Combinatorics

Dan Hefetz, Richard Mycroft (lecturer)

These are notes of the lectures given by Richard Mycroft for the course Communication Theory and Combinatorics (3P16/4P16) at the University of Birmingham, on from January to March 2015.

These notes have been typed by students, who are solely responsible for the mistakes below (we claim no credit for the material). We will gladly correct any errors you find in the text.

150114

Notation

Some of the notation we use is fairly standard, but not all of it.

$\mathbb{N}$	the natural numbers $1, 2, \ldots$
$\eqqcolon$	equality by definition, e.g.: $\mathbb{N} \eqqcolon \{1, 2, 3, \ldots\}$.
$[n]$	the cannonical set with n elements: $[n] \eqqcolon \{1, 2, \ldots, n\}$.

1 Ramsey theory

Ramsey theory deals (morally) with existence arguments that some substructure is unavoidable in (large) combinatorial objects. One could say Ramsey-type questions seek order in (large) chaos.

A rudimentary example: let a and b be positive numbers, with $a+b=c$. Then $\max\{a,b\} \geq c/2$. (You may argue this is a fact of averages, and you would be right.) The next example is much less artificial.

Example In any group of six people, there are either 3 people who each know each other or three people who each do not know each other

Proof. Represent each person by a point, connect two points with a red line if those people know each other, and with a blue line if they don't. (Let us suppose we are allowed to "draw" in 3 dimensions, so that lines can only intersect at their endpoints.) Then there is a line (of some coulour) between each two points. Fix some person u. Then there are 5 lines leaving u, and at least 3 of those are of the same colour. Without loss of generality, we assume they are red. Then there are people a, b and c which are connected to u by red lines. If any of the lines ab, ac or bc is red, then with u we have formed a red triangle. Otherwise, all these three lines are blue, and they form a blue triangle, which proves the claim. □

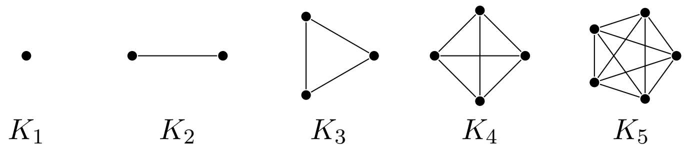
1.1 Definitions

A **graph** G is a pair of sets (V, E), where V is the **vertex set** and E is the **edge set**, and the elements of E are unnordered pairs $\{u, v\}$ of vertices (elements of V). We usually abbreviate the edge $\{u, v\}$ as uv (and this is the same edge as vu). If uv is an edge, we say that v is a **neighbour** of u. The **degree** $d(u)$ of a vertex u is $d(u) \eqqcolon |\{uv \colon uv \in E\}|$.

The **complete graph** K_n has n vertices, each pair of which forms an edge. See Figure 1.

Let C be a set of colours. A **colouring** of a set X with C iw a function $f \colon X \to C$, i.e., an assignment of a colour in C to each member of X. An edge-colouring of a graph with C is a colouring of the edge set (and no restrictions on which edge gets some colour in C). Given a colouring of the edges of a graphm, we shall use specialized notions of some concepts. For

Figure 1: Some complete graphs.



instance, **blue neighbour** of a vertex u is a neighbor of u connected to u by a blue edge, and the **blue degree** of u is the number of blue neighbors of u. (Similarly to other colours.)

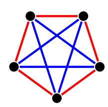
With this terminology we can restate our initial example as follows: in any edge colouring of K_6 with colours red ad blue, there is a monochromatic copy of K_3 (i.e., a subgraph isomorphic to K_3 with all edges red or all edge blue).

1.2 Ramsey theory for finite graphs

For ant $t \in \mathbb{N}$, the **diagonal Ramsey number** $R(t)$ is the smallest integer n such that whenever the edges of K_n are coloured red and blue there exists a moochromatic copy of K_t. For any $s, t \in \mathbb{N}$, the **Ramsey number** $R(s,t)$ is the smallest integer s such that whenever the edges of K_n are coloured red and blue there is either a red K_s or a blue K_t.

