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Quasi-P_*-Maps, $P(\tau, \alpha, \beta)$-Maps, Exceptional Family of Elements, and Complementarity Problems[1]

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Abstract. Quasi-P_*-maps and $P(\tau, \alpha, \beta)$-maps defined in this paper are two large classes of nonlinear mappings which are broad enough to include P_*-maps as special cases. It is of interest that the class of quasi-P_*-maps also encompasses quasimonotone maps (in particular, pseudomonotone maps) as special cases. Under a strict feasibility condition, it is shown that the nonlinear complementarity problem has a solution if the function is a nonlinear quasi-P_*-map or $P(\tau, \alpha, \beta)$-map. This result generalizes a classical Karamardian existence theorem and a recent result concerning quasimonotone maps established by Hadjisawas and Schaible, but restricted to complementarity problems. A new existence result under an exceptional regularity condition is also established. Our method is based on the concept of exceptional family of elements for a continuous function, which is a powerful tool for investigating the solvability of complementarity problems.

Key Words. Nonlinear complementarity problems, exceptional family of elements, quasi-P_*-maps, $P(\tau, \alpha, \beta)$-maps, exceptional regularity.

1. Introduction

Let $f: R^n \to R^n$ be a continuous function. The complementarity problem is to determine a vector $x \in R^n$ such that

$$f(x) \geq 0, \qquad x \geq 0, \qquad x^T f(x) = 0. \tag{1}$$

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This problem is a fundamental and interesting problem of mathematical programming. It provides a general framework for studying linear and quadratic programming problems, bimatrix games, as well as more general equilibrium problems emerging in physics, engineering, and economics. It is also viewed as an important special case of finite-dimensional variational inequality problems (Refs. 1–3).

Given a complementarity problem, the existence of a solution is not always ensured. Because of this aspect, a large number of classes of functions have been defined and a variety of existence theorems have been established by many authors (Refs 1–16). Many of these results assume the monotonicity or some generalized monotonicity of the function f. In recent years, the generalization of monotonicity has been investigated extensively in the literature (Refs. 17–20). Generalized monotonicity concepts emerged from the concept of a pseudomonotone map introduced by Karamardian (Ref. 13), who showed that the nonlinear complementarity problem over a pointed, solid, closed convex cone K in R^n [i.e., K a nonempty closed set with the properties $K+K\subseteq K$, $\lambda K\subseteq K$ for all $\lambda\geq 0$, and $K\cap(-K)=\{0\}$] has a solution if f is a continuous, pseudomonotone function and a strict feasibility condition is satisfied; i.e., there exists an element $x\in K$ such that $f(x)$ is an interior point of K^*, the dual cone of K. Cottle and Yao (Ref. 4) generalized the Karamardian result to the complementarity problem over a solid closed convex cone in Hilbert space. Harker–Pang (Ref. 3) and Yao (Refs. 21–22) extended the Karamardian result to variational inequalities. The class of quasimonotone maps is larger than the class of pseudomonotone maps. Hadjisavvas and Schaible (Ref. 7) established an existence result for a variational inequality with a quasimonotone map in a reflexive Banach space. Restricted to the complementarity problem (1), their result can be stated as follows: for any continuous quasimonotone function f, if a strict feasibility condition holds [i.e., if there exists a vector $u\geq 0$ such that $f(u)>0$], then the complementarity problem has a solution. Recently, Zhao and Han (Ref. 23) extended the concept of P_*-matrix (Ref. 24) to a nonlinear P_*-map which is also a generalization of monotone mapping. However, as shown in Section 2, a P_*-map need not be necessarily a quasimonotone map, and vice versa. Under a strict feasibility condition, Zhao and Han (Ref. 23) showed that the nonlinear complementarity problem with a P_*-map has a solution.

In this paper, we define two large classes of nonlinear maps. The first class includes quasi-P_*-mappings, a class significantly larger than the class of quasimonotone maps and that of P_*-maps. The second class includes nonlinear $P(\tau,\alpha,\beta)$-maps, which also includes the class of P_*-maps as a particular case. We prove that, for the two new classes of nonlinear maps, the complementarity problem has a solution if a strict feasibility condition holds. For the complementarity problem (1), the aforementioned existence theorems of Karamardian (Ref. 13), Hadjisavvas–Schaible (Ref. 7), and Zhao–Han (Ref. 23) all follow from our main results. On the other hand, under positive homogeneity and exceptional regularity assumptions, we will develop an existence theorem for nonlinear complementarity problems without a strict feasibility condition. This result is quite different from the result proved by Karamardian (Ref. 12). The argument used in the proof of our results is based on the concept of exceptional family of elements of a continuous function, which was introduced by Iasc, Bulavski, and Kalashnikov (Ref. 8). This concept provides a new method for studying the solvability of complementarity problems (Refs. 8–9, 23).

In Section 2, we introduce the concept of quasi-P_*-map and prove an existence theorem for the corresponding complementarity problem. In Section 3, we define the notion of $P(\tau, \alpha, \beta)$-map and obtain an existence result for such class of complementarity problems. We also give the equivalent definitions for P_*-maps and P_*-matrices. The solvability of the complementarity problem (1) under positive homogeneity and exceptional regularity assumptions is discussed in Section 4. Final remarks are given in Section 5.

