The cone of $Z$-transformations on Lorentz cone

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THE CONE OF \(Z\)-TRANSFORMATIONS ON THE LORENTZ CONE\(^*\)

SÁNDOR ZOLTÁN NÉMETH\(^{†}\) AND MUDDAPPA SEETHARAMA GOWDA\(^{‡}\)

Abstract. In this paper, the structural properties of the cone of \(Z\)-transformations on the Lorentz cone are described in terms of the semidefinite cone and copositive/completely positive cones induced by the Lorentz cone and its boundary. In particular, its dual is described as a slice of the semidefinite cone as well as a slice of the completely positive cone of the Lorentz cone. This provides an example of an instance where a conic linear program on a completely positive cone is reduced to a problem on the semidefinite cone.

Key words. \(Z\)-transformation, Lorentz cone, Semidefinite cone, Copositive cone, Completely positive cone.

AMS subject classifications. 90C33, 15A48.

1. Introduction. Given a proper cone \(\mathcal{K}\) in a finite dimensional real Hilbert space \((\mathcal{H}, \langle \cdot, \cdot \rangle)\), a linear transformation \(A : \mathcal{H} \to \mathcal{H}\) is said to be a \(Z\)-transformation on \(\mathcal{K}\) if

\[
[ x \in \mathcal{K}, y \in \mathcal{K}^*, \text{ and } \langle x, y \rangle = 0 ] \Rightarrow \langle Ax, y \rangle \leq 0,
\]

where \(\mathcal{K}^*\) denotes the dual of \(\mathcal{K}\) in \(\mathcal{H}\). Such transformations appear in various areas including economics, dynamical systems, optimization, see e.g., [2, 3, 9, 12] and the references therein. When \(\mathcal{H}\) is \(\mathbb{R}^n\) and \(\mathcal{K}\) is the nonnegative orthant, \(Z\)-transformations become \(Z\)-matrices, which are square matrices with nonpositive off-diagonal entries.

The set \(Z(\mathcal{K})\) of all \(Z\)-transformations on \(\mathcal{K}\) is a closed convex cone in the space of all (bounded) linear transformations on \(\mathcal{H}\). Given their appearance and importance in various areas, describing/characterizing elements of \(Z(\mathcal{K})\) and its interior, boundary, dual, etc., is of interest. An early result of Schneider and Vidyasagar [16] asserts that \(A\) is a \(Z\)-transformation on \(\mathcal{K}\) if and only if \(e^{-tA}(\mathcal{K}) \subseteq \mathcal{K}\) for all \(t \geq 0\); consequently,

\[
Z(\mathcal{K}) = \mathbb{R}I - \pi(\mathcal{K}),
\]

where \(\pi(\mathcal{K})\) denotes the set of all linear transformations that leave \(\mathcal{K}\) invariant, \(I\) denotes the identity transformation, and overline denotes the closure. To see another description of \(Z(\mathcal{K})\), let \(\text{LL}(\mathcal{K}) := Z(\mathcal{K}) \cap -Z(\mathcal{K})\) denote the lineality space of \(Z(\mathcal{K})\), the elements of which are called Lyapunov-like transformations. Then the inclusions

\[
\mathbb{R}I - \pi(\mathcal{K}) \subseteq \text{LL}(\mathcal{K}) - \pi(\mathcal{K}) \subseteq Z(\mathcal{K}) = \mathbb{R}I - \pi(\mathcal{K})
\]

imply that

\[
Z(\mathcal{K}) = \overline{\text{LL}(\mathcal{K}) - \pi(\mathcal{K})}.
\]

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As the cones $\mathcal{Z}(K)$, $\pi(K)$, and $LL(K)$ are generally difficult to describe for an arbitrary proper cone $K$, we consider special cases. When $K$ is the nonnegative orthant, $\mathcal{Z}(K)$ consists of square matrices with nonpositive off-diagonal entries, $\pi(K)$ consists of nonnegative matrices, and $LL(K)$ consists of diagonal matrices. In this paper, we focus on the Lorentz cone (also called the ice-cream cone or the second-order cone as it is induced by the 2-norm) in the Euclidean space $\mathbb{R}^n$, $n > 1$, defined by:

\begin{equation}
\mathcal{L} := \{(t, u)^\top : t \in \mathbb{R}, u \in \mathbb{R}^{n-1}, t \geq ||u||\}.
\end{equation}

