THE EXISTENCE OF DESIGNS VIA ITERATIVE ABSORPTION

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Abstract. In a recent breakthrough, Keevash proved the Existence conjecture for combinatorial designs, which has its roots in the 19th century. We give a new proof, based on the method of iterative absorption. Our main result concerns $K_q^{(r)}$-decompositions of hypergraphs whose clique distribution fulfils certain uniformity criteria. These criteria offer considerable flexibility. This enables us to strengthen the results of Keevash as well as to derive a number of new results, for example a resilience version and minimum degree version.

1. Introduction

The term 'Combinatorial design' usually refers to a system of finite sets which satisfy some specified balance or symmetry condition. Some well known examples include balanced incomplete block designs, projective planes, Latin squares and Hadamard matrices. These have applications in many areas such as finite geometry, statistics, experiment design and cryptography.

1.1. **Block designs and Steiner systems.** In this paper, an (n, q, r, λ)-*design* is a set X of q-subsets (often called 'blocks') of some n-set V, such that every r-subset of V belongs to exactly λ elements of X. (Note that this makes only sense if $q > r$, which we assume throughout the paper.) In the case when $r = 2$, this coincides with the notion of balanced incomplete block designs. An $(n, q, r, 1)$-design is also called an (n, q, r)-*Steiner system*. There are some obviously necessary 'divisibility conditions' for the existence of a design: consider some subset S of V of size $i < r$ and assume that X is an (n, q, r, λ)-design. Then the number of elements of X which contain S is $\lambda\binom{n-i}{r-i}/\binom{q-i}{r-i}$. We say that the necessary divisibility conditions are satisfied if $\binom{q-i}{r-i}$ divides $\lambda\binom{n-i}{r-i}$ for all $0 \le i < r$.

The 'Existence conjecture' states that for given q, r, λ, the necessary divisibility conditions are also sufficient for the existence of an (n, q, r, λ)-design, except for a finite number of exceptional n. Its roots can be traced back to work of e.g. Plücker, Kirkman and Steiner in the 19th century. Over a century later, a breakthrough result of Wilson [42, 43, 44] resolved the graph case $r = 2$.

For $r \ge 3$, much less was known until relatively recently. In 1963, Erdős and Hanani [12] proposed an approximate version of the Existence conjecture for the case of Steiner systems. More precisely, they asked whether one can find blocks which cover every r-set at most once and cover all but $o(n^r)$ of the r-sets, as n tends to infinity. This was proved in 1985 by Rödl [34] via his celebrated 'nibble' method, and the bounds were subsequently improved by increasingly sophisticated randomised techniques (see e.g. [1, 41]). Ferber, Hod, Krivelevich and Sudakov [13] recently observed that this method can be used to obtain an 'almost' Steiner system in the sense that every r-set is covered by either one or two q-sets.

Teirlinck [40] was the first to prove the existence of designs for arbitrary $r \ge 6$, via an ingenious recursive construction based on the symmetric group (this however requires $q = r+1$ and λ large compared to q). Kuperberg, Lovett and Peled [27] proved a 'localized central limit theorem' for rigid combinatorial structures, which implies the existence of designs for arbitrary

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q and r, but again for large λ. There are many constructions resulting in sporadic and infinite families of designs (see e.g. the handbook [9]). However, the set of parameters they cover is very restricted. In particular, even the existence of infinitely many Steiner systems with $r \geq 4$ was open until recently.

In a recent breakthrough, Keevash [20] proved the Existence conjecture in general, based on the method of 'Randomised algebraic constructions'. This method is inspired by Wilson's algebraic approach to the graph case as well as results on integral designs by Graver and Jurkat [15].

In the current paper, we provide a new proof of the Existence conjecture based on the method of iterative absorption. In fact, our main theorem (Theorem 3.7) is considerably more general than this (as well as the results in [20]): it implies a number of new results about designs in the 'incomplete setting', that is, when only a given subset E of all the possible r-sets of V are allowed in the blocks.

The method of iterative absorption was initially introduced in [23, 26] to find Hamilton decompositions of graphs. In the meantime it has been successfully applied to verify the Gyárfás-Lehel tree packing conjecture for bounded degree trees [19], as well as to find decompositions of dense graphs into a given graph F [5, 6, 14]. We believe that the present paper will pave the way for further applications beyond the graph setting.

1.2. **Designs in hypergraphs.** We will study designs in a hypergraph setting. Here a *hypergraph* H is a pair (V, E), where V is the vertex set and the edge set E is a set of subsets of V. We identify H with E. In particular, we let $|H| := |E|$. We say that H is an *r-graph* if every edge has size r. We let $K_n^{(r)}$ denote the complete r-graph on n vertices.

Let H be some r-graph. A *$K_q^{(r)}$-decomposition* of H is a collection $\mathcal{K}$ of copies of $K_q^{(r)}$ in H such that every edge of H is contained in exactly one of these copies. More generally, a *(q, r, λ)-design* of H is a collection $\mathcal{K}$ of distinct copies of $K_q^{(r)}$ in H such that every edge of H is contained in exactly λ of these copies. Note that a (q, r, λ)-design of $K_n^{(r)}$ is equivalent to an (n, q, r, λ)-design.