2. Nonlinear Quasi-P_*-Maps

In this section, we establish an existence theorem for nonlinear complementarity problems with quasi-P_*-map.

Definition 2.1. A nonlinear mapping $f: R^n \to R^n$ is said to be a quasi-P_*-mappng if there exists a constant $\tau \geq 0$ such that the following implication holds:

$$f(y)^T(x-y) - \tau \sum_{i \in I_+(x,y)} (x_i - y_i)[f_i(x) - f_i(y)] > 0 \Rightarrow f(x)^T(x-y) \geq 0, \tag{2}$$

for all distinct points x, y in R^n, where

$$I_+(x,y) = \{i: (x_i - y_i)(f_i(x) - f_i(y)) > 0\}. \tag{3}$$

Definition 2.2. See Ref. 23. A mapping $f: R^n \to R^n$ is said to be a P_*-map if there exists a scalar $\kappa \geq 0$ such that, for any distinct points x, y in R^n, we have

$$(1+\kappa) \sum_{i \in I_+(x,y)} (x_i - y_i)[f_i(x) - f_i(y)] + \sum_{i \in I_-(x,y)} (x_i - y_i)[f_i(x) - f_i(y)] \geq 0, \tag{4}$$

where $I_+(x, y)$ is defined by (3) and

$$I_-(x, y) = \{i: (x_i - y_i)[f_i(x) - f_i(y)] \leq 0\}.$$

Recall that a matrix M is said to be a P_*-matrix (Ref. 24) if there exists a nonnegative number κ such that

$$(1 + \kappa) \sum_{i \in I_+} x_i (Mx)_i + \sum_{i \in I_-} x_i (Mx)_i \geq 0,$$

where

$$I_+ = \{i: x_i (Mx)_i > 0\}, \qquad I_- = \{i: x_i (Mx)_i \leq 0\}.$$

Clearly, an affine map $f = Mx + q$, where $M \in R^{n \times n}$ and $q \in R^n$, is a P_*-map if and only if M is a P_*-matrix. It is known that the class of positive semidefinite matrices and the class of P-matrices are included in the class of P_*-matrices. Väliaho (Ref. 25) showed that the class of P_*-matrices coincides with the class of sufficient matrices introduced by Cottle, Pang, and Venkateswaran (Ref. 26). In recent years, linear P_*-matrix complementarity problems have gained more attention in the field of interior-point algorithms; see for example Refs. 24 and 27–30.

It is easy to see that (4) can be written as

$$(x - y)^T [f(x) - f(y)] + \kappa \sum_{i \in I_+(x,y)} (x_i - y_i)[f_i(x) - f_i(y)] \geq 0.$$

Thus, a P_*-map must be a quasi-P_*-map; but the converse is not true (see the following example).

Example 2.1. Let $x = (x_1, x_2)^T \in R^2$ and

$$f(x) = (-2x_1^2, 0)^T.$$

We show that $f(x)$ is a quasi-P_*-map, but is not a P_*-map. Indeed, for any x, y in R^2, we have

$$f(y)^T (x - y) = -2y_1^2 (x_1 - y_1),$$

$$f(x)^T (x - y) = -2x_1^2 (x_1 - y_1).$$

Clearly,

$$f(y)^T (x - y) > 0 \text{ implies } f(x)^T (x - y) \geq 0.$$

Thus, $f(x)$ is a quasimonotone function, and hence $f(x)$ is a quasi-P_*-map with constant $\tau = 0$. However, for $x > 0$, $y > 0$, and $x_1 \neq y_1$, we have

$$(x_i - y_i)[f_i(x) - f_i(y)] = \begin{cases} -2(x_1 - y_1)^2 (x_1 + y_1), & i = 1, \\ 0, & i = 2, \end{cases}$$

which implies that

$$I_+(x,y) = \{i: (x_1 - y_1)[f_i(x) - f_i(y)] > 0\} = \emptyset,$$

$$I_-(x,y) = \{1, 2\}.$$

Thus, there exists no $\kappa \geq 0$ such that the relation (4) holds, and hence $f(x)$ is not a P_*-map. This example shows also that a quasimonotone map need not be necessarily a P_*-map.

We recall that a map is said to be quasimonotone (Ref. 17) if, for any x, y in R^n,

$$f(y)^T(x-y) > 0 \text{ implies } f(x)^T(x-y) \geq 0.$$

Clearly, a quasimonotone map, which corresponds to the case $\tau = 0$ in (2), must be a quasi-P_*-map. However, the converse is not true. See the next example. Thus, the class of quasi-P_*-maps is larger than the union of P_*-maps and quasimonotone maps.

Example 2.2. Let $x = (x_1, x_2)^T \in R^2$ and

$$f(x) = (-2x_1 + 2x_2, -x_1 + x_2)^T.$$

We show that $f(x)$ is a quasi-P_*-map, but it is not quasimonotone. Indeed, consider the function

$$g(x) = (g_1(x), g_2(x))^T = (x_1 - x_2, -x_1 + x_2)^T.$$

Since $g(x)$ is a monotone map and

$$f(x) = (-2g_1(x), g_2(x))^T$$

[i.e., $f(x)$ is a scaled map of $g(x)$], we deduce that $f(x)$ is a P_*-map (Proposition 3.1, Ref. 23), and hence is a quasi-P_*-map.

