Example Set #4

Definition: If $K \subseteq L$ is a field extension, $G = Gal(L/K)$, and $H \subseteq G$ is a subset of G then $Fix_L(H) = \{a \in L \mid h(a) = a \text{ for all } h \in H\}$. If $H = \{h_1, h_2, \ldots, h_k\}$ is a finite set then we write $Fix_L(h_1, h_2, \ldots, h_k)$ instead of $Fix_L(\{h_1, h_2, \ldots, h_k\})$.

1. Prove that $F = Fix_L(H)$ is a subfield of L and that $K \subseteq F$.

2. Prove that $H \subseteq Gal(L/F)$, where $F = Fix_L(H)$.

3. Do the following:

   (a) Let $L = \mathbb{Q}(\sqrt{3})$ and $g \in Aut(L)$ such that $g(a + b\sqrt{3}) = a - b\sqrt{3}$ for all $a, b \in \mathbb{Q}$. Determine $Fix_L(g)$.

   (b) More generally, suppose $K \subseteq L$ is a field extension such that $[L : K] = p$, a prime. Suppose $g \in Gal(L/K)$, $g \neq 1$. Prove that $Fix_L(g) = K$.

4. For a subset $\{g_1, \ldots, g_k\}$ of G, let $H = \langle g_1, \ldots, g_k \rangle$ be the subgroup of G generated by the subset. Prove that $Fix_L(g_1, \ldots, g_k) = Fix_L(H)$.

5. Prove that $Fix_L(g_1, \ldots, g_k) = \cap_{i=1}^{k} Fix_L(g_i)$.

6. Consider $g \in G$ as a K-linear mapping on the vector space L. Prove that $Fix_L(g)$ is the eigenspace for g corresponding to the eigenvalue 1.

7. Let L be the splitting field (in $\mathbb{C}$) of the polynomial $f = x^3 - 3$ over $\mathbb{Q}$.

   (a) Determine $[L : \mathbb{Q}]$ and find a basis of L over $\mathbb{Q}$.

   (b) Prove that $G = Gal(L/\mathbb{Q}) \cong Sym(3)$, the symmetric group on three letters.

   (c) For $g \in G$ sending $\sqrt[3]{3} \to \sqrt[3]{3}\xi \to \sqrt[3]{3}\xi^2 \to \sqrt[3]{3}$, where ξ is the primitive cubic root of unity, find the explicit action of g on the basis of L.

   (d) Determine $Fix_L(g)$.

8. For L as in Problem 7, find all subgroups of $G = Gal(L/\mathbb{Q})$ and determine the corresponding fixed subfields.

9. Repeat Problem 7, parts (a) and (b), and Problem 8 with L being the splitting field of the polynomial $f = x^8 - 1$.

10. Do the same for $f = x^4 - 7$.