x_{i_0}^{a_j} - y_{i_0}\right)\left(F_{i_0}(x^{\alpha_j}) - F_{i_0}(y)\right) \geqslant c\|x^{\alpha_j} - y\|_{\infty}^2.$$

By the same argument as in (a_1), a contradiction will be obtained. This completes the proof.

The problem NCP(F)

$$x \geqslant 0, \quad F(x) \geqslant 0, \quad x^T F(x) = 0$$

is a special case of VI(S, F). It corresponds to the case $A = -I$, $b = 0$. The following consequence is immediate from proposition 3.3.

COROLLARY 3.6. *When F is a uniform P-function or uniformly diagonally dominant function, the NCP(F) has no exceptional family.*

In proposition 3.3, we only show the case that A is diagonal, but it is conjectured that the assumption can eventually be eliminated.

The coercivity property of the mapping has played a very important role in existence theory of variational inequality. Indeed, the variational problem VI(X, F) has a nonempty compact solution set provided that F is coercive with respect to X (see, [1], theorem 3.2). It is easy to see that if F is strongly monotone over X, i.e., there is a $\alpha > 0$ such that $(F(x) - f(y))^T(x - y) \geqslant \alpha\|x - y\|^2$ holds for all x, y in X, then F is coercive with respect to X. It is also evident that when $0 \in X$ and F is strongly copositive with respect to X, i.e., if there exists a scalar $\alpha > 0$ such that $(F(x) - F(0))^T x \geqslant \alpha\|x\|_2^2$ holds for all $x \in X$, then F is coercive with respect to X. In what follows, we will show that for VI(S, F) the sufficient condition "nonexistence of exceptional family" is weaker than coercive condition. The result generalizes the results in [13] and Isac, et al. The definition of coercivity property can be found in [1], [8], [13] and Isac, et al., but for completeness, we state it as follows.

DEFINITION 3.3. A function F: $S \to R^n$ is said to be coercive with respect to S if there exists some $x^0 \in S$ such that

$$\lim_{x \in S, \|x\| \to \infty} \frac{\left(F(x) - F(x^0)\right)^T (x - x^0)}{\|x - x^0\|} = +\infty. \tag{3.8}$$

PROPOSITION 3.4. If F is coercive with respect to S, then the variational problem VI(S, F) has no exceptional family.

PROOF. Let $x^0 \in S$ such that (3.8) holds. Assume that there is exceptional family $\{x^\alpha\}_{\alpha>0}$, let $\lambda^\alpha \in R_+^m$ and $\mu_\alpha > 0$ be defined as in definition 2.1, then by (2.17) and (2.18) for sufficiently large $\alpha > 0$, we have

$$\frac{(F(x^\alpha) - F(x^0))^T (x^\alpha - x^0)}{\|x^\alpha - x^0\|}$$

$$= \frac{F(x^\alpha)^T x^\alpha - F(x^\alpha)^T x^0}{\|x^\alpha - x^0\|} - \frac{F(x^0)^T (x^\alpha - x^0)}{\|x^\alpha - x^0\|}$$

$$= \frac{-\mu_\alpha (x^\alpha - x^0)^T x^\alpha + (1/2)\big((\lambda^\alpha)^T Ax^0 - ((\lambda^\alpha)^T Ax^\alpha\big)}{\|x^\alpha - x^0\|}$$

$$- \frac{F(x^0)^T (x^\alpha - x^0)}{\|x^\alpha - x^0\|}$$

$$\leq \frac{-\mu_\alpha \|x^\alpha\|(\|x^\alpha\| - \|x^0\|) + (1/2)(Ax^0 - b)^T \lambda^\alpha}{\|x^\alpha - x^0\|} + \|F(x^0)\|$$

$$\leq \frac{-\mu_\alpha (\|x^\alpha\|(\|x^\alpha\| - \|x^0\|)}{\|x^\alpha - x^0\|} + \|F(x^0)\|$$

$$\leq \|F(x^0)\|$$

which implies that F is noncoercive. This completes the proof.

The above proposition asserts that coercive condition implies that nonexistence of exceptional family for VI(S, F), but the converse is not necessarily true. In fact, Isac, et al. give an example concerning NCP(F) to show that the condition "nonexistence of exceptional family" can not imply the coercive condition.

4. CONCLUSION

The concept of exceptional family for variational inequality introduced in this paper can be viewed as an extension of the concepts of exceptional sequence and exceptional family of elements introduced by Smith and Isac, respectively. It provides a new analytical tool for investigating the existence theory of variational inequality. We conclude that it also opens an interesting research direction in variational inequality. Considering our discussion in section 3, it is interesting to note that our concept of exceptional family has closed relations to optimality condition for convex programming.

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