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Exceptional Family of Elements and the Solvability of Variational Inequalities for Unbounded Sets in Infinite Dimensional Hilbert Spaces

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1. INTRODUCTION

It is well known that the theory of *variational inequalities* is now very well developed. The development of this theory has been stimulated by the diversity of applications in physics, mechanics, elasticity, fluid mechanics, economics, engineering, etc. [19]. The number of papers published on this subject is impressive. The solvability of variational inequalities has been studied by several methods using, for example, *coercivity conditions*, *compactness*, *fixed-point theory*, *KKM-mappings*, *min-max theory*, and many other topological methods. The relations between variational inequalities and complementarity problems are well known. A new method is now used in the study of solvability of complementarity problems. This new method is based on the concept of *exceptional family of elements* and it is related to the *topological degree* to the concept of *zero-epi mapping* and to the *Leray–Schauder alternative* [4, 5, 8, 10–18, 23, 26, 27–29].

In [20], T. E. Smith used the notion of an exceptional sequence of elements, which is more restrictive than our notion and is not related to

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the topological degree. The efficiency of the method based on the concept of *exceptional family of elements* can be measured by the number of papers based on this notion written only in the last three years [4, 5, 8, 10–18, 21–29].

In 1997 and 1998, Y. B. Zhao adapted the concept of exceptional family of elements to the study of solvability of variational inequalities in Euclidean spaces [21, 22].

Now, in this paper we will extend the concept of exceptional family of elements to the study of variational inequalities in arbitrary infinite dimensional Hilbert spaces. The concept obtained this way is appropriated for variational inequalities defined on unbounded sets. To realize this extension we replace the topological degree by *the Leray–Schauder type alternatives*. This paper opens a new research direction in the theory variational inequalities.

2. PRELIMINARIES

Let $(H, \langle \cdot , \cdot \rangle$ be a Hilbert space and $\Omega \subset H$ a non-empty closed convex subset. Since Ω is closed and convex, then the *projection operator* onto Ω, denoted by P_Ω, is well defined for every $x \in H$. We have that for every $x \in H$, $P_\Omega(x)$ is the unique element in Ω such that

$$\|x - P_\Omega(x)\| = \min_{y \in \Omega} \|x - y\|.$$

Given a mapping $f\colon H \to H$, we consider the following *variational inequality* defined by f and Ω,

$$\mathrm{VI}(f, \Omega)\colon \begin{cases} \text{find } x_* \in \Omega & \text{such that} \\ \langle x - x_*, f(x_*)\rangle \geq 0, & \text{for all } x \in \Omega. \end{cases} \tag{2.1}$$

It is well known [19] that the solvability of variational inequality (2.1) is equivalent to the solvability in H of the equation

$$x = P_\Omega(x - f(x)). \tag{2.2}$$

For any real number $r > 0$, we denote $\overline{B}_r = \{x \in H \mid \|x\| \leq r\}$ and $\Omega_r = \Omega \cap \overline{B}_r$. The following lemma is useful for our results.

LEMMA 2.1 [19]. *Let $f\colon H \to H$ be a continuous mapping and $\Omega \subset H$ a non-empty closed convex subset. The variational inequality* $\mathrm{VI}(f, \Omega)$ *has a solution if and only if there exists some $r > 0$ such that the variational inequality* $\mathrm{VI}(f, \Omega_r)$ *has a solution x_r with $\|x_r\| < r$.*

If $X \subset H$ is an arbitrary non-empty subset, we denote by ∂X the boundary of X, by $\operatorname{int}(X)$ the interior of X, and by $\operatorname{cl} X$ the closure of X. We say that a subset $\mathbf{K} \subset H$ is a *cone* if $\lambda \mathbf{K} \subseteq \mathbf{K}$ for all $\lambda \in \mathbf{R}_+$ and we say that $\mathbf{K}$ is *a convex cone* if (1) $\lambda \mathbf{K} \subseteq \mathbf{K}$ and (2) $\mathbf{K} + \mathbf{K} \subseteq \mathbf{K}$.