For a set $S \subseteq V$ with $0 \leq |S| \leq r$, the $(r-|S|)$-graph $H(S)$ has vertex set $V \setminus S$ and contains all $(r-|S|)$-subsets of $V \setminus S$ that together with S form an edge in H. ($H(S)$ is often called the *link graph of S*.) We say that H is *(q, r, λ)-divisible* if for every $S \subseteq V$ with $0 \leq |S| \leq r-1$, we have that $\binom{q-|S|}{r-|S|}$ divides $\lambda |H(S)|$. Similarly to Section 1.1, this is a necessary condition for the existence of a (q, r, λ)-design of H. We say that H is *$K_q^{(r)}$-divisible* if H is $(q, r, 1)$-divisible.

We let $\delta(H)$ and $\Delta(H)$ denote the minimum and maximum $(r-1)$-degree of an r-graph H, respectively, that is, the minimum/maximum value of $|H(S)|$ over all $S \subseteq V(H)$ of size $r-1$. The following result guarantees designs not just for $K_n^{(r)}$, but also for r-graphs which are allowed to be far from complete in the sense that they only have large minimum degree.

Theorem 1.1 (Minimum degree version)**.** *For all $q > r \geq 2$ and $\lambda \in \mathbb{N}$, there exists an $n_0 \in \mathbb{N}$ such that the following holds for all $n \geq n_0$. Let*

$$c^{\diamond}_{q,r} := \frac{r!}{3 \cdot 14^r q^{2r}}.$$

Suppose that G is an r-graph on n vertices with $\delta(G) \geq (1-c^{\diamond}_{q,r})n$. Then G has a (q, r, λ)-design if it is (q, r, λ)-divisible.

The main result of [20] implies a weaker version where $c^{\diamond}_{q,r}$ is replaced by some non-explicit $\varepsilon \ll 1/q$.

Note that Theorem 1.1 implies that whenever X is a partial (n, q, r)-Steiner system (i.e. a set of edge-disjoint $K_q^{(r)}$ on n vertices) and $n^* \geq \max\{n_0, n/c^{\diamond}_{q,r}\}$ satisfies the necessary divisibility conditions, then X can be extended to an (n^*, q, r)-Steiner system. For the case of Steiner triple systems (i.e. $q = 3$ and $r = 2$), Bryant and Horsley [8] showed that one can take $n^* = 2n + 1$, which proved a conjecture of Lindner.

Theorem 1.1 motivates the following very challenging problem regarding the decomposition threshold $c_{q,r}$ of $K_q^{(r)}$.

Problem 1.2. *Determine the supremum $c_{q,r}$ of all $c \in [0,1]$ with the following property: There exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, every $K_q^{(r)}$-divisible r-graph on n vertices with $\delta(G) \geq (1-c)n$ has a $K_q^{(r)}$-decomposition.*

Theorem 1.1 implies that $c_{q,r} \geq c^{\diamond}_{q,r}$. It is not clear what the correct value should be. We note that for all $r, q, n_0 \in \mathbb{N}$, there exists an r-graph G_n on $n \geq n_0$ vertices with $\delta(G_n) \geq (1 - b_r \frac{\log q}{q^{r-1}})n$ such that G_n does not contain a single copy of $K_q^{(r)}$, where $b_r > 0$ only depends on r. This can be seen by adapting a construction from [24] as follows. Without loss of generality, we may assume that $1/q \ll 1/r$. By a result of [37], for every $r \geq 2$, there exists a constant b_r such that for any large enough q, there exists a partial $(N, r, r-1)$-Steiner system S_N with independence number $\alpha(S_N) < q/(r-1)$ and $1/N \leq b_r \log q / q^{r-1}$. This partial Steiner system can be 'blown up' (cf. [24]) to obtain arbitrarily large r-graphs H_n on n vertices with $\alpha(H_n) < q$ and $\Delta(H_n) \leq n/N \leq b_r n \log q / q^{r-1}$. Then the complement G_n of H_n is $K_q^{(r)}$-free and satisfies $\delta(G_n) \geq (1 - b_r \frac{\log q}{q^{r-1}})n$.

We now consider the graph case $r = 2$. A famous conjecture by Nash-Williams [31] on the decomposition threshold of a triangle would imply that $c_{3,2} = 3/4$. Until recently, the best bound for the problem was by Gustavsson [16], who claimed that $c_{q,2} \geq 10^{-37} q^{-94}$. Iterated absorption methods have led to significant progress in this area. For instance, the results in [14] imply that $c_{q,2} = c^*_{q,2}$, where $c^*_{q,r}$ denotes the fractional version of the decomposition threshold (the triangle case $q = 3$ was already obtained in [5]). This in turn has resulted in significantly improved explicit bounds on $c_{q,2}$, via results on fractional decompositions obtained in [4, 11]. In particular, the results from [4, 14] imply that $\frac{1}{10^4 q^{3/2}} \leq c_{q,2} \leq \frac{1}{q+1}$, where the upper bound is conjectured to be the correct value. The results in [5, 14] make (implicit) use of Szemerédi's regularity lemma, whereas our proof avoids this, resulting in much more moderate requirements on n.

