Example Set #1

(If you take this course for credit please submit the solutions of Problems 2, 5 and 6 .)

The first three problems have to do with finitely generated abelian groups. Let G be such a group and let $\mathrm{rk}(G)$ denote the free rank of G, that is, the number of factors $\mathbb{Z}$ in the direct decomposition of G with cyclic factors. It was proven in the notes that if $H \leq G$ then H is also finitely generated and furthermore $\mathrm{rk}(H) \leq \mathrm{rk}(G)$.

1. Let $N \leq G_T$. Prove that $\mathrm{rk}(G/N) = \mathrm{rk}(G)$.

2. Suppose $H \leq G$ and $[G : H] < \infty$. Prove that $\mathrm{rk}(H) = \mathrm{rk}(G)$. (Hint: Let $n = [G : H]$ and consider $K = \mathrm{im}\phi$ where $\phi : G \to H$ is given by $g \mapsto ng$. Use Problem 1!)

3. Conversely, suppose $H \leq G$ and $\mathrm{rk}(H) = \mathrm{rk}(G)$. Prove that G/H is a finite group. (Hint: By contradiction, if G/H contains an element of infinite order then construct $K \leq G$ with $H \leq K$ and $\mathrm{rk}(K) = \mathrm{rk}(H) + 1$.)

The remaining problems explore the concepts of norm and trace and the ring of algebraic integers.

4. Let m be a nonzero integer. Let n^2 is the largest square that divides m and call $m' = m/n^2$ the square-free part of m. Show that $K = \mathbb{Q}(\sqrt{m}) = \mathbb{Q}(\sqrt{m'})$. Conclude that $[K : \mathbb{Q}] = 2$ unless $m' = 1$

5. Let m be square-free integer and let $K = \mathbb{Q}[\sqrt{m}]$.

   (a) Determine $\mathrm{norm}_{K/\mathbb{Q}}(\alpha)$ and $\mathrm{Tr}_{K/\mathbb{Q}}(\alpha)$ for all $\alpha \in K$.

   (b) Show that if $m' \not\equiv 1 \bmod 4$ then the ring of algebraic integers of K is $\{a + b\sqrt{m'} \mid a, b \in \mathbb{Z}\}$.

   (c) Determine the ring of algebraic integers of K when $m' \equiv 1 \bmod 4$.

6. Compute the trace and norm for the extension $K = \mathbb{Q} \subset \mathbb{Q}(\sqrt{2}, i)$ of $\mathbb{Q}$. Determine the ring of algebraic integers of K.