This, being an example of a symmetric cone, appears prominently in conic optimization [1]. For this cone, Stern and Wolkowicz [17] have shown that $A \in \mathcal{Z}(\mathcal{L})$ if and only if for some real number $\gamma$, the matrix $\gamma J - (JA + A^\top J)$ is positive semidefinite, where $J$ is the diagonal matrix $\text{diag}(1, -1, -1, \ldots, -1)$. Another result of Stern and Wolkowicz ([18], Theorem 4.2) asserts that

\begin{equation}
\mathcal{Z}(\mathcal{L}) = LL(\mathcal{L}) - \pi(\mathcal{L}).
\end{equation}

(Going in the reverse direction, in a recent paper, Kuzma et al., [13] have shown that for an irreducible symmetric cone $K$, the equality $\mathcal{Z}(K) = LL(K) - \pi(K)$ holds only when $K$ is isomorphic to $\mathcal{L}$.) Characterizations of $\pi(\mathcal{L})$ and $LL(\mathcal{L})$ appear, respectively, in [14] and [20].

In this paper, we describe $\mathcal{Z}(\mathcal{L})$ and its interior, boundary, and dual in terms of the semidefinite cone and the so-called copositive and completely positive cones induced by $\mathcal{L}$ (or its boundary $\partial(\mathcal{L})$), see below for the definitions. In particular, we describe the dual of $\mathcal{Z}(\mathcal{L})$ as a slice of the semidefinite cone and also of the completely positive cone of $\mathcal{L}$. This provides an example of an instance where a conic linear optimization problem over a completely positive cone is reduced to a semidefinite problem. To elaborate, consider $\mathbb{R}^n$, the Euclidean $n$-space of (column) vectors with the usual inner product, $\mathbb{R}^{n \times n}$, the space of all real $n \times n$ matrices with the inner product $\langle X, Y \rangle = \text{tr}(X^\top Y)$, and $S^n$, the subspace of all real $n \times n$ symmetric matrices in $\mathbb{R}^{n \times n}$. Corresponding to a closed cone $\mathcal{C}$ (which is not necessarily convex) in $\mathbb{R}^n$, let

$$
\mathcal{E}_\mathcal{C} := \text{copos}(\mathcal{C}) := \{A \in S^n : x^\top Ax \geq 0 \text{ for all } x \in \mathcal{C}\}
$$

denote the copositive cone of $\mathcal{C}$ and

$$
\mathcal{K}_\mathcal{C} := \text{compos}(\mathcal{C}) := \left\{\sum_{u \in U} uu^\top : U \text{ is a finite subset of } \mathcal{C}\right\}
$$

denote the completely positive cone of $\mathcal{C}$. When $\mathcal{C} = \mathbb{R}^n$, these two cones coincide with the semidefinite cone $S^n_+$ (consisting of all real $n \times n$ symmetric positive semidefinite matrices); when $\mathcal{C} = \mathbb{R}^n_+$, these reduce, respectively, to the (standard) copositive cone and completely positive cone. All these cones appear prominently in conic optimization. A result of Burer [5] (see also, [4, 7]) says that any nonconvex quadratic programming problem over a closed cone with additional linear and binary constraints can be reformulated as a linear program over a suitable completely positive cone. For this and other reasons, there is a strong interest in understanding copositive and completely positive cones. For the closed convex cones $\mathcal{E}_\mathcal{C}$ and $\mathcal{K}_\mathcal{C}$, various structural properties (such as the interior, boundary) as well as duality, irreducibility, and homogeneity properties, have been investigated in the literature, see for example, [19, 6, 8, 11]. Taking $\mathcal{C}$ to be one of $\mathbb{R}^n$, $\mathcal{L}$, or $\partial(\mathcal{L})$, we show that