If $\mathbf{K} \subset H$ is a cone, its dual is (by definition) $\mathbf{K}^* = \{y \in H | \langle x, y \rangle \geq 0$ for all $x \in \mathbf{K}\}$. We can show that $\mathbf{K}^*$ is a convex cone. If $D \subset H$ is a non-empty convex set and $x \in \operatorname{cl} D$, then (by definition) the *normal cone* of D at the point x is

$$N_D(x) = \{\xi \in H | \langle \xi, y - x \rangle \leq 0 \text{ for all } y \in D\}$$

or $N_D(x) = -[T_D(x)]^*$, where $T_D(x)$ is the tangent cone of D at the point x, i.e., $T_D(x) = \operatorname{cl} \bigcup_{\lambda > 0} \lambda(D - x)$.

The next result is a classical known result (see, for example, [6]).

PROPOSITION 2.2. *For each $x \in H$, $y = P_\Omega(x)$ if and only if $x \in y + N_\Omega(y)$.*

We need also to recall the definition of the subdifferential of a convex function f: $H \to \mathbf{R}$, at a point $x \in H$. This is the set

$$\partial f(x) = \{u \in H | f(y) - f(x) \geq \langle u, y - x \rangle \text{ for all } y \in H\}.$$

THEOREM 2.3. *Let $g_1, g_2, \ldots, g_m$ be m continuous convex functions defined on a Banach space $(E, \| \ \|)$, and let f: $E \to \mathbf{R} \cup \{+\infty\}$ be a proper convex function. Suppose that there exists a point $\bar{x} \in E$ such that $f(\bar{x})$ is a finite real number and $g_i(\bar{x}) < 0$ for all $i = 1, 2, \ldots, m$. Then $x_0 \in D = \{x \in E | g_i(x) \leq 0;\ i = 1, 2, \ldots, m\}$ is a minimum point of f with respect to D, if and only if there exist a vector $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbf{R}_+^m$ and $x_* \in \partial f(x_0)$ such that*

(i) $-x_* \in \sum_{i=1}^{m} \lambda_i \partial g_i(x_0)$,

(ii) $\lambda_i g_i(x_0) = 0$, *for $i = 1, 2, \ldots, m$.*

Proof. A proof of this result is in [1]. ∎

3. LERAY–SCHAUDER TYPE ALTERNATIVES

Let $(H, \langle \cdot, \cdot \rangle$ be a Hilbert space and $X, Y \subset H$ non-empty subsets. Denote by $\operatorname{co}(X)$ the convex hull of X and by $\mathscr{P}(Y)$ the family of all non-empty subsets of Y.

Let f: $X \to Y$ be a set-valued mapping, i.e., f: $X \to \mathscr{P}(Y)$. The mapping f is said to be *upper semi-continuous* (u.s.c.) on X if the set

$\{x \in X | f(x) \subset V\}$ is open in X, whenever V is open in Y. The set-valued mapping f is said to be *compact* if $f(X)$ is contained in a compact subset of Y. A subset D of H is called *contractible* if there is a continuous mapping h: $D \times [0, 1] \to D$ such that for all $x \in D$ we have $h(x, 0) = x$ and $h(x, 1) = x_0$ for some $x_0 \in D$.

We note that if D is convex, then it is contractible, since for any $x_0 \in D$ the mapping $h(x, t) = tx_0 + (1 - t)x$ satisfies the definition of a contractible set. Also, a set starshaped at a point x_0 is also contractible to x_0. We say that a set-valued mapping f: $H \to H$ with non-empty values is *completely upper semi-continuous* if it is upper semi-continuous and for any bounded set $B \subset H$, we have that $f(B) = \bigcup_{x \in B} f(x)$ is relatively compact. In particular, a mapping f: $H \to H$ is called *completely continuous* if f is continuous and for any bounded set $B \subset H$, $f(B)$ is relatively compact.

We say that f: $H \to H$ *is a completely continuous field* (resp. *completely upper semi-continuous field*) if f has a decomposition of the form $f(x) = x - T(x)$, where T: $H \to H$ is a completely continuous mapping (resp. completely upper semi-continuous set-valued mapping).

We will use in this paper the following two Leray–Schauder type alternatives.