However, for any distinct points x, y in R^n, we have

$$f(y)^T(x-y) = (y_2 - y_1)[2(x_1 - y_1) + x_2 - y_2],$$

$$f(x)^T(x-y) = (x_2 - x_1)[2(x_1 - y_1) + x_2 - y_2].$$

Clearly, in general

$$f(y)^T(x-y) > 0 \text{ cannot imply } f(x)^T(x-y) \geq 0.$$

Therefore, the function f is not a quasimonotone map. This example shows also that a P_*-map need not be necessarily quasimonotone.

The following notion is useful for the study of solvability of nonlinear complementarity problems. Note that the notion has been extended to nonlinear variational inequality problems (Refs. 23, 31–32).

Definition 2.3. See Ref. 8. A set of points $\{x^r\}_{r>0} \subset R^n_+$ is an exceptional family of elements for the continuous function f if $\|x^r\| \to \infty$ as $r \to \infty$ and, for each $r > 0$, there exists a scalar $\mu_r > 0$ such that

$$f_i(x^r) = -\mu_r x_i^r, \qquad \text{if } x_i^r > 0, \tag{5}$$

$$f_i(x^r) \geq 0, \qquad \text{if } x_i^r = 0. \tag{6}$$

The following result is crucial for our subsequent discussion.

Lemma 2.1. See Ref. 8. For any continuous function $f: R^n \to R^n$, there exists either a solution to the corresponding complementarity problem or an exceptional family of elements for f.

According to the above lemma, to prove that a complementarity problem possesses a solution, it is sufficient to show that the function has no exceptional family of elements. We utilize this fact to prove the following existence result.

Theorem 2.1. Let f: $R^n \to R^n$ be a continuous quasi-P_*-mapping. If there exists a point $u \geq 0$ such that $f(u) > 0$, then the complementarity problem (1) has a solution.

Proof. Assume the contrary, that is, (1) has no solution. By Lemma 2.1, there must exist an exceptional family of elements for f, denoted by $\{x^r\}_{r>0}$. Then, there exists a positive sequence $\{\mu_r\}_{r>0}$ such that (5) and (6) hold: hence, for each $i \in \{1, 2, \ldots, n\}$, we have

$$(x_i^r - u_i)[f_i(x^r) - f_i(u)] = (u_i - x_i^r)[f_i(u) + \mu_r x_i^r], \qquad \text{if } x_i^r > 0, \tag{7}$$

$$(x_i^r - u_i)[f_i(x^r) - f_i(u)] \leq u_i f_i(u), \qquad \text{if } x_i^r = 0. \tag{8}$$

Therefore,

$$(x_i^r - u_i)[f_i(x^r) - f_i(u)] \leq u_i[f_i(u) + \mu_r x_i^r], \tag{9}$$

for all $i \in \{1, 2, \ldots, n\}$.

Since $\{x^r\} \subset R^n_+$ and $\|x^r\| \to \infty$, there must be an index i_0 such that $x_{i_0}^r \to \infty$ as $r \to \infty$; and since

$$\mu_r x_{i_0}^r + f_{i_0}(u) \geq f_{i_0}(u) > 0,$$

it follows from (7) that

$$(x_{i_0}^r - u_{i_0})[f_{i_0}(x^r) - f_{i_0}(u)] \to -\infty, \qquad \text{as } r \to \infty, \tag{10}$$

which implies the cardinality $|I_+(x^r, u)| \leq n-1$ for sufficiently large r. Two cases are possible:

Case 1. The sequence $\{\mu_r\}_{r>0}$ is bounded. We consider the following two subcases.

Subcase 1a. There exists some r_0 such that

$$I_+(x^r, u) = \emptyset, \qquad \text{for all } r \geq r_0.$$

In this case, we have

$$f(u)^T(x^r - u) = \sum_{i \in \{i: x_i^r > 0\}} f_i(u)(x_i^r - u) + \sum_{i \in \{x: x_i^r = 0\}} (-u_i f_i(u)).$$

The right-hand side tends to ∞, since $x_{i_0}^r \to \infty$; thus,

$$f(u)^T(x^r - u) > 0, \qquad \text{for all sufficiently large } r.$$

Subcase 1b. There exists a subsequence $\{r_j\} \to \infty$, $j = 1, 2, \ldots$, such that $I_+(x^{r_j}, u)$ is not empty for all j. We have known that

$$|I_+(x^{r_j}, u)| \leq n-1,$$

for all sufficently large j. There must be a subsequence of $\{x^{r_j}\}$, denoted also by $\{x^{r_j}\}$, such that, for some fixed $p \in \{1, \ldots, n\}$ and all j, we have

$$\begin{aligned}
&(x_p^{r_j} - u_p)[f_p(x^{r_j}) - f_p(u)] \\
&= \max_{1 \leq i \leq n} (x_i^{r_j} - u_i)[f_i(x^{r_j}) - f_i(u)] \\
&\geq (1/(n-1)) \sum_{i \in I_+(x^{r_j}, u)} (x_i^{r_j} - u_i)[f_i(x^{r_j}) - f_i(u)].
\end{aligned} \tag{11}$$