1.3. **Resilience and typicality.** An important trend in probabilistic combinatorics has been to study the resilience of r-graph properties with respect to local and global perturbations. This was first systematically approached in an influential paper of Sudakov and Vu [39]. Recent highlights include the transference results for random hypergraphs by Conlon and Gowers [10] as well as Schacht [38].

Our main result implies the following local resilience version of the existence of designs, which is new even in the graph case. Given two r-graphs H and L, define $H \triangle L$ to be the r-graph on $V(H) \cup V(L)$ whose edge set is $(H \setminus L) \cup (L \setminus H)$. Let $\mathcal{H}_r(n,p)$ denote the random binomial r-graph on $[n]$ whose edges appear independently with probability p.

Theorem 1.3 (Resilience version)**.** *Let $p \in (0,1]$ and $q, r, \lambda \in \mathbb{N}$ with $q > r$ and let*

$$c(q,r,p) := \frac{r! p^{2^r \binom{q+r}{r}}}{3 \cdot 14^r q^{2r}}.$$

Then the following holds whp for $H \sim \mathcal{H}_r(n,p)$. For every r-graph L on $[n]$ with $\Delta(L) \leq c(q,r,p)n$, $H \triangle L$ has a (q,r,λ)-design whenever it is (q,r,λ)-divisible.

Note that the case $p = 1$ implies Theorem 1.1.

The main result of [20] is actually a decomposition result for 'typical' hypergraphs (and complexes). Here an r-graph H on n vertices is called (c,h,p)-*typical* if for any set A of $(r-1)$-subsets of $V(H)$ with $|A| \leq h$ we have $|\bigcap_{S \in A} H(S)| = (1 \pm c)p^{|A|}n$. (So in the 2-graph case for example, this requires that any set of up to h vertices have roughly as many common neighbours as one would expect in a binomial random graph of density p.) The main result in [20] requires (c,h,p)-typicality for $c \ll 1/h \ll 1/q, 1/\lambda$. We can relax this to more moderate requirements on c and h.

Theorem 1.4. *For all $q, r, \lambda \in \mathbb{N}$ and $c, p \in (0, 1]$ with $q > r$ and*

$$c \le p^{2^r \binom{q+r}{r}}/(q^r 8^q), \tag{1.1}$$

there exists $n_0 \in \mathbb{N}$ such that the following holds for all $n \ge n_0$. Suppose that H is a $(c, 2^r\binom{q+r}{r}, p)$-typical r-graph on n vertices. Then H has a (q, r, λ)-design if it is (q, r, λ)-divisible.

Note that whenever H is (c, h, p)-typical and $\Delta(L) \le \gamma n$ with $V(L) = V(H)$, then $H \triangle L$ is $(c+hp^{-h}\gamma, h, p)$-typical. Thus, the above theorem can also be applied to obtain a (q, r, λ)-design of $H \triangle L$, with $c + 2^r\binom{q+r}{r}p^{-2^r\binom{q+r}{r}}\gamma$ playing the role of c in (1.1).

1.4. **Matchings and further results.** As another illustration, we now state a consequence of our main result which concerns perfect matchings in hypergraphs that satisfy certain uniformity conditions on their edge distribution. Note that the conditions are much weaker than any standard pseudorandomness notion.

Theorem 1.5. *For all $q \ge 2$ and $\xi > 0$ there exists $n_0 \in \mathbb{N}$ such that the following holds whenever $n \ge n_0$ and $q \mid n$. Let G be a q-graph on n vertices which satisfies the following properties:*

- *for some $d \ge \xi$, $|G(v)| = (d \pm 0.01\xi)n^{q-1}$ for all $v \in V(G)$;*
- *every vertex is contained in at least ξn^q copies of $K^{(q)}_{q+1}$;*
- *$|G(v) \cap G(w)| \ge \xi n^{q-1}$ for all $v, w \in V(G)$.*

Then G has $0.01\xi n^{q-1}$ edge-disjoint perfect matchings.

Note that for $G = K^{(q)}_n$, this is strengthened by Baranyai's theorem [3], which states that $K^{(q)}_n$ has a decomposition into $\binom{n-1}{q-1}$ edge-disjoint perfect matchings. More generally, the interplay between designs and the existence of (almost) perfect matchings in hypergraphs has resulted in major developments over the past decades, e.g. via the Rödl nibble. For more recent progress on results concerning perfect matchings in hypergraphs and related topics, see e.g. the surveys [35, 45, 46].

We discuss further applications of our main result in Section 3, e.g. to partite graphs (see Example 3.10) and to (n, q, r, λ)-designs where we allow any $\lambda \le n^{q-r}/(11 \cdot 7^r q!)$, say (under more restrictive divisibility conditions, see Corollary 3.13). We also note that, in a similar way as discussed in [20, 21], the results of this paper can be combined with 'counting versions' of the Rödl nibble (or corresponding random greedy processes) to obtain lower bounds on the number of designs with given parameters. (Linial and Luria [29] showed that one can obtain good upper bounds via entropy techniques.) These developments also make it possible to systematically study random designs (see e.g. [28]). This can be done e.g. by analysing a random approximate decomposition and showing that the leftover satisfies the conditions of Theorem 1.4.