THEOREM 3.1 (The Nonlinear Alternative). *Let $C \subseteq H$ be a convex subset and $U \subset C$ a non-empty subset. Suppose that U is open in C and $0 \in U$. Then each continuous compact mapping f: $\overline{U} \to C$ has at least one of the following properties*:

(1) *f has a fixed point*,

(2) *there is an $x_* \in \partial U$ with $x_* = \lambda_* f(x_*)$ for some $\lambda_* \in]0, 1[$.*

Proof. A proof of this theorem is in [7]. ∎

THEOREM 3.2 (Leray–Schauder Type Alternative). *Let X be a closed subset of a locally convex space E such that $0 \in \operatorname{int}(X)$ and f: $X \to E$ a compact u.s.c. set-valued mapping with non-empty compact contractible values. If f is fixed-point free, then it satisfies the following Leray–Schauder condition.*

There exists $(\lambda_*, x_*) \in]0, 1[\times \partial X$ such that $x_* \in \lambda_* f(x_*)$.

Proof. This result is a part of corollary of the main theorem proved in [2]. A simple proof is also in [3]. ∎

4. MAIN RESULTS

Let $(H, \langle \cdot , \cdot \rangle$ be a Hilbert space and $\Omega \subset H$ a non-empty unbounded closed convex set. We will study in this section the solvability of a general variational inequality defined by the set Ω and *a completely continuous field* $f(x) = x - T(x)$, where T: $H \to H$ and $x \in H$. First we introduce the following definition.

DEFINITION 4.1. We say that $\{x_r\}_{r>0} \subset H$ is an exceptional family of elements for a completely continuous field $f(x) = x - T(x)$ defined on H, with respect to the subset Ω, if the following conditions are satisfied:

(1) $\|x_r\| \to +\infty$ as $r \to +\infty$.

(2) for any $r > 0$ there exists a real number $\mu_r > 1$ such that $\mu_r x_r \in \Omega$ and $T(x_r) - \mu_r x_r \in N_\Omega(\mu_r x_r)$,

where $N_\Omega(\mu_r x_r)$ is the normal cone of Ω at the point $\mu_r x_r$.

Remark. The above concept is inspired by the concept of "exceptional family of elements" used recently in complementarity theory [4, 5, 10–18, 21–29].

Also the concept defined in Definition 4.1 can be considered as a generalization of the notion of exceptional family of elements used by Zhao and Sun [28] and Zhao [22] in the Euclidean spaces.

As an application of this notion we have the following result.

THEOREM 4.1. *Let* $(H, \langle \cdot , \cdot \rangle$ *be a Hilbert space*, $\Omega \subset H$ *an arbitrary unbounded closed convex set, and* f: $H \to H$ *a completely continuous field with the representation* $f(x) = x - T(x)$, *where* T: $H \to H$ *is a completely continuous mapping* (*linear or nonlinear*).

Then the problem $\mathrm{VI}(f, \Omega)$ *has at least one of the following properties*:

(1) $\mathrm{VI}(f, \Omega)$ *has a solution*,

(2) *the completely continuous field* f *has an exceptional family of elements with respect to* Ω.

Proof. We know that the problem $\mathrm{VI}(f, \Omega)$ has a solution if and only if the mapping

$$\Phi(x) = P_\Omega(x - f(x)) = P_\Omega(T(x)); \qquad x \in H,$$

has a fixed point (in H) [19].

Obviously, this fixed point must be in Ω. We observe that the mapping Φ is completely continuous. The set $\overline{B}_r$ has a non-empty interior and $0 \in \mathrm{int}(\overline{B}_r)$.

About the problem $\mathrm{VI}(f, \Omega)$ we have two situations:

(I) The problem $\mathrm{VI}(f, \Omega)$ has a solution. If this is the case the proof is finished.

(II) The problem $\mathrm{VI}(f, \Omega)$ is without solution.

In this case the mapping Φ is fixed-point free with respect to any set $\overline{B}_r$, because if Φ restricted to a set $\overline{B}_r$ has a fixed point in $\overline{B}_r$ we have that the problem (VI(f, Ω) has a solution which is a contradiction.

Now, since $\overline{B}_r$ is bounded and Φ is completely continuous, we have that Φ restricted to $\overline{B}_r$ is a compact continuous mapping. The assumptions of Theorem 3.1 are satisfied taking $C = H$ and $U = \operatorname{int}(\overline{B}_r)$.

Therefore there is an element $x_r \in \partial \overline{B}_r$ such that $x_r = \lambda_r P_\Omega(T(x_r))$ for some $\lambda_r \in]0, 1[$.

By Proposition 2.2 we have that $T(x_r) \in (1/\lambda_r)x_r + N_\Omega((1/\lambda_r)x_r)$.