On the other hand, if $x_i^{r_j} > u_i$ for each i then (7) implies

$$(x_i^{r_j} - u_i)[f_i(x^{r_j}) - f_i(u)] < 0.$$

Since

$$(x_p^{r_j} - u_p)[f_p(x^{r_j}) - f_p(u)] \geq 0,$$

we deduce that

$$0 \leq x_p^{r_j} \leq u_p.$$

Thus by (9), we have

$$\begin{aligned}
(x_p^{r_j} - u_p)[f_p(x^{r_j}) - f_p(u)] &\leq u_p[f_p(u) + \mu_{r_j} x_p^{r_j}] \\
&\leq u_p[f_p(u) + \mu_{r_j} u_p].
\end{aligned} \tag{12}$$

Therefore, for sufficiently large j, by (11) and (12), we have

$$\begin{aligned} & f(u)^T(x^{r_j}-u)-(1+\tau)\sum_{i\in I_+(x^{r_j},u)}(x_i^{r_j}-u_i)[f_i(x^{r_j})-f_i(u)] \\ & \geq f(u)^T(x^{r_j}-u)-(1+\tau)(n-1)(x_p^{r_j}-u_p)[f_p(x^{r_j})-f_p(u)] \\ & =\sum_{i\in\{i:x_i^{r_j}>0\}} f_i(u)^T(x_i^{r_j}-u_i)-\sum_{i\in\{i:x_i^{r_j}=0\}} u_i f_i(u) \\ & \quad -(1+\tau)(n-1)[u_p f_p(u)+\mu_{r_j}u_p^2] \\ & >0. \end{aligned}$$

The last inequality follows from the facts that $x_{i_0}^{r_j}\to\infty$. $f(u)>0$, and $\{\mu_{r_j}\}$ is bounded.

Combining Subcases 1a and 1b, we have

$$f(u)^T(x^{r_j}-u)-(1+\tau)\sum_{i\in I_+(x^{r_j},u)}(x_i^{r_j}-u_i)[f_i(x^{r_j})-f_i(u)]>0, \tag{13}$$

for all sufficiently large j. Since f is a quasi-P_*-mapping, (13) implies

$$f(x^{r_j})^T(x^{r_j}-u)\geq 0, \qquad \text{for all sufficiently large } j. \tag{14}$$

However, by (5) and (6), we deduce that

$$\begin{aligned} & f(x^{r_j})^T(x^{r_j}-u) \\ & =\sum_{i\in\{i:x_i^{r_j}>0\}} -\mu_{r_j}[(x_i^{r_j})^2-u_i x_i^{r_j}]+\sum_{i\in\{i:x_i^{r_j}=0\}} -u_i f_i(x^{r_j}) \\ & \leq \sum_{i\in\{i:x_i^{r_j}>0\}} -\mu_{r_j}[(x_i^{r_j})^2-u_i x_i^{r_j}] \\ & <0, \end{aligned} \tag{15}$$

for all sufficiently large j. The last inequality follows from the fact that there exists at least one component, for instance $x_{i_0}^{r_j}$, such that $x_{i_0}^{r_j}\to\infty$. Clearly, (15) is in contradiction with (14).

Case 2. The sequence $\{\mu_r\}_{r>0}$ is unbounded. Without loss of generality, we assume that $\mu_r\to\infty$ as $r\to\infty$. Similar to Case 1, we have two possible subcases.

Subcase 2a. There exists some r_0 such that

$$I_+(x^r,u)=\varnothing, \qquad \text{for all } r\geq r_0.$$

In this case,

$$
\begin{aligned}
& f(x^r)^T(u-x^r) \\
&= \sum_{i\in\{i:x_i^{r_j}>0\}} -\mu_r x_i^r(u_i-x_i^r) + \sum_{i\in\{i:x_i^{r_j}=0\}} f_i(x^r)u_i \\
&\geq \mu_r \sum_{i\in\{i:x_i^{r_j}>0\}} [(x_i^r)^2-u_ix_i^r].
\end{aligned}
$$

Since at least one component x_i^r tends to ∞, the above inequality implies that

$$f(x^r)^T(u-x^r)>0,$$

for sufficiently large r.

Subcase 2b. There exists a subsequence $\{r_j\}\to\infty$, $j=1,2,\ldots,$ such that $I_+(x^{r_j},u)$ is not empty for all j. Similarly, there must be a subsequence of $\{x^{r_j}\}$, denoted also by $\{x^{r_j}\}$, such that, for some fixed index p, the inequalities (11) and (12) remain valid for this case. Thus, for sufficiently large j, we have