\begin{equation}
\mathcal{Z}(\mathcal{L})^* = \{B \in \mathbb{R}^{n \times n} : \langle B, I \rangle = 0, -JB \in \mathcal{K}_\mathcal{C}\}
\end{equation}

and deduce the equality of slices

\begin{equation}
\{X \in \mathbb{R}^{n \times n} : \langle J, X \rangle = 0, X \in S^n_+\} = \{X \in \mathbb{R}^{n \times n} : \langle J, X \rangle = 0, X \in \mathcal{K}_\mathcal{C}\}.
\end{equation}
2. Preliminaries. In a (finite dimensional real) Hilbert space $\mathcal{H} = (\mathcal{H}, \langle \cdot, \cdot \rangle)$, a nonempty set $\mathcal{C}$ is said to be a closed cone if it closed and $tx \in \mathcal{C}$ whenever $x \in \mathcal{C}$ and $t \geq 0$ in $\mathbb{R}$. Throughout this paper, $\mathcal{C}$ denotes a closed cone. A nonempty set $\mathcal{K}$ is said to be a closed convex cone if it is a closed cone which is also convex. Such a cone is said to be proper if $\mathcal{K} \cap -\mathcal{K} = \{0\}$ and has nonempty interior. Corresponding to a closed convex cone $\mathcal{K}$, we define its dual in $\mathcal{H}$ as the set

$$\mathcal{K}^* = \{x \in \mathcal{H} : \langle x, y \rangle \geq 0, \forall y \in \mathcal{K}\}.$$ 

We say that a linear transformation $A : \mathcal{H} \to \mathcal{H}$ is copositive on $\mathcal{K}$ if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{K}$. We also let $\pi(K) = \{A : A(K) \subseteq K\}$, where $A$ denotes a linear transformation on $\mathcal{H}$. For a set $S$ in $\mathcal{H}$, we denote the closure, interior, and the boundary by $\overline{S}$, $S^0$, and $\partial(S)$, respectively.

We will be considering closed convex cones in the space $\mathcal{H} = \mathbb{R}^n$ which carries the usual inner product and in the space $\mathbb{R}^{n \times n}$ which carries the inner product $\langle X, Y \rangle := \text{tr}(X^T Y)$, where the trace of a square matrix is the sum of its diagonal entries. In $\mathbb{R}^{n \times n}$, $\mathcal{S}^n$ denotes the subspace of all symmetric matrices and $\mathcal{A}^n$ denotes the subspace of all skew-symmetric matrices. We note that $\mathbb{R}^{n \times n}$ is the orthogonal direct sum of $\mathcal{S}^n$ and $\mathcal{A}^n$.

We recall some (easily verifiable) properties of the Lorentz cone $\mathcal{L}$ given by (1.2). $\mathcal{L}$ is a self-dual cone in $\mathbb{R}^n$, that is, $\mathcal{L}^* = \mathcal{L}$; its interior and boundary are given, respectively, by

$$\mathcal{L}^0 = \{(t, u)^T : t > \|u\|\},$$

$$\partial(\mathcal{L}) = \{(t, u)^T : t = \|u\|\} = \{\alpha (1, u)^T : \alpha \geq 0, \|u\| = 1\}.$$ 

We also have

$$[0 \neq x, y \in \mathcal{L}, \langle x, y \rangle = 0] \Rightarrow x = \alpha (1, u)^T \text{ and } y = \beta (1, -u)^T,$$

for some $\alpha, \beta > 0$ and $\|u\| = 1$.

For a closed cone $\mathcal{C}$ in $\mathbb{R}^n$, we consider the copositive cone $\mathcal{E}_C$ and the completely positive cone $\mathcal{K}_C$ (defined in the introduction). Note that these are cones of symmetric matrices.