If we denote by $\mu_r = 1/\lambda_r$ for all $r > 0$, then we have

(i) $\|x_r\| = r$ and $\mu_r > 1$ for all $r > 0$,

(ii) $\mu_r x_r \in \Omega$ for all $r > 0$,

(iii) $T(x_r) - \mu_r x_r \in N_\Omega(\mu_r x_r)$ for all $r > 0$. Since $\|x_r\| \to +\infty$ as $r \to +\infty$, we deduce that $\{x_r\}_{r>0}$ is an exceptional family of elements for f with respect to Ω. ∎

COROLLARY 4.2. *Let $(H, \langle \cdot, \cdot \rangle$ be a Hilbert space, $\Omega \subset H$ an arbitrary unbounded closed convex set, and $f(x) = x - T(x)$ a completely continuous field on H. If f is without exceptional family of elements with respect to Ω, then the problem VI(f, Ω) is solvable.*

Theorem 4.1 can be extended to variational inequalities for set-valued mappings in the following manner.

Let f: $H \to H$ be a set-valued mapping and $\Omega \subset H$ a non-empty unbounded closed convex set. Consider the problem

$$\mathrm{VI}(h, \Omega): \begin{cases} \text{find } (x_*, y_*) \in \Omega \times H \text{ such that} \\ y_* \in f(x_*) \qquad \text{and} \qquad \langle u - x_*, y_* \rangle \geq 0 \text{ for all } u \in \Omega. \end{cases}$$

It is known that the problem VI(h, Ω) is solvable if and only if the set-valued mapping $P_\Omega(x - f(x))$ has a fixed point in H; i.e., there exists an element $x_* \in H$, such that $x_* \in P_\Omega(x_* - f(x_*))$. In this case there exists $y_* \in f(x_*)$ such that $x_* = P_\Omega(x_* - y_*)$, which implies that (x_*, y_*) is a solution of the problem VI(h, Ω).

Now, suppose that f: $H \to H$ is a set-valued mapping of the form $f(x) = x - T(x)$, where T: $H \to H$ is a set-valued mapping. We introduce the following definition.

DEFINITION 4.2. We say that $\{x_r\}_{r>0} \subset H$ is an exceptional family of elements for the set-valued mapping $f(x) = x - T(x)$, with respect to the subset Ω, if the following conditions are satisfied:

(1) $\|x_r\| \to +\infty$ as $r \to +\infty$,

(2) for any $r > 0$ there exists a real number $\mu_r > 1$ and an element $y_r \in T(x_r)$ such that $\mu_r x_r \in \Omega$ and $y_r - \mu_r x_r \in N_\Omega(\mu_r x_r)$, where $N_\Omega(\mu_r x_r)$ is the normal cone of Ω at the point $\mu_r x_r$.

THEOREM 4.3. *Let $(H, \langle \cdot , \cdot \rangle)$ be a Hilbert space, $\Omega \subset H$ an arbitrary unbounded closed convex set, and f: $H \to H$ a completely upper semi-continuous field with the representation $f(x) = x - T(x)$, where T: $H \to H$ is a completely upper semi-continuous set-valued mapping with non-empty compact contractible values. Then the problem* VI(f, Ω) *has at least one of the following properties*:

(1) VI(f, Ω) *has a solution*,

(2) *the completely upper semi-continuous field f has an exceptional family of elements with respect to Ω, in the sense of Definition* 4.2.

Proof. The proof is similar to the proof of Theorem 4.1. Consider the set-valued mapping $\Phi(x) = P_\Omega(x - f(x)) = P_\Omega(T(x))$. As in our paper [14] we can show that $P_\Omega(T(x))$ is a set-valued mapping with compact contractible values. The proof follows the proof of Theorem 4.1 but Theorem 3.1 is replaced by Theorem 3.2. ▮

A consequence of Theorem 4.3 is the following result.

COROLLARY 4.4. *Let $(H, \langle \cdot , \cdot \rangle)$ be a Hilbert space and $\Omega \subset H$ an arbitrary unbounded closed convex set. Let $f(x) = x - T(x)$ be a completely upper semi-continuous field where T: $H \to H$ is with non-empty compact contractible values. If f is without exceptional family of elements with respect to Ω in the sense of Definition* 4.2, *then the problem* VI(f, Ω) *is solvable*.