$$
\begin{aligned}
& f(x^{r_j})^T(u-x^{r_j}) - (1+\tau)\sum_{i\in I_+(x^{r_j},u)} (u_i-x_i^{r_j})[f_i(u)-f_i(x^{r_j})] \\
&\geq f(x^{r_j})^T(u-x^{r_j}) - (1+\tau)(n-1)\max_{1\leq i\leq n}(u_i-x_i^{r_j})[f_i(u)-f_i(x^{r_j})] \\
&\geq f(x^{r_j})^T(u-x^{r_j}) - (1+\tau)(n-1)(u_p-x_p^{r_j})[f_p(u)-f_p(x^{r_j})] \\
&\geq \sum_{i\in\{i:x_i^{r_j}>0\}} \mu_{r_j}[(x_i^{r_j})^2-u_ix_i^{r_j}] + \sum_{i\in\{i:x_i^{r_j}=0\}} f_i(x^{r_j})u_i \\
&\quad -(1+\tau)(n-1)[\mu_{r_j}u_p^2+u_pf_p(u)] \\
&\geq \mu_{r_j}\left\{-(1+\tau)(n-1)u_p^2 + \sum_{i\in\{i:x_i^{r_j}>0\}} [(x_i^{r_j})^2-u_ix_i^{r_j}]\right\} \\
&\quad -(1+\tau)(n-1)u_pf_p(u) \\
&>0.
\end{aligned}
$$

The last inequality above follows from the facts that $\mu_{r_j}\to\infty$ and $x_{i_0}^{r_j}\to\infty$.

Combining Subcases 2a and 2b, we have

$$f(x^{r_j})^T(u-x^{r_j}) - (1+\tau)\sum_{i\in I_+(x_i^{r_j},u)} (u_i-x_i^{r_j})[f_i(u)-f_i(x^{r_j})]>0,$$

for all sufficiently large j. Thus, by the quasi-P_*-property of the function f, we deduce that

$$f(u)^T(u-x^{r_j})\geq 0, \tag{16}$$

for all sufficiently large j. On the other hand, since $x_{i_0}^{r_j} \to \infty$, we have

$$f(u)^T(u - x^{r_j}) = \sum_{i \in \{i: x_i^{r_j} > 0\}} f_i(u)(u_i - x_i^{r_j}) + \sum_{i \in \{i: x_i^{r_j} = 0\}} f_i(u) u_i < 0,$$

for all sufficiently large j. This is in contradiction with (16). □

Since each quasimonotone function is a quasi-P_*-map, the following corollary follows from Theorem 2.1.

Corollary 2.1. See Ref. 7. If f is a quasimonotone map, and if there exists a point $u \geq 0$ such that $f(u) > 0$, then the complementarity problem (1) has a solution.

The above result extends the Karamardian result involving pseudomonotonicity to quasimonotone maps. The next result is also an immediate consequence of Theorem 2.1.

Corollary 2.2. See Ref. 23. If f is a nonlinear P_*-map, and if there exists a point $u \geq 0$ such that $f(u) > 0$, then the complementarity problem (1) has a solution.

Remark 2.1. Megiddo (Ref. 33) gave an example to show that a monotone complementarity problem with a feasible solution [i.e., there exists a point $u \geq 0$ such that $f(u) \geq 0$] does not possess a solution. Since the class of nonlinear quasi-P_* maps includes the class of monotone maps as a special case, the strict feasibility condition [that is, $u \geq 0$ and $f(u) > 0$] of Theorem 2.1 cannot be replaced by the feasibility condition to ensure the same result.

3. Nonlinear P(τ, α, β)-Map

In this section, we prove an existence theorem based on the notion of P(τ, α, β)-map, which is defined below.

Definition 3.1. A mapping $f: R^n \to R^n$ is said to be a P(τ, α, β)-map if there exist constants $\tau \geq 0$, $\alpha \geq 0$, and $0 \leq \beta < 1$ such that the following inequality holds:

$$(1+\tau) \max_{1 \leq i \leq n} (x_i - y_i)[f_i(x) - f_i(y)] + \min_{i \leq i \leq n} (x_i - y_i)[f_i(x) - f_i(y)]$$
$$\geq -\alpha \|x - y\|^\beta, \tag{17}$$

for any distinct points x, y in R^n.

Remark 3.1. If $\beta = 0$, then (17) reduces to

$$(1+\tau).\max_{1\le i\le n}(x_i-y_i)[f_i(x)-f_i(y)]+\min_{1\le i\le n}(x_i-y_i)[f_i(x)-f_i(y)]$$
$$\ge -\alpha. \tag{18}$$

We call a function satisfying (18) a P(τ, α)-map. P(τ, α)-maps are the same as P(τ, α, 0)-maps. Furthermore, if $a = 0$, then (18) reduces to

$$(1+\tau)\max_{1\le i\le n}(x_i-y_i)[f_i(x)-f_i(y)]+\min_{1\le i\le n}(x_i-y_i)[f_i(x)-f_i(y)]$$
$$\ge 0. \tag{19}$$

We will show that this class of functions coincides with the class of P_*-maps. Thus, (19) gives an equivalent definition of a P_*-map.

Proposition 3.1. The union of all P(τ, 0)-maps, $\tau \ge 0$, coincides with the class of P_*-maps.