It is perhaps not immediately obvious that these numbers exist; but Ramsey's theorem, proved by Frank Plumpton Ramsey in 1930 states that they do.

Our earlier example shows that $R(3) \leq 6$. The following example shows that $R(3) > 5$.



150116

Some simple facts about the Ramsey numbers (exercises!).

1. $R(t) = R(t,t)$.
2. $R(s,t) = R(t,s)$ for any $s, t \in \mathbb{N}$.
3. $R(1,t) = 1$ for any $t \in \mathbb{N}$.
4. $R(2,t) = t$ for any $t \in \mathbb{N}$.

Theorem 1.1 (Ramsey theorem)**.** *$R(s,t)$ exists for any $s, t \in \mathbb{N}$. Moreover, if $s, t \geq 2$ then*

$$R(s,t) \leq R(s-1,t) + R(s,t-1).$$

Proof by Erdős and Szekeres, 1935. Proof by induction on $s+t$. Note that if $s \leq 2$ or $t \leq 2$ then $R(s,t)$ exists by the facts above. So we only need do consider $s, t \geq 3$.

Base case: if $s+t \leq 5$ then either $s \leq 2$ or $t \leq 2$, so $R(s,t)$ exists.

Inductive step: fix $s, t \geq 3$. By the inductive hypothesis, $R(i,j)$ exists whenever $i+j < s+t$. In particular, $R(s-1,t)$ and $R(s,t-1)$ both exist.

Let $n = R(s-1,t) + R(s,t-1)$. We will show that whenever the edges of K_n are coloured red and green there is a red K_s or a green K_t. This will prove that $R(s,t)$ exists and that $R(s,t) \leq n$.

Suppose that the edges of K_n are coloured red and green and choose a vertex v arbitrarily. Then v has degree $n-1$, and has either red degree $R(s-1,t)$ or green degree $R(s,t-1)$.

In the former case, the "red neighbors" of v have either a red copy of K_{s-1} or a green copy of K_t, which (together with v) demonstrate that K_n has the desired monochromatic complete graph. The latter case is symmetric. $\square$

Generated February 6, 2015, at 12 36.

Corollary 1.2. *For any $s, t \in \mathbb{N}$ we have $R(s,t) \le \binom{s+t-2}{s-1}$. In particular, $R(t,t) \le \binom{2t-2}{t-1} < 2^{2t-2} = 4^{t-1}$.*

Proof. The proof is by induction on $s+t$. First note that $R(s,t) \le \binom{s+t-2}{s-1}$. If $s \le 2$ or $t \le 2$ (see Problem sheet 1). This includes our base case when $s+t \le 5$. Now suppose $s, t \ge 3$. Then

$$R(s-1,t) \le \binom{s-1+t-2}{s-1} = \binom{s+t-3}{s-2} \quad \text{and} \quad R(s,t-1) \le \binom{s+t-1-2}{s-1} = \binom{s+t-3}{s-1}.$$

And then

$$R(s,t) \le R(s-1,t) + R(s,t-1) \le \binom{s+t-3}{s-2} + \binom{s+t-3}{s-1} = \binom{s+t-2}{s-1},$$

where the first inequality follows from Theorem 1.1 and the second from the induction hypothesis.

For the second part of the claim, note that $R(t,t) \le \binom{2t-2}{t-1} \le 2^{2t-2} = 4^{t-1}$. □

150121

Proposition 1.3. *For $t \in \mathbb{N}$ we have $R(t,t) \le 2^{2t-1}$.*

Proof. We begin with the following observation.

(1) *If n is even and the edges of K_n are coloured red and blue then any of its vertices has either at least $n/2$ red neighbours or at least $n/2$ blue neighbours (as it has $n-1 > 2(n/2-1)$ neighbours in total).*

Consider a red/blue edge colouring of $K_{2^{2t-1}}$. Then it suffices to find either a red K_t or a blue K_t. We do this as follows.

