Example Set #2

1. Use Eisenstein's Criterion to show that $f = x^5 - 6x^4 - 15x^3 + 18x^2 + 6x - 12 \in \mathbb{Q}[x]$ is irreducible.

2. Suppose K is a field and $f \in K[x]$.

   (a) For $a \in K$, set $g = f(x + a)$. Prove that f is irreducible in $K[x]$ if and only if g is irreducible in $K[x]$.

   (b) For $a \in K$, $a \neq 0$, set $g = f(ax)$. Prove that f is irreducible in $K[x]$ if and only if g is irreducible in $K[x]$.

3. Use Eisenstein's Criterion and Problem 2(a) to show that $f = x^4 - 4x^3 + 6x^2 - 7x + 10 \in \mathbb{Q}[x]$ is irreducible.

4. Suppose $K \subseteq L$ is a field extension and $u \in L$. Let $p = \min_{u,K} \in K[x]$.

   (a) For $a \in K$, set $v = u + a$ and $q = p(x - a)$. Prove that q is the minimal polynomial of v over K, that is, $q = \min_{v,K}$.

   (b) For $a \in K$, $a \neq 0$, set $v = au$ and $q = a^n p(\frac{x}{a})$, where $n = \deg(p)$. Prove that q is the minimal polynomial of v over K.

5. Suppose that $u \in \mathbb{C}$ is a root of the polynomial $f = x^2 - x + 2$.

   (a) Prove that f is the minimal polynomial of u over $\mathbb{Q}$.

   (b) Compute the minimal polynomials of the numbers $v = u - 1$ and $w = \frac{u}{2}$.

   (c) Compute the minimal polynomial of $z = 2u + 1$.

6. Suppose u is a root of $2x^3 - x^2 + 3x + 1$. Determine the minimal polynomial of u over $\mathbb{Q}$.

7. Suppose again $u \in \mathbb{C}$ is a root of the polynomial $f = x^2 - x + 2$.

   (a) Use f to express u^2 as a linear combination of 1 and u.

   (b) Using (a), express u^3 and u^4 as linear combinations of 1 and u.

8. For $w = 2u + 1$, using the expression for u^2 from Problem 7(a), write w^2 as a linear combination of 1 and u. Then express w^2 as a linear combination of 1 and w. Does your result agree with Problem 5(c)?

9. Try a slightly different approach:

(a) Using the expression for w^2 and w in terms of 1 and u from Problem 8, express $f(w)$, where $f = a_2x^2 + a_1x + a_0 \in \mathbb{Q}[x]$, in terms of 1 and u.

(b) Determine all triples $a_0, a_1, a_2 \in \mathbb{Q}$, such that $f(w) = 0$. (This should lead to a system of two linear equations with a_0, a_1, and a_2 as the unknowns—solve this system!).

(c) Is there such a triple with $a_2 = 1$ (monic polynomial f)?

10. Suppose u is a root of $2x^3 - x^2 + 3x + 1$. Determine the minimal polynomial of $u^2 + 1$ over $\mathbb{Q}$. (Methods of Problem 9 generalize! The coefficients of the polynomial may not be pretty—sorry!)