In the Hilbert space $\mathcal{S}^n$ (which carries the inner product from $\mathbb{R}^{n \times n}$), the following hold.

1. $\mathcal{K}_C$ is the dual cone of $\mathcal{E}_C$ [19].
2. When $\mathcal{C} = \mathbb{R}^n$, both $\mathcal{E}_C$ and $\mathcal{K}_C$ are proper cones ([10], Proposition 2.2). In particular, this holds when $\mathcal{C}$ is one of $\mathbb{R}^n$, $\mathcal{L}$, or $\partial(\mathcal{L})$.
3. We have $\mathcal{S}^n_+ = \mathcal{E}_{\mathbb{R}^n} \subseteq \mathcal{E}_{\mathcal{L}} \subseteq \mathcal{E}_{\partial(\mathcal{L})}$, or equivalently, $\mathcal{K}_{\partial(\mathcal{L})} \subseteq \mathcal{K}_{\mathcal{L}} \subseteq \mathcal{K}_{\mathbb{R}^n} = \mathcal{S}^n_+$.

3. Main results. In this section, we provide a closure-free description of $\mathcal{Z}(\mathcal{L})$ and, additionally, describe the dual, interior, and the boundary of $\mathcal{Z}(\mathcal{L})$. We recall that $J = \text{diag}(1, -1, -1, \ldots, -1)$ and $\mathcal{A}^n$ denotes the set of all skew-symmetric matrices in $\mathbb{R}^{n \times n}$.

**Theorem 3.1.** Let $\mathcal{C}$ denote one of $\mathbb{R}^n$, $\mathcal{L}$, or $\partial(\mathcal{L})$. Then,

$$\mathcal{Z}(\mathcal{L}) = \mathbb{R} I - J(\mathcal{E}_C + \mathcal{A}^n).$$
Proof. Let $A \in Z(\mathcal{L})$. From the result of Stern and Wolkowicz [17] mentioned in the introduction, we have

$$2\gamma J - (JA + A^\top J) = 2P$$

for some $\gamma \in \mathbb{R}$ and $P \in S^+_n$. Hence, $JA + (JA)^\top = 2(\gamma J - P)$, which implies

$$(3.2) \quad 2JA = JA + (JA)^\top - [(JA)^\top - JA] = 2(\gamma J - P) - 2Q,$$

where $2Q = (JA)^\top - JA$ is skew-symmetric. Since $J^2 = I$, this leads to

$$A = \gamma I - J(P + Q),$$

where $P \in S^+_n$ and $Q \in A^n$. As $S^+_n \subset \mathcal{E}_\mathcal{L} \subset \mathcal{E}_{\partial(\mathcal{L})}$, this proves that

$$(3.3) \quad Z(\mathcal{L}) \subseteq \mathbb{R}I - J(S^+_n + A^n) \subseteq \mathbb{R}I - J(\mathcal{E}_\mathcal{L} + A^n) \subseteq \mathbb{R}I - J(\mathcal{E}_{\partial(\mathcal{L})} + A^n).$$

Now, to see the reverse inclusions, suppose $A = \gamma I - J(P + Q)$ for some $\gamma \in \mathbb{R}$, $P \in \mathcal{E}_{\partial(\mathcal{L})}$, and $Q$ skew-symmetric. Let $0 \neq x, y \in \mathcal{L}$ with $(x, y) = 0$. By (2.1), and $y$ are in $\partial(\mathcal{L})$, and $Jy$ is a positive multiple of $x$. Hence, $(Px, Jy) \geq 0$ as $P \in \mathcal{E}_{\partial(\mathcal{L})}$ and $(Qx, Jy) = 0$ as $Q$ is skew-symmetric. Thus,

$$\langle Ax, y \rangle = \gamma \langle x, y \rangle - (JPx, y) + (JQx, y) = -(Px, Jy) + (Qx, Jy) \leq 0.$$

This shows that $A \in Z(\mathcal{L})$ and so, inclusions in (3.3) turn into equalities. Thus, we have (3.1). \qed

