Heuristic Optimisation

Problem sheet 1

Brief solutions

1. Mr. Smith and his wife invited four other couples for a party. When everyone arrived, some of the people in the room shook hands with some of the others. Of course, nobody shook hands with their spouse and nobody shook hands with the same person twice.

   After that Mr. Smith asked everyone how many times they shook someone's hand. He received different answers from everybody.

   How many times did Mrs. Smith shake someone's hand?

   Solution

   This seems a difficult problem. However, by choosing a good representation, it becomes quite easy. Represent the problem as a graph. The nodes of the graph are people given by their number of handshakes. The arcs represent the handshakes between people.

   - With 10 people in a room, the number of possible handshakes for one person can range between 0 and 8.
   - So, the 9 people gave as answers the numbers $0, 1, \ldots, 8$.
   - The person shaking hands 8 times must have shaken hands with everybody, except their spouse. As there is one person who did not shake hands at all, that person must be 8's spouse.
   - Person 7 could not shake hands with 0 or 1 (because 1 has already shaken hands with 8), so he/she must have shaken hands with all the others. 1 and 7 are a couple.
   - Person 6 could not shake hands with 0 or 1 or 2 (because 1 has already shaken hands with 8, and 2 has already shaken hands with 8 and 7), so he/she must have shaken hands with all the others. 2 and 6 are a couple.
   - ...
   - At the end, Mrs. Smith must be the person shaking 4 hands.

2. Discuss possible representations, size of search space, and possible evaluation functions for Rubik's cube. You may want to check a Rubik's cube application on your smart phone or an online virtual cube, such as

   `http://www.speedcubing.com/CubeSolver/CubeSolver.html`

   `http://www.mathplayground.com/rubikscube.html`

   and play a little bit with it before thinking on this.

   Solution

   Let's look at two possible representations:

   - Use the colour of the squares on each face of the cube in a given order.

     The middle square has fixed colour, which can give the order for listing the faces: white, blue, red, yellow, green, orange.

     For each face we have to record 8 squares (no need for the middle one), for each square there are 6 possible colours, and there are 6 faces, so the size of the search space is $(6^8)^6$.

     Note that if we consider the fact that we can actually have exactly 8 squares of each colour, we have a ***feasible region*** *of the search space of a much smaller size: we have 8 squares on each sides and 6 sides, so 48 squares alltogether. Each square can be theoretically in any position, which gives 48! possibilities. As there are 8 squares of the same colour and 6 colours, there will be* $(8!)^6$ *positions which look the same, so the size of the search space is* $48!/[(8!)^6]$. *In this counting we did not consider the fact that a square which is in a corner cannot appear on an edge and viceversa. Taking this constraint into account might reduce the size of the feasible region at the cost of extra calculations!*

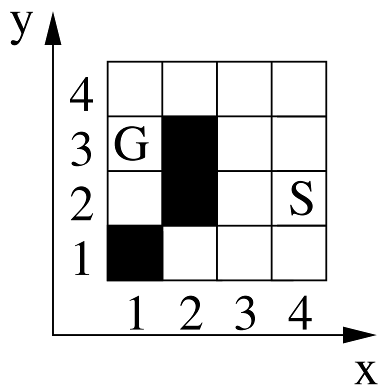
     Evaluation function: how many squares are of the right colour?

   - Use the small cubes. There are 20 cubes to record: 12 "edges" and 8 "corners".

     Each "corner" could be in any "corner" position, and it can have 3 possible orientations, so the number of possible options for corners is $8! \times 3^8$, similarly for "edges", $12! \times 2^{12}$, the size of the feasible region is $8! \times 3^8 \times 12! \times 2^{12}$.

     Evaluation function: number of small cubes in right position.

3. Consider the maze given below.



The black squares are obstacles. The objective is to get from the start node S to the goal node G, by moving horizontally and vertically. Use hill-climbing to solve this problem.

Solution

The evaluation function f is the Manhattan distance from G=(1,3) which should be minimised.

We start at S=(4,2) with an evaluation value of 4.

At next step we can go to (4,1), (4,3), or (3,2). We have $f(4,1) = 5$, $f(4,3) = f(3,2) = 3$. We have to choose the smallest value, that is, to go to either (4,3) or (3,2). Let us go to (4,3) with $f(4,3) = 3$.