1.5. **Structure of the paper.** In the following section, we will introduce our basic terminology. In Section 3 we introduce supercomplexes and state our main theorem (Theorem 3.7). We also give several applications. Section 4 is devoted to a brief outline of our proof method. In Sections 5 and 6 we collect tools (which are mainly probabilistic) and observations for later use. In particular, we prove the Boost lemma (Lemma 6.3), which allows us to 'boost' our regularity parameters.

In Section 7 we introduce vortices (which form the framework for our iterated absorption) and state the Cover down lemma (Lemma 7.4). The latter is the main engine behind the iterative absorption process – it allows us to reduce the current decomposition problem to a significantly smaller one in each iteration. We then construct absorbers (which deal with the leftover from the iterative process) in Section 8. We combine all these results in Section 9 to prove our main theorem (Theorem 3.7) and also deduce Theorems 1.3, 1.4, 1.5 and 3.14. Finally, in Section 10 we prove the Cover down lemma.

2. Notation

2.1. **Basic terminology.** We let $[n]$ denote the set $\{1,\dots,n\}$, where $[0]:=\emptyset$. Moreover, $[n]_0 := [n]\cup\{0\}$ and $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$. As usual, $\binom{n}{i}$ denotes the binomial coefficient, where we set $\binom{n}{i}:=0$ if $i>n$ or $i<0$. Moreover, given a set X and $i\in\mathbb{N}_0$, we write $\binom{X}{i}$ for the collection of all i-subsets of X. Hence, $\binom{X}{i}=\emptyset$ if $i>|X|$. If F is a collection of sets, we define $\bigcup F := \bigcup_{f\in F} f$.

We write $X\sim B(n,p)$ if X has binomial distribution with parameters n,p, and we write $bin(n,p,i):=\binom{n}{i}p^i(1-p)^{n-i}$. So by the above convention, $bin(n,p,i)=0$ if $i>n$ or $i<0$.

We say that an event holds *with high probability (whp)* if the probability that it holds tends to 1 as $n\to\infty$ (where n usually denotes the number of vertices). If we use a probabilistic argument in a proof in order to show the existence of a certain object, and a bounded number of properties of this random object hold whp then we can assume that for large enough n there is such an object that has all the desired properties.

We write $x\ll y$ to mean that for any $y\in(0,1]$ there exists an $x_0\in(0,1)$ such that for all $x\le x_0$ the subsequent statement holds. Hierarchies with more constants are defined in a similar way and are to be read from the right to the left. We will always assume that the constants in our hierarchies are reals in $(0,1]$. Moreover, if $1/x$ appears in a hierarchy, this implicitly means that x is a natural number. More precisely, $1/x\ll y$ means that for any $y\in(0,1]$ there exists an $x_0\in\mathbb{N}$ such that for all $x\in\mathbb{N}$ with $x\ge x_0$ the subsequent statement holds.

We write $a=b\pm c$ if $b-c\le a\le b+c$. Equations containing $\pm$ are always to be interpreted from left to right, e.g. $b_1\pm c_1=b_2\pm c_2$ means that $b_1-c_1\ge b_2-c_2$ and $b_1+c_1\le b_2+c_2$. We will often use the fact that for all $0<x<1$ and $n\in\mathbb{N}$ we have $(1\pm x)^n=1\pm 2^n x$.

2.2. **Hypergraphs and complexes.** Let H be an r-graph. Note that $H(\emptyset)=H$. For a set $S\subseteq V(H)$ with $|S|\le r$ and $L\subseteq H(S)$, let $S\uplus L:=\{S\cup e\,:\,e\in L\}$. Clearly, there is a natural bijection between L and $S\uplus L$.

For $i\in[r-1]_0$, we define $\delta_i(H)$ and $\Delta_i(H)$ as the minimum and maximum value of $|H(S)|$ over all i-subsets S of $V(H)$, respectively. As before, we let $\delta(H):=\delta_{r-1}(H)$ and $\Delta(H):=\Delta_{r-1}(H)$. Note that $\delta_0(H)=\Delta_0(H)=|H(\emptyset)|=|H|$.

For two r-graphs H and H', we let $H-H'$ denote the r-graph obtained from H by deleting all edges of H'.

Definition 2.1. A *complex* G is a hypergraph which is closed under inclusion, that is, whenever $e'\subseteq e\in G$ we have $e'\in G$. If G is a complex and $i\in\mathbb{N}_0$, we write $G^{(i)}$ for the i-graph on $V(G)$ consisting of all $e\in G$ with $|e|=i$. We say that a complex is empty if $\emptyset\notin G^{(0)}$, that is, if G does not contain any edges.

Suppose G is a complex and $e\subseteq V(G)$. Define $G(e)$ as the complex on vertex set $V(G)\setminus e$ containing all sets $f\subseteq V(G)\setminus e$ such that $e\cup f\in G$. Clearly, if $e\notin G$, then $G(e)$ is empty. Observe that if $|e|=i$ and $r\ge i$, then $G^{(r)}(e)=G(e)^{(r-i)}$. We say that G' is a *subcomplex* of G if G' is a complex and a subhypergraph of G.