Remarks. From the above theorem, we have

$$\mathbb{R}I - J(S^+_n + A^n) = \mathbb{R}I - J(\mathcal{E}_\mathcal{L} + A^n) = \mathbb{R}I - J(\mathcal{E}_{\partial(\mathcal{L})} + A^n).$$

Multiplying throughout by $J$ and noting $-A^n = A^n$, we get the equality of sets

$$(\mathbb{R}J - S^+_n) + A^n = (\mathbb{R}J - \mathcal{E}_\mathcal{L}) + A^n = (\mathbb{R}J - \mathcal{E}_{\partial(\mathcal{L})}) + A^n,$$

where each set is a sum of $A^n$ and a subset of $S^n$. Since $\mathbb{R}^{n \times n} = S^n + A^n$ is an (orthogonal) direct sum decomposition, we see that

$$(3.4) \quad \mathbb{R}J - S^+_n = \mathbb{R}J - \mathcal{E}_\mathcal{L} = \mathbb{R}J - \mathcal{E}_{\partial(\mathcal{L})}.$$

These equalities can also be established via different arguments. A result of Loewy and Schneider [14] asserts that $A$ symmetric matrix $X$ is copositive on $\mathcal{L}$ if and only if there exists $\mu \geq 0$ such that $X - \mu J \in S^+_n$. (This is essentially a consequence of the so-called S-Lemma [15]: If $A$ and $B$ are two symmetric matrices with $\langle Ax_0, x_0 \rangle > 0$ for some $x_0$ and $\langle Ax, x \rangle \geq 0 \Rightarrow \langle Bx, x \rangle \geq 0$, then there exists $\mu \geq 0$ such that $B - \mu A$ is positive semidefinite.) This result gives the equality

$$\mathcal{E}_\mathcal{L} = S^+_n + \mathbb{R}_+ J,$$

and consequently, $\mathbb{R}J - S^+_n = \mathbb{R}J - \mathcal{E}_\mathcal{L}$. The equality

$$\mathcal{E}_{\partial(\mathcal{L})} = S^+_n + \mathbb{R}J$$

can be seen via an application of Finsler’ theorem [15] that says that if $A$ and $B$ are two symmetric matrices with $[x \neq 0, \langle Ax, x \rangle = 0] \Rightarrow \langle Bx, x \rangle > 0$, then there exists $\mu \in \mathbb{R}$ such that $B + \mu A$ is positive semidefinite. (For $M \in \mathcal{E}_{\partial(\mathcal{L})}$ and vectors $u, v \in \mathcal{L}^\circ$, one has $\langle Jx, x \rangle = 0 \Rightarrow \langle Mk, x \rangle > 0$, where $k$ is a natural number and $M_k := M + \frac{1}{k} uu^\top$. When $M_k + \mu_k J$ is positive semidefinite for all $k$, it follows that the sequence $\mu_k$ is bounded. One can then use a limiting argument.) From this equality, one gets $\mathbb{R}J - S^+_n = \mathbb{R}J - \mathcal{E}_{\partial(\mathcal{L})}$. 

Our next result deals with the dual of $Z(\mathcal{L})$. 

Sándor Zoltán Németh and Muddappa Seetharama Gowda 390
The Cone of \( \mathcal{Z} \)-Transformations on the Lorentz Cone

**Theorem 3.2.** Let \( \mathcal{C} \) denote one of \( \mathbb{R}^n, \mathcal{L}, \) or \( \partial(\mathcal{L}) \). Then,

\[
\mathcal{Z}(\mathcal{L})^* = \{ B \in \mathbb{R}^{n \times n} : \langle B, I \rangle = 0, -JB \in \mathcal{K}_C \}.
\]

In particular, (1.5) holds.