In many applications of variational inequalities the set Ω is described by a system of inequalities or equalities. Therefore we consider the case

$$\Omega = \{x \in H | g_1(x) \leq 0, \dots, g_m(x) \leq 0\}, \tag{4.1}$$

where $g_1, \dots, g_m$: $H \to \mathbf{R}$ are continuous real-valued convex functions. In this case we introduce the following notion of exceptional family of elements.

DEFINITIONS 4.3. If f: $H \to H$ has the form $f(x) = x - T(x)$, where T: $H \to H$, then we say that $\{x_r\}_{r>0} \subset H$ is an exceptional family of elements for f with respect to the subset $\Omega = \{x \in H | g_1(x) \leq 0, \dots, g_m(x) \leq 0\}$ supposed to be unbounded, if the following conditions are satisfied:

(1) $\|x_r\| \to +\infty$ as $r \to +\infty$ and

(2) for any $r > 0$ there exist a real number $\mu_r > 1$ and a vector $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbf{R}_+^m$ such that $\mu_r x_r \in \Omega$ and

(i) $T(x_r) - \mu_r x_r \in \sum_{i=1}^{m} \lambda_i \partial g_i(\mu_r x_r)$

(ii) $\lambda_i g_i(\mu_r x_r) = 0$, $i = 1, 2, \dots, m$.

Using the above concept we have the following result.

THEOREM 4.5. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $\Omega \subset H$ a closed unbounded convex set defined by $\Omega = \{x \in H | g_1(x) \leq 0, \dots, g_m(x) \leq 0\}$ where $g_1, g_2, \dots, g_m$: $H \to \mathbf{R}$ are continuous convex functions. Suppose that there exists a point $\bar{x} \in H$ such that $g_i(\bar{x}) < 0$ for all $i = 1, 2, \dots, m$. If f: $H \to H$ is a completely continuous field which has the representation $f(x) = x - T(x)$, then the problem* VI(f, Ω) *has at least one of the following properties*:

(1) VI(f, Ω) *has a solution*,

(2) *the field f has an exceptional family of elements with respect to Ω in the sense of Definition* 4.3.

Proof. About the problem VI(f, Ω) we have two situations:

(I) The problem VI(f, Ω) has a solution. In this case the proof is finished.

(II) The problem VI(f, Ω) is without solution.

In this case as in the proof of Theorem 4.1, the completely continuous mapping $\Phi(x) = P_\Omega(x - f(x)) = P_\Omega(T(x))$ is fixed-point free with respect to any set $\overline{B}_r = \{x \in H | \|x\| \leq r\}$, $r > 0$.

Since all the assumptions of Theorem 3.1 are satisfied there is an element $x_r \in \partial \overline{B}_r$ such that $x_r = \lambda_r P_\Omega(T(x_r))$ for some $\lambda_r \in]0, 1[$. If we denote by $\mu_r = 1/\lambda_r$ we have that $\mu_r > 1$ and because $\mu_r x_r = P_\Omega(T(x_r))$ we have that $\mu_r x_r$ is the unique solution of the convex program

$$\begin{cases} \min \mathscr{Q}(y) = \dfrac{1}{2} \| y - T(x_r) \|^2 \\ y \in \Omega. \end{cases}$$

By Theorem 2.3, there exists a vector $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbf{R}_+^m$ such that

$$\begin{cases} 0 \in \mu_r x_r - T(x_r) + \sum_{i=1}^{m} \lambda_i \partial g_i(\mu_r x_r) \text{ and} \\ \lambda_i g_i(\mu_r x_r) = 0, \qquad i = 1, 2, \dots, m. \end{cases}$$

Since $\|x_r\| = r \to +\infty$ as $r \to +\infty$, the desired result is proved. ∎

We proved Theorem 4.1 and 4.5 using the *nonlinear Leray–Schauder alternative* (Theorem 3.1) but the same theorems can be proved using the topological degree.

Now, we will show that with a small modification of the concept of exceptional family of elements, introduced by Definition 4.3 we can prove Theorem 4.5 using only the optimization.