For a set U, define $G[U]$ as the complex on $U\cap V(G)$ containing all $e\in G$ with $e\subseteq U$. Moreover, for an r-graph H, let $G[H]$ be the complex on $V(G)$ with edge set

$$G[H]:=\{e\in G\,:\,\binom{e}{r}\subseteq H\},$$

and define $G-H:=G[G^{(r)}-H]$. So for $i\in[r-1]$, $G[H]^{(i)}=G^{(i)}$. For $i>r$, we might have $G[H]^{(i)}\subsetneq G^{(i)}$. Moreover, if $H\subseteq G^{(r)}$, then $G[H]^{(r)}=H$. Note that for an r_1-graph H_1 and an r_2-graph H_2, we have $(G[H_1])[H_2]=(G[H_2])[H_1]$. Also, $(G-H_1)-H_2=(G-H_2)-H_1$, so we may write this as $G-H_1-H_2$.

If G_1 and G_2 are complexes, we define $G_1\cap G_2$ as the complex on vertex set $V(G_1)\cap V(G_2)$ containing all sets e with $e\in G_1$ and $e\in G_2$. We say that G_1 and G_2 are *i-disjoint* if $G_1^{(i)}\cap G_2^{(i)}$ is empty.

For any hypergraph H, let $H^{\leq}$ be the complex on $V(H)$ *generated by* H, that is,

$$H^{\leq} := \{e \subseteq V(H) : \exists f \in H \text{ such that } e \subseteq f\}.$$

For an r-graph H, we let $H^{\leftrightarrow}$ denote the complex on $V(H)$ that is *induced by* H, that is,

$$H^{\leftrightarrow} := \{e \subseteq V(H) : \binom{e}{r} \subseteq H\}.$$

Note that $H^{\leftrightarrow(r)} = H$ and for each $i \in [r-1]_0$, $H^{\leftrightarrow(i)}$ is the complete i-graph on $V(H)$. We let K_n denote the the complete complex on n vertices.

3. Supercomplexes and the main theorem

3.1. **Supercomplexes.** Our main theorem is a statement about 'supercomplexes', which we now define. The definition involves three properties: regularity, density, extendability. We require regularity primarily to apply the Rödl nibble (via Theorem 6.1). Moreover, we need the density notion for our 'Boost lemma' (Lemma 6.3). Finally, extendability is needed to find a special r-graph in $G^{(r)}$ that we need to build absorbers.

Definition 3.1. Let G be a complex on n vertices, $q \in \mathbb{N}$ and $r \in [q-1]_0$, $0 \leq \varepsilon, d, \xi \leq 1$. We say that G is

(i) (ε, d, q, r)-*regular*, if for all $e \in G^{(r)}$ we have

$$|G^{(q)}(e)| = (d \pm \varepsilon)n^{q-r};$$

(ii) (ξ, q, r)-*dense*, if for all $e \in G^{(r)}$, we have

$$|G^{(q)}(e)| \geq \xi n^{q-r};$$

(iii) (ξ, q, r)-*extendable*, if $G^{(r)}$ is empty or there exists a subset $X \subseteq V(G)$ with $|X| \geq \xi n$ such that for all $e \in \binom{X}{r}$, there are at least ξn^{q-r} $(q-r)$-sets $Q \subseteq V(G) \setminus e$ such that $\binom{Q \cup e}{r} \setminus \{e\} \subseteq G^{(r)}$.

We say that G is a *full* (ε, ξ, q, r)-*complex* if G is

- (ε, d, q, r)-regular for some $d \geq \xi$,
- $(\xi, q+r, r)$-dense,
- (ξ, q, r)-extendable.

We say that G is an (ε, ξ, q, r)-*complex* if there exists a q-graph Y on $V(G)$ such that $G[Y]$ is a full (ε, ξ, q, r)-complex. Note that $G[Y]^{(r)} = G^{(r)}$.

The additional flexibility offered by considering (ε, ξ, q, r)-complexes rather than full (ε, ξ, q, r)-complexes is key to proving our minimum degree and resilience results (via the 'boosting' step discussed below). We also note that for the scope of this paper, it would be sufficient to define extendability more restrictively, by letting $X := V(G)$. However, for future applications, it might turn out to be useful that we do not require $X = V(G)$.

Fact 3.2. *Note that G is an $(\varepsilon, \xi, q, 0)$-complex if and only if G is empty or $|G^{(q)}| \geq \xi n^q$. In particular, every $(\varepsilon, \xi, q, 0)$-complex is a $(0, \xi, q, 0)$-complex.*

Definition 3.3. Let G be a complex. We say that G is an (ε, ξ, q, r)-*supercomplex* if for every $i \in [r]_0$ and every set $F \subseteq G^{(i)}$ with $1 \leq |F| \leq 2^i$, we have that $\bigcap_{f \in F} G(f)$ is an $(\varepsilon, \xi, q-i, r-i)$-complex.