At next step we can go to (4,4) or (3,3) (there is no need to go back to (4,2)). We have $f(4,4) = 4$ and $f(3,3) = 2$. We have to choose the smallest value, that is, to go to $(3,3)$ with $f(3,3) = 2$.

At next step we can go to (3,2) or (3,4). We have $f(3,2) = f(3,4) = 3$. But this value is worse than $f(3,3) = 2$. Hence the algorithm finishes at (3,3) with a Manhattan distance of 2 from the goal.

4. Consider the function $f(x,y) = x^2 + xy + y^2 + 3x + 4y$ defined on points of integer coordinates in the box $[-10,10] \times [-10,10]$. Use the hill-climbing algorithm to find the function's minimum value on this box.

First define the search space, then the neighbourhood. Try to find the minimum using the starting point $(x,y) = (0,0)$.

Solution

The search space is the set

$$S = \{-10, -9, \dots, -1, 0, 1, \dots, 9, 10\} \times \{-10, -9, \dots, -1, 0, 1, \dots, 9, 10\}.$$

Define the neighbourhood of a point (x,y) by

$$N(x,y) = \{(z,t) \in S : \sqrt{(z-x)^2 + (t-y)^2} \leq 1\}.$$

The neighbourhood of $(0,0)$ is

$$N(0,0) = \{(0,0), (-1,0), (1,0), (0,-1), (0,1)\}.$$

We have $f(0,0) = 0$, $f(-1,0) = -2$, $f(1,0) = 4$, $f(0,-1) = -3$ and $f(0,1) = 5$. Since we are looking for the minimum, hill-climbing will select $(x,y) = (0,-1)$. The neighbourhood of $(0,-1)$ is

$$N(0,-1) = \{(0,-1), (-1,-1), (1,-1), (0,-2), (0,0)\}.$$

We have $f(0,-1) = -3$, $f(-1,-1) = -4$, $f(1,-1) = 0$, $f(0,-2) = -4$ and $f(0,0) = 0$. Since we are looking for the minimum, hill-climbing will select $(x,y) = (-1,-1)$ or $(x,y) = (0,-2)$. Suppose the algorithm has selected $(x,y) = (-1,-1)$. The neighbourhood of $(-1,-1)$ is

$$N(-1,-1) = \{(-1,-1), (-2,-1), (0,-1), (-1,-2), (-1,0)\}.$$

We have $f(-1,-1) = -4$, $f(-2,-1) = -3$, $f(0,-1) = -3$, $f(-1,-2) = -4$ and $f(-1,0) = -2$, so there is no further improvement in the value of f. Since we are looking for the minimum, hill-climbing will stop and give the answer $(x,y) = (-1,-1)$, with the minimum $f(-1,-1) = -4$.

5. Consider the following 6 dimensional SAT problem.

$$F(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1 \bigvee \overline{x}_2 \bigvee x_3) \bigwedge (\overline{x}_2 \bigvee \overline{x}_6) \bigwedge (x_3 \bigvee x_6))$$

(a) Consider the incomplete solutions given by the binary strings of length 6 that start with 01. This property determines a subspace of the search space. Does there exist a solution of the problem which shares this property? If yes, then what should the value of x_3 and x_6 be for such a solution?

(b) Consider the incomplete solutions given by the binary strings **0**1 (i.e, binary strings of length 6 with $x_3 = 0$, $x_6 = 1$). This property determines a subspace of the search space. Does there exist a solution of the problem which shares this property? If yes, then what should the value of x_2 be for such a solution?

(c) Consider the incomplete solutions given by the binary strings *1*011. This property determines a subspace of the search space. Determine all solutions of the problem which share this property.

Solution

(a) In this subspace: The first clause is $0 \bigvee 0 \bigvee x_3$ and the second clause is $0 \bigvee \overline{x}_6$. In order to have a solution, the value of the first and second clause must be 1, hence $x_3 = 1$ and $x_6 = 0$, respectively. Since $x_3 = 1$, the value of the third clause is 1. Thus, we got a solution which shares the property of the subspace.

(b) In this subspace: The value of the third clause is 1, because $x_6 = 1$. The second clause is $\overline{x}_2 \bigvee 0$. In order to have a solution, the value of the second clause must be 1, hence $x_2 = 0$. Since $x_2 = 0$, the value of the first clause is 1. Thus, we got a solution which shares the property of the subspace.

(c) In this subspace: The second clause is false, because $x_2 = 1$ and $x_6 = 1$. Hence, there is no solution which shares the property of the subspace.