Heuristic Optimisation

Part 7: Properties of A*

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The properties to be studied

- Completeness
- Admissibility
- Monotonicity
- Informedness

Completeness and optimality

- Greedy search:
  minimizes the estimated cost to the goal, $h(n)$

  $\Rightarrow$ neither optimal nor complete,
  but low search cost

- A*:
  minimizes $f(n) = g(n) + h(n)$

  $\Rightarrow$ optimal and complete with a restriction on h

Properties of *h*

- If *h* is a perfect estimator of the distance from the current node to the goal node, A* will never leave the optimal path.

- The better *h* estimates the real distance, the closer A* is to the ”direct” path.

- If *h* never overestimates the real distance, A* is guaranteed to find the optimal solution.

- If *h* may overestimate the real distance, the optimal path can be found only if ...

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- If *h* never overestimates the real distance, A* is guaranteed to find the optimal solution.

- If *h* may overestimate the real distance, the optimal path can be found only if ... all paths in the search graph longer than the optimal solution are expanded.

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So A* will find the optimal solution.

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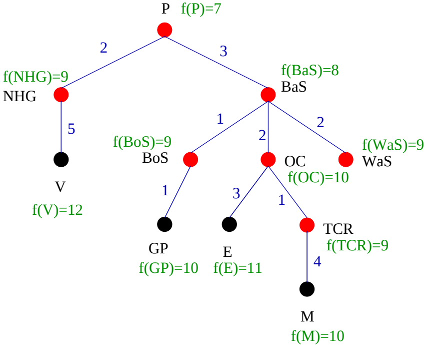
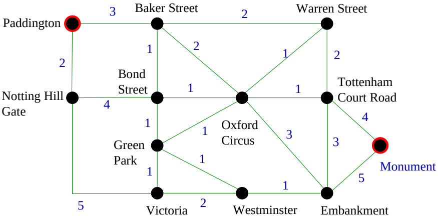
Example admissible heuristic: straight-line distance.

A* example





OPEN=[M,GP,V,E]

CLOSED=[P,BaS,NHG,BoS,WaS,OC,TCR]

h-values:

P	BaS	WaS	NHG	BoS	OC
7	5	4	7	5	5

TCR	GP	V	We	E	M
3	5	5	4	3	0

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- A* expands no nodes with $f(n) > f^*$

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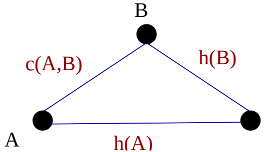
A* expands the nodes in order of increasing f, so it will eventually reach the goal state if the number of nodes with $f(n) < f^*$ is finite.

Monotonicity (Consistency)

If along any path in the search tree of A* the *f*-cost never decreases then the heuristic is said to be monotonic.

$$f(A) \leq f(B) \iff h(A) \leq h(B) + c(A, B)$$

(Can you prove this?)



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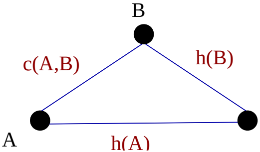
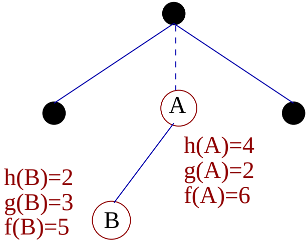


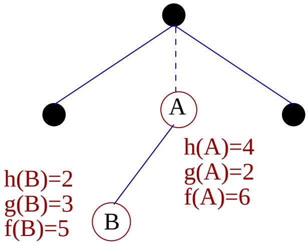
Can we transform a nonmonotonic heuristic into a monotonic heuristic?

Restoring monotonicity



$\Rightarrow$ f is a nonmonotonic heuristic.

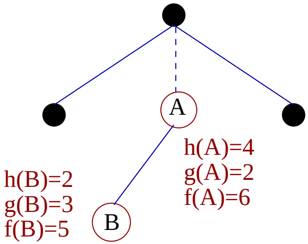
Restoring monotonicity



⇒ f is a nonmonotonic heuristic.

But any path through B passes through its parent A, too.

Restoring monotonicity



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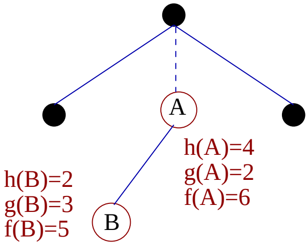

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$f(B) = \max(f(A), g(B) + h(B))$ is never decreasing along any path from the root if h is admissible.

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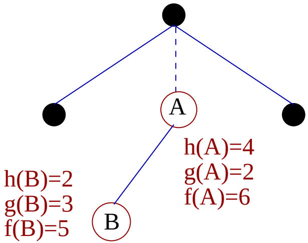

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This is called the pathmax equation.

Efficiency of A*

A* is optimally efficient for any consistent *h* function:

No other optimal algorithm is guaranteed to expand fewer nodes than A* (except possibly nodes with $f(n) = f^*$)

Monotonicity theorem

If a monotonic heuristic *h* is used, when A* expands a node *n*, it has already found an optimal path to *n*.

Important: in this case, the current path to a node will never be better than some previously found path.

Consequences:

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- No modifications will be made after a node is expanded. (step 2.(b)ii. not needed)
- Searching a graph will be the same as searching a tree.

Informedness

If two versions of A*, A_1^* and A_2^*, differ only in $h_1 < h_2$ for all nodes that are not goal nodes then A_2^* is more informed than A_1^*.

Example – the 8-puzzle

Compare A* based on the Manhattan distance and A* based on the number of tiles in wrong position.

Which one is more informed?

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If A_2^* is more informed than A_1^*, then at the termination of their searches on any graph having a path from the start node to a goal node, every node expanded by A_2^* is also expanded by A_1^*.

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- A_1^* expands at least as many nodes as A_2^*.
- A more informed algorithm is more efficient.
- The most efficient heuristic is the perfect estimator!

Recommended reading

N. Nilsson: Artificial Intelligence, A New Synthesis
Section 9.2

S. Russell and P. Norvig: Artificial Intelligence, A Modern Approach
Section 4.1

E. Rich and K. Knight: Artificial Intelligence
Section 3.3