In particular, taking $i = 0$ and $F = \{\emptyset\}$ implies that every (ε, ξ, q, r)-supercomplex is also an (ε, ξ, q, r)-complex. Moreover, the above definition ensures that if G is a supercomplex and $S, S' \in G^{(i)}$, then $G(S) \cap G(S')$ is also a supercomplex (cf. Proposition 5.1). This is crucial for the construction of our absorbers in Section 8 and is the reason why we consider (ε, ξ, q, r)-supercomplexes rather than (ε, ξ, q, r)-complexes.

In the next subsection, we will give some examples of supercomplexes. Note also that the parameters ε and ξ are monotone in that whenever $\varepsilon' \geq \varepsilon$ and $\xi' \leq \xi$, every (ε, ξ, q, r)-supercomplex is also an $(\varepsilon', \xi', q, r)$-supercomplex. The next lemma shows that we can in fact significantly improve on ε (make it smaller) if ξ is allowed to decrease as well. We call this 'boosting' (see Section 6.2 for the proof). We achieve this by restricting each $\bigcap_{f\in F} G(f)^{(q-i)}$ to a suitable $(q-i)$-graph Y_F (independently of each other), as permitted in the definition of an (ε,ξ,q,r)-complex.

Lemma 3.4. *Let $1/n \ll \varepsilon,\xi,1/q$ and $r\in[q-1]$ with $2(2\sqrt{\mathrm{e}})^r\varepsilon \le \xi$. Let $\xi' := 0.9(1/4)^{\binom{q+r}{q}}\xi$. If G is an (ε,ξ,q,r)-complex on n vertices, then G is an $(n^{-1/3},\xi',q,r)$-complex. In particular, if G is an (ε,ξ,q,r)-supercomplex, then it is a $(2n^{-1/3},\xi',q,r)$-supercomplex.*

3.2. **The main theorem.** Before we can state the main result of this paper (Theorem 3.7), we first need to define the notions of $K_q^{(r)}$-decomposition and $K_q^{(r)}$-divisibility for complexes.

Definition 3.5. Let G be a complex. A $K_q^{(r)}$*-packing in G* is a subcomplex $\mathcal{K}\subseteq G$ for which the following hold:

- $\mathcal{K}$ is generated by some $Y\subseteq G^{(q)}$, that is $\mathcal{K}=Y^{\le}$;
- for all $K,K'\in\mathcal{K}^{(q)}=Y$, we have $|K\cap K'|<r$.

A $K_q^{(r)}$*-decomposition of G* is a $K_q^{(r)}$-packing $\mathcal{K}$ in G with $\mathcal{K}^{(r)}=G^{(r)}$.

Note that a $K_q^{(r)}$-packing $\mathcal{K}$ in G can be viewed as a $K_q^{(r)}$-packing in $G^{(r)}$ (i.e. a collection of edge-disjoint copies of $K_q^{(r)}$ in $G^{(r)}$) with the additional property that the vertex set of every copy of $K_q^{(r)}$ in the packing belongs to $G^{(q)}$. Moreover, $\mathcal{K}^{(r)}$ is the r-subgraph of $G^{(r)}$ containing all covered edges (in the usual sense).

Definition 3.6. A complex G is called $K_q^{(r)}$*-divisible* if $G^{(r)}$ is $K_q^{(r)}$-divisible.

The following theorem is our main theorem, which we will prove by induction on r in Section 9.

Theorem 3.7 (Main theorem)**.** *For all $r\in\mathbb{N}$, the following is true.*

$(*)_r$ *Let $1/n\ll\varepsilon\ll\xi,1/q$, where $q>r$. Let G be a $K_q^{(r)}$-divisible (ε,ξ,q,r)-supercomplex on n vertices. Then G has a $K_q^{(r)}$-decomposition.*

Note that in light of Lemma 3.4, $(*)_r$ already holds if $\varepsilon\le\frac{\xi}{2(2\sqrt{\mathrm{e}})^r}$.

3.3. **Applications.** As the definition of a supercomplex covers a broad range of settings, we give some applications here. We will use Examples 3.8, 3.9 and 3.11 in Section 9 to prove Theorem 1.3 (and thus Theorem 1.1) as well as Theorems 1.4 and 1.5. We will also see that random subcomplexes of a supercomplex are again supercomplexes with appropriately adjusted parameters (see Corollary 5.16).

Example 3.8. Let $1/n\ll1/q$ and $r\in[q-1]$. It is straightforward to check that the complete complex K_n is a $(0,0.99/q!,q,r)$-supercomplex.

Recall that (c,h,p)-typicality was defined in Section 1.3.

Example 3.9 (Typicality)**.** Suppose that $1/n\ll c,p,1/q$, that $r\in[q-1]$ and that G is a $(c,2^r\binom{q+r}{r},p)$-typical r-graph on n vertices. Then $G^{\leftrightarrow}$ is an (ε,ξ,q,r)-supercomplex, where

$$\varepsilon := 2^{q-r+1}c/(q-r)! \quad\text{and}\quad \xi := (1-2^{q+1}c)p^{2^r\binom{q+r}{r}}/q!.$$