Choose a vertex x_1 arbitrarily. Then x_1 has at least 2^{2t-2} red neighbours or at least 2^{2t-2} blue neighbours, by (1). In the first case, mark x_1 "red" and let N_1 be a set of 2^{2k-2} red neighbours of x_1. In the second case, mark x_1 "blue" and let N_1 be a set of 2^{2t-2} blue neighbours of x_1.

And repeat the following.

At step i, choose a vertex x_i in N_{i-1} arbitrarily. Then x_i has at least 2^{2t-1-i} red neighbours or at least 2^{2t-1-i} blue neighbours, by (1). In the first case, mark x_i "red" and let N_i be a set of 2^{2k-1-i} red neighbours of x_i. In the second case, mark x_i "blue" and let N_i be a set of 2^{2t-1-i} blue neighbours of x_i.

As N_i has half the size of N_{i-1}, and we start with 2^{2t-1} vertices, we may continue for $2t-1$ steps to obtain $x_1, x_2, \ldots, x_{2t-1}$. These vertices have the property that for any $i < j$ the edge $x_i x_j$ has the colour we used to mark x_i.

Finally, since there are $2t-1$ marked vertices, some t of them have been marked with the same colour, and these form a monochromatic K_t. □

In particular, since we may mark the last vertex of the sequence with either colour, we can actually obtain a better bound (by a factor of 2).

1.3 Ramsey theory with more than 2 colours

For $k \in \mathbb{N}$ and $s_1, s_2, \ldots, s_k \in \mathbb{N}$ we define $R_k(s_1, s_2, \ldots, s_k)$ to be the smallest natural number n such that whenever the edges of $K_n$are coloured with colours $c_1, c_2, \ldots, c_k$ there exists a copy of K_{s_i} in colour c_i, for some i.

For instance, $R_2(s_1, s_2) = R(s_1, s_2)$ and $R1(s_1) = s_1 \ne R(s_1) = R_2(s_1, s_1)$.

Table 1: Known Ramsey numbers $R(s,t)$ for $s,t \geq 3$.

	3	4	5	6	7	8	9
3	6	9	14	18	23	28	36
4	9	18	25				

Theorem 1.4 (Ramsey, 1930)**.** *Let $k \in \mathbb{N}$, and $s_1, s_2, \ldots, s_k \in \mathbb{N}$. Then the Ramsey number $R_k(s_1, s_2, \ldots, s_k)$ exists.*

One possible proof mimicks the proof of existence of $R_2(s,t)$ by Erdős and Szekeres. Find another (exercise!). *[Hint: pretend you can't distinguish between two of the colours.]*

1.4 Small Ramsey numbers

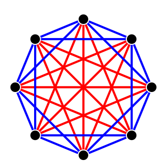
Not much is known about the Ramsey numbers $R(s,t)$ in general. For $t \in \mathbb{N}$, we know $R(1,t) = R(t,1) = 1$, and $R(2,t) = R(t,2) = r$. Otherwise, the known Ramsey numbers are in Table 1. There are, however, general bounds on the Ramsey numbers (their gap is large!). E.g., $43 \leq R(5,5) \leq 49$.

150123

Example We will see that $R(3,4) = 9$. To begin, we show that any colouring of K_9 has the sought monochromatic subgraph. To prove this, consider some red/green colouring of the edes of K_9, and consider 3 cases.
1) some vertex has at least 4 red neighbors;
2) some vertex has at most 2 red neighbors;
3) every vertex has exactly 3 red neighbors.
Which are left as exercises. The first case should be clear; check cases 2) *[hint: use $R(3,3)$]* and 3) *[hint: count the edges!]*.

To conclude, one must show that $R(3,4) > 8$ (see figure 2).

Figure 2: Colouring of the edges of $R(3,4)$ without red K_3 or blue K_4.



1.5 Lower bounds for Ramsey numbers

Our aim in this section is to give lower bounds on $R(t,t)$ in terms of t.