DEFINITION 4.4. Let $f(x) = x - T(x)$ be a mapping from H into H and $\Omega \subset H$ an unbounded closed convex set given by $\Omega = \{x \in H \mid g_1(x) \leq 0, \ldots, g_m(x) \leq 0\}$, where $g_1, g_2, \ldots, g_m \colon H \to \mathbf{R}$ are convex continuous functions. Suppose the existence of an element $\bar{x} \in H$ such that $g_i(\bar{x}) < 0$ for all $i = 1, 2, \ldots, m$. In this case, we say that $\{x_r\}_{r>0} \subset \Omega$ is an exceptional family of elements for f with respect to Ω if the following conditions are satisfied:

(1) $\|x_r\| \to +\infty$ as $r \to +\infty$,

(2) for any $r > 0$ there exist a scalar $\mu_r > 0$ and a vector $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbf{R}^m_+$ such that $T(x_r) - x_r \in \mu_r(x_r - \bar{x}) + \sum_{i=1}^m \lambda_i \partial g_i(x_r)$ and $\lambda_i g_i(x_r) = 0$, $i = 1, 2, \ldots, m$.

Before proving the next alternative we need to prove the following lemmas.

LEMMA 4.6. *Let $g_1, g_2, \ldots, g_m \colon H \to \mathbf{R}$ be continuous convex functions such that there exists $\bar{x} \in H$ with the property $g_i(\bar{x}) < 0$ for all $i = 1, 2, \ldots, m$. For an arbitrary real number $r > 0$ consider the set $\Omega_r = \{x \in H \mid \frac{1}{2}\|x - \bar{x}\|^2 \leq \frac{1}{2}r^2$ and $g_1(x) \leq 0,\ i = 1, 2, \ldots, m\}$. Then $x_r \in \Omega_r$ solves the problem $\mathrm{VI}(f, \Omega_r)$ if and only if there exist a real number $\mu_r \geq 0$ and a vector $\lambda^r = (\lambda^r_1, \ldots, \lambda^r_m) \in \mathbf{R}^m_+$ such that*:

(1) $T(x_r) - x_r \in \mu_r(x_r - \bar{x}) + \sum_{i=1}^m \lambda^r_i \partial g_i(x_r)$,

(2) $\lambda^r_i g_i(x_r) = 0$ *for all* $i = 1, 2, \ldots, m$,

(3) $\mu_r(\|x_r - \bar{x}\|^2 - r^2) = 0$.

Proof. Indeed, $x_r \in \Omega_r$ solves the problem $\mathrm{VI}(f, \Omega_r)$ if and only if x_r is the unique solution of the problem

$$\min_{y \in \Omega_r} \frac{1}{2}\| y - T(x_r)\|^2.$$

By Theorem 2.3 there exists $\mu_r \geq 0$ and $\lambda^r = (\lambda_1^r, \dots, \lambda_m^r) \in \mathbf{R}_+^m$ such that

$$T(x_r) - x_r \in \mu_r(x_r - \bar{x}) + \sum_{i=1}^{m} \lambda_i^r \partial g_i(x_r),$$

$$\lambda_i^r g_i(x_r) = 0 \text{ for all } i = 1, 2, \dots, m$$

and

$$\mu_r \left(\frac{1}{2} \|x_r - \bar{x}\|^2 - \frac{1}{2} r^2 \right) = 0.$$

Obviously $\mu_r(\|x_r - \bar{x}\|^2 - r^2) = 0$ and we have the desired result. ∎

LEMMA 4.7. *Let $g_1, g_2, \dots, g_m$: $H \to R$ be continuous convex functions such that there exists $\bar{x} \in H$ with the property that $g_i(\bar{x}) < 0$ for all $i = 1, 2, \dots, m$. Consider the closed convex set $\Omega = \{x \in H | g_i(x) \leq 0,$ and $i = 1, 2, \dots, m\}$ and the mapping $f(x) = x - T(x)$ with T: $H \to H$.*

Then $x_* \in \Omega$ solves the problem $\mathrm{VI}(f, \Omega)$ *if and only if there exists a vector $\lambda^* = (\lambda_1^*, \dots, \lambda_m^*) \in \mathbf{R}_+^m$ such that*

$$T(x_*) - x_* \in \sum_{i=1}^{m} \lambda_i^* \partial g_i(x_*) \qquad \textit{and}$$

$$\lambda_i^* g_i(x_*) = 0 \textit{ for all } i = 1, 2, \dots, m.$$

Proof. The proof is similar to the proof of Lemma 4.6. ∎

Now, we can prove the following alternative.

