Example Set #1

1. Prove that 1 and $\sqrt{3}$ constitute a basis in $\mathbb{Q}(\sqrt{3})$ viewed as a vector space over $\mathbb{Q}$.

2. What is the degree of the extension $\mathbb{Z}_7 \subseteq \mathbb{Z}_7(\sqrt{\bar{2}})$? Here $\bar{2}$ is the congruence class of 2 modulo 7.

3. Suppose F is an extension of $\mathbb{Z}_2$ of degree three. Prove that F is a finite field. What is the size of F?

4. Prove that $\mathbb{Q}(\sqrt{2}+i) = \mathbb{Q}(\sqrt{2}, i)$.

5. Find the degree of the extension $\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}, i)$ and a basis for this extension.

6. Suppose $\mathbb{Q} \subseteq F \subseteq \mathbb{C}$ is a tower of extensions and suppose that $i \in F$. If the extension $\mathbb{Q} \subseteq F$ is finite, show that $[F : \mathbb{Q}]$ is even.

7. Consider a finite filed extension $F \subseteq L$ and let K be an intermediate field (that is, $F \subseteq K \subseteq L$). Show that both $[K : F]$ and $[L : K]$ divide $[L : F]$. In particular, both $F \subseteq K$ and $K \subseteq L$ are finite field extensions of degree no greater than $[L : F]$.

8. Consider a finite field extension $F \subseteq M$ and two intermediate fields K and L (that is, $F \subset K, L \subseteq M$). If $K \subseteq L$, show that $[L : K] \leq [M : F]$. (In fact, $[L : K]$ divides $[M : F]$.)

9. Suppose $K_0 \subseteq K_1 \subseteq \ldots \subseteq K_m$ is a tower of (finite) field extensions. Show that

$$[K_m : K_0] = \prod_{i=1}^{m} [K_i : K_{i-1}].$$

10. Consider a field extension $K \subseteq L$. Let R be a subring in L such that $K \subseteq R$. Prove that if R has finite dimension as a vector space over K then R is a field.

11. Suppose $F \subseteq K \subseteq L$ and $a \in L$. If $F(a)$ is a finite extension of F, use Problem 10 to show that $K(a)$ is a finite extension of K and $[K(a) : K] \leq [F(a) : F]$.