Proof. Let $i\in[r]_0$ and $F\subseteq G^{\leftrightarrow(i)}$ with $1\le|F|\le2^i$. Let $G_F := \bigcap_{f\in F} G^{\leftrightarrow}(f)$ and $n_F := |V(G)\setminus\bigcup F|$. Let $e\in G_F^{(r-i)}$. To estimate $|G_F^{(q-i)}(e)|$, we let $\mathcal{Q}_e$ be the set of ordered $(q-r)$-tuples $(v_1,\dots,v_{q-r})$ consisting of distinct vertices in $V(G)\setminus(e\cup\bigcup F)$ such that for all $f\in F$, $\binom{f\cup e\cup\{v_1,\dots,v_{q-r}\}}{r}\subseteq G$. Note that $|G_F^{(q-i)}(e)|=|\mathcal{Q}_e|/(q-r)!$. We estimate $|\mathcal{Q}_e|$ by picking $v_1,\dots,v_{q-r}$ sequentially. So let $j\in[q-r]$ and suppose that we have already chosen $v_1,\dots,v_{j-1}\notin e\cup\bigcup F$ such that $\binom{f\cup e\cup\{v_1,\dots,v_{j-1}\}}{r}\subseteq G$ for all $f\in F$. Let $D_j=\bigcup_{f\in F}\binom{f\cup e\cup\{v_1,\dots,v_{j-1}\}}{r-1}$. Thus the possible candidates for v_j are precisely the vertices in

$\bigcap_{S\in D_j} G(S)$. Note that $d_j := |D_j| \le |F|\binom{r+j-1}{r-1}$, and that d_j only depends on the intersection pattern of the $f \in F$, but not on our previous choice of e and $v_1, \dots, v_{j-1}$. Since G is typical, we have $(1 \pm c)p^{d_j} n$ choices for v_j. We conclude that

$$|\mathcal{Q}_e| = (1 \pm c)^{q-r} p^{\sum_{j=1}^{q-r} d_j} n^{q-r} = (1 \pm 2^{q-r+1}c) d_F (q-r)! n_F^{q-r},$$

where $d_F := p^{\sum_{j=1}^{q-r} d_j}/(q-r)!$. Thus, G_F is $(2^{q-r+1}cd_F, d_F, q-i, r-i)$-regular. Since $\sum_{j=1}^{q-r}\binom{r+j-1}{r-1} = \binom{q}{r} - 1$ we have $1/(q-r)! \ge d_F \ge p^{|F|(\binom{q}{r}-1)}/(q-r)! \ge p^{2^r\binom{q}{r}}/(q-r)!$. Similarly, we deduce that G_F is $((1-2^{q-r+1}c)d_F, q-i, r-i)$-extendable. Moreover, we have

$$|G_F^{(q+r-2i)}(e)| \ge \frac{(1-2^{q-i+1}c)p^{2^r\binom{q+r-i}{r}}}{(q-i)!} n_F^{q-i} \ge \xi n_F^{q-i}.$$

Thus, G_F is $(\xi, q+r-2i, r-i)$-dense. We conclude that G_F is an $(\varepsilon, \xi, q-i, r-i)$-complex. □

Example 3.10 (Partite graphs)**.** Let $1/N \ll 1/k$ and $2 = r < q \le k-6$. Let $V_1, \dots, V_k$ be vertex sets of size N each. Let G be the complete k-partite 2-graph on $V_1, \dots, V_k$. It is straightforward to check that $G^{\leftrightarrow}$ is a $(0, k^{-q}, q, 2)$-supercomplex. Thus, using Theorem 3.7, we can deduce that G has a $K_q^{(2)}$-decomposition if it is $K_q^{(2)}$-divisible. To obtain a resilience version (and thus also a minimum degree version) along the lines of Theorems 1.3 and 1.1, one can argue similarly as in the proof of Theorem 1.3 (cf. Section 9).

Results on (fractional) decompositions of dense q-partite 2-graphs into q-cliques are proved in [6, 7, 30]. These have applications to the completion of partial (mutually orthogonal) Latin squares.

Example 3.11 (The matching case)**.** Consider $1 = r < q$. Let G be a q-graph on n vertices such that the following conditions hold for some $0 < \varepsilon \le \xi \le 1$:

- for some $d \ge \xi - \varepsilon$, $|G(v)| = (d \pm \varepsilon)n^{q-1}$ for all $v \in V(G)$;
- every vertex is contained in at least ξn^q copies of $K_{q+1}^{(q)}$;
- $|G(v) \cap G(w)| \ge \xi n^{q-1}$ for all $v, w \in V(G)$.

Then $G^{\leftrightarrow}$ is an $(\varepsilon, \xi - \varepsilon, q, 1)$-supercomplex.

3.4. **Designs.** Recall that a $K_q^{(r)}$-decomposition of an r-graph is a $(q, r, 1)$-design. We now discuss consequences of our main theorem for general (q, r, λ)-designs. We can deduce from Theorem 3.7 that there are many q-disjoint $K_q^{(r)}$-decompositions (which we will also require during our induction proof). Clearly, any complex G on n vertices can have at most $n^{q-r}/(q-r)!$ q-disjoint $K_q^{(r)}$-decompositions.

