Example Set #2

(If you take this course for credit please submit the solutions of Problems 2, 7, 9, *and* 10.)

1. Determine the absolute discriminant d_k, where $k = \mathbb{Q}(\sqrt{5})$.

2. Prove that the absolute discriminant d_k is an integer for all number fields k. (Hint: Show that the matrix $M^T M$ involved in the definition of d_k is integral.)

3. Prove that the value of the absolute discriminant d_k does not depend on the choice of base in $\mathfrak{o}_k$.

4. Suppose a ring R is Noetherian and I is an ideal of R.

   (a) Prove that I is Notherial as an R-module.

   (b) Prove that R/I is a Noetherian ring.

5. Conversely, suppose R is a ring and I is an ideal of R. Prove that if I is Noetherian as an R-module and R/I is Noetherian then R is Noetherian.

6. This will be needed for Problem 4: Let p be a prime and $n \geq 2$. Prove that $\frac{1}{\sqrt[n]{p}}$ is not an algebraic integer.

7. Prove that the ring $\mathfrak{O}$ of all algebraic integers is not Noetherian. Namely, take a prime number p and let $I_n = (\sqrt[n]{p})$ for all $n \geq 1$. Show that $\sqrt[n+1]{p} \notin I_n$ and hence this chain of ideals in strictly increasing.

8. Prove that $\alpha \in \mathfrak{o}_k$, where k is a number field, is a unit if and only if $\mathrm{norm}_{k/\mathbb{Q}}(\alpha) = \pm 1$.

9. Prove that $\mathfrak{o}_k$, where $k = \mathbb{Q}(\sqrt{-13})$, is not a unique factorization domain.

10. Find a non-principal ideal in $\mathfrak{o}_{\mathbb{Q}(\sqrt{-13})}$.