Proof. For $n = 1$, the assertion of the statement is trivial. Consider the case $n \ge 2$. Assume that f is a P_*-mapping; i.e., there exists a scalar $\kappa \ge 0$ such that (4) holds. We can show that (19) holds for

$$\tau = (1-\kappa)(n-1)-1.$$

If $I_-(x, y) = \emptyset$, then (4) reduces to

$$(1+\kappa)(x-y)[f(x)-f(y)] \ge 0;$$

thus,

$$0 \le (x-y)[f(x)-f(y)]$$
$$\le (n-1)\max_{1\le i\le n}(x_i-y_i)[f_i(x)-f_i(y)]$$
$$+\min_{1\le i\le n}(x_i-y_i)[f_i(x)-f_i(y)]. \tag{20}$$

If $I_-(x, y) \ne \emptyset$, then

$$0 \le (1+\kappa)\sum_{i\in I_+(x,y)}(x_i-y_i)[f_i(x)-f_i(y)]$$
$$+\sum_{i\in I_-(x,y)}(x_i-y_i)[f_i(x)-f_i(y)]$$
$$\le (1+\kappa)(n-1)\max_{1\le i\le n}(x_i-y_i)[f_i(x)-f_i(y)]$$
$$+\min_{1\le i\le n}(x_i-y_i)[f_i(x)-f_i(y)]. \tag{21}$$

Setting

$$\tau = (1+\kappa)(n-1) - 1,$$

and combining (20) and (21), we obtain (19).

Conversely, assuming that (19) holds, we claim that f is a P_*-mapping with constant

$$\kappa = (n-1)(1+\tau) - 1.$$

Indeed, if

$$\min_{1 \le i \le n} (x_i - y_i)[f_i(x) - f_i(y)] < 0,$$

then for $q = |I_-(x, y)|$, we have

$$q \le n-1,$$

and for each $j \in I_-(x, y)$, we have

$$(x_j - y_j)[f_j(x) - f_j(y)] \ge \min_{1 \le i \le n} (x_i - y_i)[f_i(x) - f_i(y)].$$

By (19), for all $j \in I_-(x, y)$, we have

$$(1+\tau) \max_{1 \le i \le n} (x_i - y_i)[f_i(x) - f_i(y)] + (x_j - y_j)[f_j(x) - f_j(y)] \ge 0.$$

Adding these inequalities, we obtain

$$q(1+\tau) \max_{1 \le i \le n} (x_i - y_i)[f_i(x) - f_i(y)] + \sum_{i \in I_-(x,y)} (x_i - y_i)[f_i(x) - f_i(y)] \ge 0.$$

Noting that $q \le n-1$ and

$$\max_{1 \le i \le n} (x_i - y_i)[f_i(x) - f_i(y)] \le \sum_{i \in I_+(x,y)} (x_i - y_i)[f_i(x) - f_i(y)],$$

we have

$$(n-1)(1+\tau) \sum_{i \in I_+(x,y)} (x_i - y_i)[f_i(x) - f_i(y)] + \sum_{i \in I_+(x,y)} (x_i - y_i)[f_i(x) - f_i(y)] \ge 0.$$

If

$$\min_{1 \le i \le n} (x_i - y_i)[f_i(x) - f_i(y)] \ge 0,$$

the above inequality holds trivially. Thus, f is a P_*-mapping with constant

$$\kappa = (n-1)(1+\tau) - 1.$$

The proof is complete. □

From the above proof, we deduce that a P_*-map with $\kappa \geq 0$ must be a $P(\tau, 0)$-map with

$$\tau = (\kappa + 1)(n-1) - 1.$$

Since a monotone mapping is a P_*-map with $\kappa = 0$, we have the following result.

Corollary 3.1. Any monotone mapping is a $P(\tau, 0)$-map, where

$$\tau = \begin{cases} n-2, & \text{for } n \geq 2, \\ 0, & \text{for } n = 1. \end{cases}$$

Since an affine function $f = Mx + q$ is a P_*-map if and only if M is a P_*-matrix, from Proposition 3.1 we have the following equivalent definition of a P_*-matrix.

Definition 3.2. A matrix $M \in R^{n \times n}$ is said to be a P_*-matrix if there exists a scalar $\kappa \geq 0$ such that

$$(1+\kappa) \max_{1 \leq i \leq n} u_i(Mu)_i + \min_{1 \leq i \leq n} u_i(Mu)_i \geq 0, \qquad \text{for } u \in R^n.$$

The following theorem is the main result of this section.

Theorem 3.1. Let $f: R^n \to R^n$ be a continuous $P(\tau, \alpha, \beta)$-map. If there exists a point $u \geq 0$ such that $f(u) > 0$, then problem (1) has a solution.

Proof. Assume the contrary, that is, (1) has no solution. Then, by Lemma 2.1, there exists an exceptional family of elements for f, denoted by $\{x^r\}_{r>0}$. Therefore, there is a positive number $\mu_r > 0$ such that, for each x^r, (5) and (6) hold. Clearly, the relations (7)–(9) remain valid.