150128

Theorem 1.5 (Erdős, 1947)**.** *For $n, t \in \mathbb{N}$, if $\binom{n}{t} 2^{1-\binom{t}{2}} < 1$ then $R(t) > n$.*

Key observation: if X is a random variable whose value must be a nonnegative integer and such that $\mathbb{E}(X) < 1$ then $\Pr(X = 0) > 0$.

Proof. Fix such n and t and colour the edges of K_n randomly, with colour choices made independently of one another: an edge is coloured red with probability $1/2$ and is coloured blue with the same probability. Now let X be the number of monochromatic copies of K_t in the colouring. So X is a random variable whose value must be a nonnegative integer. Since there are $\binom{n}{t}$ sets of t vertices of K_n. For any such set T, the probability that T induces a red K_t is $2^{-\binom{t}{2}}$, and the same holds for the probability that T induces a blue K_t. So $\mathbb{E}(X) = \binom{n}{t}2^{1-\binom{t}{2}} < 1$ and therefore $\Pr(X = 0) > 0$. This, in turn, implies that there exists a colouring of the edges of K_n with no monochromatic K_t. □

Proof. There are $2^{\binom{n}{2}}$ possible ways to colour the edges of K_n with red and blue. A set T of t vertices monochromatic in $2^{\binom{n}{2}-\binom{t}{2}+1}$ of them. So the number of colourings without monochromatic K_t is at least

$$2^{\binom{n}{2}} - \binom{n}{t}2^{\binom{n}{2}-\binom{t}{2}+1} = 2^{\binom{n}{2}}\left(1 - \binom{n}{t}2^{1-\binom{t}{2}}\right) = 2^{\binom{n}{2}}\left(1 - \binom{n}{t}2^{1-\binom{t}{2}}\right) > 0.$$

□

Corollary 1.6. *For all $t \in \mathbb{N}$ with $t \geq 4$ we have $R(t,t) > 2^{\lfloor t/2 \rfloor}$.*

Proof. Let $n = 2^{\lfloor t/2 \rfloor}$. Then by 1.5 it suffices to prove that $\binom{n}{t}2^{1-\binom{t}{2}} < 1$. Indeed,

$$\binom{n}{t} \leq \frac{n^t}{t!} \cdot 2^{1-\frac{t(t-1)}{2}} \leq \frac{2^{t^2/2}}{t!} \cdot 2^{1-\frac{t^2-t}{2}} = \frac{2^{1+t/2}}{t!} \leq \frac{2^{t-2}}{t!} < 1,$$

where we use that $t - 2 \geq 1 + t/2$ whenever $t \geq 4$. □

150130

Definition 1.1. For graphs G, H, the **Ramsey number** $R(G,H)$ is the smallest n such that whenever the edges of K_n are coloured red and green there is either a red copy of G or a green copy of H.

Therefore $R(K_s, K_t) = R(s,t)$. Also, if G has s vertices and H has t vertices $R(G,H) \leq R(K_s, K_t)$, so $R(G,H)$ exists for any G, H.

1.6 Ramsey theory for numbers

We will briefly see Ramsey-type behaviour for

- systems of equations
- increasing and decreasing sequences
- arithmetic progressions

Systems of equations

Theorem 1.7 (Schur, 1916)**.** *For any $r \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that whenever $[n]$ is coloured with r colours there exists a monochromatic solution to $x + y = z$.*

Example Six numbers suffice for for $r = 2$ (check!). *[Hint: if 1 is coloured red, no consecutive numbers can be coloured red if one tries to avoid a red solution to the equation: $1, 2, 3, 4, 5, 6$].*

We note that colouring is the same as partitioning the numbers in classes (numbers in the same class receive the same colour).