**Proof.** We fix \( \mathcal{C} \). From (3.1), we see that \( B \in \mathcal{Z}(\mathcal{L})^* \) if and only if

\[
0 \leq \langle B, \gamma I - J(P + Q) \rangle
\]

for all \( \gamma \) real, \( P \) in \( \mathcal{E}_C \), and \( Q \) in \( \mathcal{A}^n \). Clearly, this holds if and only if

\[
0 = \langle B, I \rangle, \quad -JB \geq 0, \quad \text{and} \quad -JB = 0
\]

for all \( \gamma, P, \) and \( Q \) specified above. Now, with the observation that a (real) matrix is orthogonal to all skew-symmetric matrices in \( \mathbb{R}^{n \times n} \) if and only if it is symmetric, this further simplifies to

\[
0 = \langle B, I \rangle \quad \text{and} \quad -JB \in \mathcal{E}_C^*,
\]

where \( \mathcal{E}_C^* \) is the dual of \( \mathcal{E}_C \) computed in \( S^n \). Since \( \mathcal{K}_C = \mathcal{E}_C^* \) in \( S^n \), we see that \( B \in \mathcal{Z}(\mathcal{L})^* \) if and only if

\[
0 \leq \langle B, I \rangle \quad \text{and} \quad -JB \in \mathcal{K}_C.
\]

This completes the proof. \( \square \)

We remark that (1.5) can be deduced directly from (3.4) by taking the duals in \( S^n \).

In our final result, we describe the interior and boundary of \( \mathcal{Z}(\mathcal{L}) \). First, we recall some definitions from [9]. Let

\[
\Omega := \{(x, y) \in \mathcal{L} \times \mathcal{L} : ||x|| = 1 = ||y|| \text{ and } \langle x, y \rangle = 0\}.
\]

It is easy to see that \( \Omega \) is compact and, from (2.1),

\[
\Omega = \{(x, Jx) : x \in \partial(\mathcal{L}), ||x|| = 1\}.
\]

For any \( A \in \mathbb{R}^{n \times n} \), let

\[
\gamma(A) := \max \{ \langle Ax, y \rangle : (x, y) \in \Omega \}.
\]

Note that \( A \in \mathcal{Z}(\mathcal{L}) \) if and only if \( \gamma(A) \leq 0 \). We say that \( A \in \mathbb{R}^{n \times n} \) is a strict-\( \mathcal{Z} \)-transformation on \( \mathcal{L} \) if

\[
0 \neq x, y \in \mathcal{L}, \langle x, y \rangle = 0 \Rightarrow \langle Ax, y \rangle < 0.
\]

The set of all such transformations is denoted by \( \text{str}(\mathcal{Z}(\mathcal{L})) \). For \( A \in \mathbb{R}^{n \times n} \), the following statements are shown in [9], Theorem 3.1:

\[
\gamma(A) < 0 \iff A \in \mathcal{Z}(\mathcal{L})^* \iff A \in \text{str}(\mathcal{Z}(\mathcal{L}))
\]

and

\[
\gamma(A) = 0 \iff A \in \partial(\mathcal{Z}(\mathcal{L})).
\]

Recall that \( \mathcal{E}_L \) consists of all symmetric matrices that are copositive on \( \mathcal{L} \). We say that a symmetric matrix \( P \) is strictly copositive on \( \mathcal{L} \) if \( 0 \neq x \in \mathcal{L} \Rightarrow \langle Px, x \rangle > 0 \); the set of all such matrices is denoted by \( \text{str}(\mathcal{E}_L) \). Similarly, one defines \( \text{str}(\mathcal{E}_{\partial(\mathcal{L})}) \).
Corollary 3.3. The following statements hold:

\[ \mathcal{Z}(\mathcal{L})^o = \text{str}(\mathcal{Z}(\mathcal{L})) = \mathbb{R} I - J (\text{str}(\mathcal{E}\partial(\mathcal{L})) + \mathcal{A}^n) \]

and

\[ \partial(\mathcal{Z}(\mathcal{L})) = \mathbb{R} I - J (\partial_*(\mathcal{E}\partial(\mathcal{L})) + \mathcal{A}^n), \]

where \( \partial_*(\mathcal{E}\partial(\mathcal{L})) \) denotes the boundary of \( \mathcal{E}\partial(\mathcal{L}) \) in \( S^n \).