Since $\{x^r\} \subset R^n_+$ and $\|x^r\| \to \infty$, there exists a subsequence of $\{x^r\}$, with indices denoted by $\{r_j\}$, and some index i_0 such that

$$x^{r_j}_{i_0} - u_{i_0} = \max_{1 \leq i \leq n} (x^{r_j}_i - u_i) \to \infty, \qquad \text{as } j \to \infty. \tag{22}$$

Clearly,

$$x^{r_j}_{i_0} \to \infty, \qquad \text{as } j \to \infty.$$

On the other hand, there exists a subsequence of $\{x^{r_j}\}$, denoted also by $\{x^{r_j}\}$, such that, for some fixed index p and for all j, we have

$$(x_p^{r_j} - u_p)[f_p(x^{r_j}) - f_p(u)] = \max_{1 \le i \le n} (x_i^{r_j} - u_i)[f_i(x^{r_j}) - f_i(u)]. \tag{23}$$

By (9), we have

$$(x_p^{r_j} - u_p)[f_p(x^{r_j}) - f_p(u)] \le u_p[\mu_{r_j} x_p^{r_j} + f_p(u)]. \tag{24}$$

Therefore, by using (17), (23), (24), we deduce that

$$\begin{aligned}
&(x_{i_0}^{r_j} - u_{i_0})[f_{i_0}(x^{r_j}) - f_{i_0}(u)]\\
&\ge \min_{1 \le i \le n} (x_i^{r_j} - u_i)[f_i(x^{r_j}) - f_i(u)]\\
&\ge -(1+\tau) \max_{1 \le i \le n} (x_i^{r_j} - u_i)[f_i(x^{r_j}) - f_i(u)] - \alpha\|x^{r_j} - u\|^\beta\\
&= -(1+\tau)(x_p^{r_j} - u_p)[f_p(x^{r_j}) - f_p(u)] - \alpha\|x^{r_j} - u\|^\beta\\
&\ge -(1+\tau)u_p[\mu_{r_j} x_p^{r_j} + f_p(u)] - \alpha\|x^{r_j} - u\|^\beta.
\end{aligned} \tag{25}$$

Since $x_{i_0}^{r_j} \to \infty$, by (5) and (25) we obtain

$$\begin{aligned}
&(u_{i_0} - x_{i_0}^{r_j})[\mu_{r_j} x_{i_0}^{r_j} + f_{i_0}(u)]\\
&\ge -(1+\tau)u_p[f_p(u) + \mu_{r_j} x_p^{r_j}] - \alpha\|x^{r_j} - u\|^\beta.
\end{aligned}$$

Multiplying both sides by $1/(x_{i_0}^{r_j} - u_{i_0})$ and rearranging terms, we have

$$\begin{aligned}
&-\mu_{r_j}[x_{i_0}^{r_j} - (1+\tau)u_p x_p^{r_j}/(x_{i_0}^{r_j} - u_{i_0})]\\
&\ge f_{i_0}(u) - [(1+\tau)u_p f_p(u) + \alpha\|x^{r_j} - u\|^\beta]/(x_{i_0}^{r_j} - u_{i_0}).
\end{aligned} \tag{26}$$

By (22), for sufficiently large j we have

$$-1 \le -u_i/(x_{i_0}^{r_j} - u_{i_0}) \le (x_i^{r_j} - u_i)/(x_{i_0}^{r_j} - u_{i_0}) \le 1,$$

for all $i \in \{1, 2, \ldots, n\}$. Thus, for sufficiently large j, we have

$$\begin{aligned}
\|x^{r_j} - u\|^\beta/(x_{i_0}^{r_j} - u_{i_0}) &= [\|x^{r_j} - u\|^2/(x_{i_0}^{r_j} - u_{i_0})^{2/\beta}]^{\beta/2}\\
&= \left[\sum_{i=1}^n (x_i^{r_j} - u_i)^2/(x_{i_0}^{r_j} - u_{i_0})^{2/\beta}\right]^{\beta/2}\\
&= \left[\sum_{i=1}^n (x_i^{r_j} - u_i)^2/(x_{i_0}^{r_j} - u_{i_0})^2\right]^{\beta/2} / (x_{i_0}^{r_j} - u_{i_0})^{1-\beta}\\
&\le n^{\beta/2}/(x_{i_0}^{r_j} - u_{i_0})^{1-\beta}.
\end{aligned}$$

Therefore, it follows from $x_{i0}^{r_j} \to \infty$ that the right-hand side of inequality (26) tends to $f_{i_0}(u)$, a positive number. However, since $x_{i0}^{r_j} \to \infty$, we have

$$x_{i0}^{r_j} - (1+\tau)u_p x_p^{r_j}/(x_{i0}^{r_j} - u_{i_0}) \to \infty,$$

which implies that the left-hand side of (26) is negative. A contradiction ensues, and the proof is complete. □

Corollary 3.2. Under a strict feasibility condition, if f is a P(τ, α)-map, then problem (1) has a solution.

4. Case Where $G(x) = f(x) - f(0)$ Is a Positively Homogeneous Map

In this section, we establish a new existence condition for the complementarity problem where $G(x) := f(x) - f(0)$ is a positively homogeneous map.

Definition 4.1. The function $G(x) = f(x) - f(0)$ is exceptionally regular if there exists no $(x, \alpha) \in R_+^{n+1}$, where $x \in R^n$ and $\|x\| = 1$, such that

$$G_i(x)/x_i = -\alpha, \qquad \text{for } x_i > 0, \tag{27}$$

$$G_i(x) \geq 0. \qquad \text{for } x_i = 0. \tag{28}$$

The main result of this section is as follows.

Theorem 4.1. Let $G(x)$ be a positively homogeneous map, i.e., $G(\lambda x) = \lambda G(x)$ for all $\lambda \geq 0$. If $G(x)$ is exceptionally regular, then the nonlinear complementarity problem (1) has a solution.