THEOREM 4.8. *Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space and $\Omega \subset H$ an unbounded closed convex set defined by $\Omega = \{x \in H | g_i(x) \leq 0,$ and $i = 1, 2, \dots, m\}$ where $g_1, g_2, \dots, g_m$: $H \to \mathbf{R}$ are continuous convex functions. Suppose that there exists $\bar{x} \in H$ such that $g_i(\bar{x}) < 0$ for all $i = 1, 2, \dots, m$. If f: $H \to H$ is a completely continuous field which has the representation $f(x) = x - T(x)$, with T: $H \to H$, then either the problem* $\mathrm{VI}(f, \Omega)$ *has a solution or the field f has an exceptional family of elements with respect to Ω, in the sense of Definition* 4.4.

Proof. Suppose that $\mathrm{VI}(f, \Omega)$ has no solution. We show that under this assumption f must have an exceptional family of elements in the sense of Definition 4.4.

Consider again, for each $r > 0$, the closed convex set

$$\Omega_r = \left\{ x \in H \,\middle|\, \frac{1}{2} \|x - \bar{x}\|^2 \leq \frac{1}{2} r^2 \text{ and } g_1(x) \leq 0, i = 1, 2, \dots, m \right\}.$$

The set Ω_r is non-empty (since $\bar{x} \in \Omega_r$) closed convex and bounded. By Schauder's fixed point theorem the compact continuous mapping $\Phi(x) = P_{\Omega_r}(x - f(x)) = P_{\Omega_r}(T(x))$ has a fixed point in Ω_r, which implies that the problem VI(f, Ω_r) has a solution. Let x_r be this solution. Then by Lemma 2.1 we must have $\|x_r\| = r$. Thus $\|x_r\| \to +\infty$ as $r \to +\infty$. By Lemma 4.6, for each $r > 0$, since x_r solves the problem VI(f, Ω_r) there exists a scalar $\mu_r \geq 0$ and a vector $\lambda^r = (\lambda_1^r, \dots, \lambda_m^r) \in \mathbf{R}_+^m$ such that

$$T(x_r) - x_r \in \mu_r(x_r - \bar{x}) + \sum_{i=1}^{m} \lambda_i^r \partial g_i(x_r) \qquad \text{and}$$
$$\lambda_i^r g_i(x_r) = 0 \text{ for all } i = 1, 2, \dots, m.$$

To have that $\{x_r\}_{r>0}$ is an exceptional family of elements for f with respect to Ω, in the sense of Definition 4.4 it is sufficient to show that $\mu_r \neq 0$ for any $r > 0$.

Indeed, suppose that $\mu_r = 0$. In this case we have

$$T(x_r) - x_r \in \sum_{i=1}^{m} \lambda_i^r \partial g_i(x_r) \quad \text{and} \quad \lambda_i^r g_i(x_r) = 0 \text{ for all } i = 1, 2, \dots, m,$$

which implies by Lemma 4.7 that x_r is a solution of the problem VI(f, Ω), which is a contradiction and the proof is complete. ∎

5. COMMENTS

We presented in this paper some variants of the concept of exceptional family of elements, appropriated for the study of solvability of variational inequalities, with respect to unbounded sets in infinite-dimensional Hilbert spaces. Applying these notions we obtained several alternative theorems which have as consequences some sufficient existence theorems for variational inequalities.

Our results may be considered as generalizations of similar results obtained in Euclidean spaces by Y. B. Zhao [21–23], Y. B. Zhao and J. Y. Han [24], Y. B. Zhao *et al.* [25], Y. B. Zhao and D. Sun [28], and Y. B. Zhao and J. Y. Yuan [29].

The origin of the concept of exceptional family of elements for variational inequalities is the concept of exceptional family of elements for complementarity problems introduced by G. Isac *et al.* [15] and used in the papers [4, 5, 8, 10–16, 18, 28, 29]. This notion is also related to the notion of *exceptional sequence of elements* considered by T. E. Smith [20].

Open Problem. It is interesting to study under what conditions several kinds of mappings as, for example, coercive, monotone, pseudo-monotone, and other kinds of mappings considered in [10, 12, 16–18, 28, 29] are mappings without the exceptional family of elements in the sense of the definition introduced in this paper.

We will study this problem in a later paper.

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