Proof. Let $n \coloneqq R_r(3,3,\dots,3)$ and $c\colon : [n] \to [r]$ be an arbitrary colouring of $[n]$. Define a colouring c' of the edges of K_n with r colours by $c'(ij) \coloneqq c(|i-j|)$, where $c(m)$ is the colour of the integer m. By definition of n, this colouring has a monochromatic copy of K_3, say with vertices $i<j<k$. Then $c'(ij) = c'(ik) = c'(jk)$, so $c(j-i) = c(k-i) = c(k-j)$. Let $x = j-i$, $y = k-j$ and $z = k-i$: then $x+y=z$ is a monochromatic solution we sought. $\square$

Remark 1. With a bit more work one can get a monochromatic solution with x, y, z distinct.

We say that a a system of equations is **partition regular** if for all $r \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that whenever $[n]$ is r-coloured there is a monochromatic solution to the system of equations.

So Schur's theorem states that $x+y=z$ is partition regular. This is not the case in general; $x = 2y$, for instance, does not have that property. Rado's theorem characterizes precisely the partition regular systems of linear equations.

Increasing and decreasing sequences

Suppose we have a sequence $a_1, a_2, \dots, a_n$ of n distinct real numbers. Given s and t, how large must n be to guarantee either an increasing sequence of length s or a decreasing sequence of length t?

Note that $n = R(s,t)$ suffices, as we can colour the edges of K_n by colouring ij red if $a_i < a_j$ and green otherwise. By definition of n, there is either a red K_s or a green K_t in this colouring, and these are monotonic subsequences.

However, if $n = (s-1)(t-1)$, no such subsequence need exist. We will define a sequence $a_1, a_2, \dots, a_n$ of $(s-1)(t-1)$, such that none of its increasing subsequences has length s and none of its decreasing subsequences has length t.

The sequence is formed as follows: take $s-1$ decreasing sequences $A_1, A_2, \dots, A_{s-1}$ with $t-1$ numbers each, such that the numbers in A_j are greater than the numbers in A_i, whenever $1 \le i < j \le s-1$. Form the sequence by listing first the elements of A_1 (in decreasing order), followed by the elements of A_2 (in decreasing order), followed by… the elements of A_{s-1} (in decreasing order).

150202

Lemma 1.8 (Erdős and Szekeres, 1935)**.** *Let $n, s, t \in \mathbb{N}$, with $n > (s-1)(t-1)$. Any sequence of n distinct numbers contains either an increasing subsequence of at least s elements, or a decreasing subsequence of at least t elements.*

Proof. Let n, s, t be as above and let $a_1, \dots, a_n$ be a sequence of distinct numbers. Place the numbers in piles $p_1, p_2, \dots$ in the following manner ($\operatorname{top}(p_j)$ denotes the number at the top of pile p_j, or $-\infty$ if p_j is empty):

1. a_1 goes in p_1;
2. for $i > 1$, put a_i at the top of the pile pile p_j of least index such that a_i is greater than any element at p_j (that is $j \coloneqq \min\{k\colon \operatorname{top}(p_k) < a_i\}$).

Since $n > (s-1)(t-1)$, there is either a pile with more than s elements, or more than t piles, and the theorem follows (exercise!). $\square$

Proof. Observe that for any $i < j$, since either $a_i < a_j$ or $a_j < a_i$ we have that either

- the longest increasing subsequence starting with a_i is longer than the longest increasing subsequence starting with a_j, or
- the longest decreasing subsequence starting with a_i is longer than the longest decreasing subsequence starting with a_j.

Let

- $\ell_{\text{increasing}}(i)$ be the length of longest increasing subsequence starting at a_i, and

- $\ell_{\text{decreasing}}(i)$ be the length of longest decreasing subsequence starting at a_i.

Define a function $f\colon [n] \to \mathbb{N} \times \mathbb{N}$ by $f(i) \rightleftharpoons (\ell_{\text{increasing}}, \ell_{\text{decreasing}})$.

Our initial observation implies that f is injective, as we cannot have $f(i) = f(j)$ with $i \neq j$. Since $n > (s-1)(t-1) = |[s-1] \times [t-1]|$, and f is injective, there must be some $i \in [n]$ such that $f(i) \notin \{[s-1] \times [t-1]\}$. $\square$