Proof. If (1) has no solutions, by Lemma 2.1 there must exist an exceptional family of elements for f, denoted by $\{x^r\}$, which satisfying the properties

$$\{x^r\} \subset R_+^n \quad \text{and} \quad \|x^r\| \to \infty,$$

and there exists a positive sequence $\{\mu_r > 0\}$ such that (5) and (6) hold. Since G is positively homogeneous, we have

$$G(x^r) = \|x^r\| G(x^r/\|x^r\|),$$

that is,

$$f(x^r) = \|x^r\| [f(x^r/\|x^r\|) - f(0)] + f(0).$$

Without loss of generality, we assume that $x^r/\|x^r\| \to \hat{x}$; thus, $\hat{x} \in R^n_+$ and $\|\hat{x}\| = 1$. It follows from the above inequality that

$$\lim_{r\to\infty} f(x^r)/\|x^r\| = f(\hat{x}) - f(0) = G(\hat{x}). \tag{29}$$

We have only two cases.

Case 1. $\hat{x}_i > 0$. Then, $x_i^r > 0$ for sufficiently large r; thus, by (5) and (29), we have

$$\begin{aligned}\lim_{r\to+\infty} \mu_r &= \lim_{r\to+\infty} \mu_r (x_i^r/\|x^r\|)(\|x^r\|/x_i^r)\\ &= \lim_{r\to+\infty} (-f_i(x^r)/\|x^r\|)(\|x^r\|/x_i^r)\\ &= -G_i(\hat{x})/\hat{x}_i = \hat{\mu}.\end{aligned} \tag{30}$$

It follows from $\{\mu^r > 0\}$ that $\hat{\mu} \geq 0$.

Case 2. $\hat{x}_i = 0$. In this case, $x_i^r/\|x^r\| \to 0$; thus, by (30), we have $\mu_r x_i^r/\|x^r\| \to 0$. Therefore, by (29), (5), (6), we have

$$\begin{aligned}G_i(\hat{x}) &= \lim_{r\to+\infty} f_i(x^r)/\|x^r\|\\ &= \begin{cases}\lim_{r\to\infty} f_i(x^r)/\|x^r\|, & \text{if } x_i^r = 0,\\ \lim_{r\to+\infty} -\mu_r x_i^r/\|x^r\|, & \text{if } x_i^r > 0,\end{cases}\\ &\geq 0.\end{aligned}$$

So, $(\hat{x}, \hat{\mu})$ satisfies conditions (27) and (28). Therefore, G cannot be exceptionally regular. The desired result follows. □

Recall that a function f is said to be a E_0-function if the following inequality holds:

$$\max_{1\leq i\leq n} (x_i - y_i)[f_i(x) - f_i(y)] \geq 0,$$

for any vectors x, y in R^n such that $x - y \in R^n_+$ and $x \neq y$. It is easy to verify that the concept of E_0-function is a generalized version of the notion of semimonotone matrix (Ref. 1).

Corollary 4.1. Let $G(x)$ be positively homogeneous, and let f be a E_0-function satisfying the following condition: for any $x \in R^n_+$, $x \neq 0$, there exists at least an index i such that $x_i > 0$ and $f_i(x) \neq f_i(0)$. Then, problem (1) has a solution.

Proof. The above assumption implies that G is exceptionally regular; the result is straightforward from Theorem 4.1. □

5. Final Remarks

We introduced several new classes of functions such as quasi-P_*-maps, $P(\tau, \alpha, \beta)$-maps, and exceptionally regular functions. The existence theorems presented in the paper are based on the new classes of functions. The concept of exceptional family has played a key role in our analysis. Observe that, in a recent paper (Ref. 34), it is shown that, by using an appropriate notion of exceptional family of elements, one can study the feasibility of problem (1).

Let $K \subset R^n$ be a closed pointed convex cone, and let $f: R^n \to R^n$ be a continuous function.

Definition 5.1. See Ref. 34. Given a pair of real numbers (α, β) such that $0 \leq \alpha < \beta$, we say that the family of elements $\{x^r\}_{r>0} \subset R^n$ is an (α, β)-exceptional family of elements for f with respect to K if and only if $\lim_{r\to\infty} \|x^r\| \to \infty$ and, for each real number $r > 0$, there exists a $t_r \in]0, 1[$ such that the vector

$$u_r = (1/t_r - 1)x^r + (\beta - \alpha)f(x^r)$$

satisfies the following properties:

(i) $u_r \in K^*$,
(ii) $u_r^T[x^r - \alpha t_r f(x^r)] = 0$.

The importance of this notion is given by the following result proved in Ref. 34.

Theorem 5.1. Let (α, β) be a pair of real numbers such that $0 \leq \alpha < \beta$, and let $K \subset R^n$ be a closed pointed convex cone such that $K^* \subset K$ or $K^* = K$. Then, for any continuous function $f: R^n \to R^n$, either the complementarity problem associated to f and K is feasible, or there exists an (α, β)-exceptional family of elements for f with respect to K.

A natural problem is the following open problem: Is it possible to study the strict feasibility of complementarity problems by an appropriate notion of exceptional family of elements?

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