Proof. We first deal with the interior of \( \mathcal{Z}(\mathcal{L}) \). The equality

\[ \{ A \in \mathbb{R}^{n \times n} : \gamma(A) < 0 \} = \mathcal{Z}(\mathcal{L})^o = \text{str}(\mathcal{Z}(\mathcal{L})) \]

has already been observed in [9], Theorem 3.1. To see the first assertion, we show that \( \gamma(A) < 0 \) if and only if \( A = \theta I - J(P + Q) \) for some \( \theta \in \mathbb{R} \), \( P \) (symmetric) strictly copositive on \( \partial(\mathcal{L}) \), and \( Q \) skew-symmetric. Suppose \( \gamma(A) < 0 \). Then, for any \( \theta \in \mathbb{R} \),

\[ \max \left\{ \langle (A - \theta I)x, y \rangle : (x, y) \in \Omega \right\} < 0, \]

which, from (3.5) becomes

\[ \min \left\{ \langle J(\theta I - A)x, x \rangle : x \in \partial(\mathcal{L}), ||x|| = 1 \right\} > 0. \]

Now, fix \( \theta \) and let \( J(\theta I - A) = P + Q \), where \( P \in S^n \) and \( Q \in \mathcal{A}^n \). As \( \langle Qx, x \rangle = 0 \) for any \( x \), the above inequality implies that \( \min \{ \langle Px, x \rangle : x \in \partial(\mathcal{L}), ||x|| = 1 \} > 0 \). This proves that \( P \) is strictly copositive on \( \partial(\mathcal{L}) \). Rewriting \( J(\theta I - A) = P + Q \), we see that \( A = \theta I - J(P + Q) \) which is of the required form.

To see the converse, suppose \( A = \theta I - J(P + Q) \), where \( \theta \in \mathbb{R} \), \( P \) (symmetric) strictly copositive on \( \partial(\mathcal{L}) \), and \( Q \) skew-symmetric. Using (3.5), we can easily verify that \( \gamma(A) < 0 \). Thus, \( A \in \text{str}(\mathcal{Z}(\mathcal{L})) \).

An argument similar to the above will show that \( \gamma(A) = 0 \) if and only if \( A = \theta I - J(P + Q) \) for some \( \theta \in \mathbb{R} \), \( P \in \partial_*(\mathcal{E}\partial(\mathcal{L})) \), and \( Q \) skew-symmetric. This gives the statement regarding the boundary of \( \mathcal{Z}(\mathcal{L}) \). \( \square \)

We end the paper with a remark dealing with conic linear programs. Motivated by the result of Burer (mentioned in the introduction), we consider a conic linear program on a completely positive cone \( \mathcal{K}_C \) (where \( C \) is a closed cone):

\[ \min \left\{ \langle c, x \rangle : Ax = b, x \in \mathcal{K}_C \right\}. \]

While such a problem is generally hard to solve, we ask: (When) can we replace \( \mathcal{K}_C \) by \( S^n_+ \), and thus, reduce the above problem to the semidefinite programming problem \( \text{min} \left\{ \langle c, x \rangle : Ax = b, x \in S^n_+ \right\} \)? Just replacing \( \mathcal{K}_C \) by \( S^n_+ \) without handling the constraint \( Ax = b \) is not viable as \( \mathcal{K}_C = S^n_+ \) if and only if \( C \cup -C = \mathbb{R}^n \) (which fails to hold when \( n > 1 \) and \( C \) is pointed), see [11]. While we do not answer this broad question, we point out, as a consequence of (1.5), that for any \( C \in S^n_+ \),

\[ \min \left\{ \langle C, X \rangle : (X, J) = 0, X \in \mathcal{K}_C \right\} = \min \left\{ \langle C, X \rangle : (X, J) = 0, X \in S^n_+ \right\}. \]
The Cone of $\mathbb{Z}$-Transformations on the Lorentz Cone

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