Proposition 3.12. *Let $1/n \ll \varepsilon, \xi, 1/q$ and $r \in [q-1]$. Suppose that G is an (ε, ξ, q, r)-supercomplex on n vertices and $G \subseteq \tilde{G}$. Suppose that $\mathcal{K}_1, \dots, \mathcal{K}_t$ are $K_q^{(r)}$-packings in $\tilde{G}$, where $t \le \varepsilon n^{q-r}$. Then $G - \bigcup_{j\in[t]} \mathcal{K}_j^{(q)}$ is a $(2^{r+2}\varepsilon, \xi - 2^{2r+1}\varepsilon, q, r)$-supercomplex.*

Proof. Let $Y_{used} := \bigcup_{j\in[t]} \mathcal{K}_j^{(q)}$. Fix $i \in [r]_0$ and $F \subseteq G^{(i)}$ with $1 \le |F| \le 2^i$. Let $n_F := n - |\bigcup F|$, $G' := \bigcap_{f\in F} G(f)$ and $G'' := \bigcap_{f\in F}(G - Y_{used})(f)$. By assumption, there exists $Y \subseteq G'^{(q-i)}$ such that $G'[Y]$ is a full $(\varepsilon, \xi, q-i, r-i)$-complex. We claim that $G''[Y]$ is a full $(2^{r+2}\varepsilon, \xi - 2^{2r+1}\varepsilon, q-i, r-i)$-complex.

First, there is some $d \ge \xi$ such that $G'[Y]$ is $(\varepsilon, d, q-i, r-i)$-regular. Let $e \in G'^{(r-i)}$. We clearly have $|G''[Y]^{(q-i)}(e)| \le |G'[Y]^{(q-i)}(e)| \le (d+\varepsilon)n_F^{q-r}$. Moreover, for each $j \in [t]$ and $f \in F$, there is at most one q-set in $\mathcal{K}_j^{(q)}$ that contains $e \cup f$. Thus, $|G''[Y]^{(q-i)}(e)| \ge (d-\varepsilon)n_F^{q-r} - |F|t \ge (d - \varepsilon - 1.1 \cdot 2^i\varepsilon)n_F^{q-r}$. Thus, $G''[Y]$ is $(2^{r+2}\varepsilon, d, q-i, r-i)$-regular.

Next, by assumption we have that $G'[Y]$ is $(\xi, q+r-2i, r-i)$-dense. Let $e \in G'^{(r-i)}$. Fix $j \in [t]$ and $f \in F$. We claim that the number N of $(q+r-i)$-sets in $V(G)$ that contain $e \cup f$

and also contain some q-set from $\mathcal{K}_j^{(q)}$ is at most $2^r n^{r-i}$. Indeed, for any $k \in [r]_0$ and any $K \in \mathcal{K}_j^{(q)}$ with $|(e \cup f) \cap K| = k$, there are at most n^{k-i} $(q+r-i)$-sets that contain $e \cup f$ and K. Moreover, there are at most $\binom{r}{k} n^{r-k}$ q-sets $K \in \mathcal{K}_j^{(q)}$ with $|(e \cup f) \cap K| = k$ since $\mathcal{K}_j^{(q)}$ covers every r-set at most once. So $N \leq \sum_{k=0}^{r} n^{k-i} \binom{r}{k} n^{r-k} = 2^r n^{r-i}$. We then deduce that

$$|G''[Y]^{(q+r-2i)}(e)| \geq \xi n_F^{q-i} - t|F| 2^r n^{r-i} \geq \xi n_F^{q-i} - \varepsilon 2^{r+i} n^{q-i} \geq (\xi - 2^{2r+1}\varepsilon) n_F^{q-i}.$$

Finally, since $G''[Y]^{(r-i)} = G'[Y]^{(r-i)}$, $G''[Y]$ is $(\xi, q-i, r-i)$-extendable. Thus, $G - Y_{used}$ is a $(2^{r+2}\varepsilon, \xi - 2^{2r+1}\varepsilon, q, r)$-supercomplex. □

Note that if a complex G has λ q-disjoint $K_q^{(r)}$-decompositions, then $G^{(r)}$ has a (q, r, λ)-design.

Corollary 3.13. *Let $1/n \ll \varepsilon, \xi, 1/q$ and $r \in [q-1]$ with $10 \cdot 7^r \varepsilon \leq \xi$ and assume that $(*)_r$ is true. Suppose that G is a $K_q^{(r)}$-divisible (ε, ξ, q, r)-supercomplex on n vertices. Then G has εn^{q-r} q-disjoint $K_q^{(r)}$-decompositions. In particular, $G^{(r)}$ has a (q, r, λ)-design for all $1 \leq \lambda \leq \varepsilon n^{q-r}$.*

Proof. Suppose that $\mathcal{K}_1, \dots, \mathcal{K}_t$ are q-disjoint $K_q^{(r)}$-decompositions of G, where $t \leq \varepsilon n^{q-r}$. By Proposition 3.12, $G - \bigcup_{j \in [t]} \mathcal{K}_j^{(q)}$ is a $(2^{r+2}\varepsilon, \xi - 2^{2r+1}\varepsilon, q, r)$-supercomplex. Since $2(2\sqrt{\mathrm{e}})^r 2^{r+2}\varepsilon \leq \xi - 2^{2r+1}\varepsilon$, $G - \bigcup_{j \in [t]} \mathcal{K}_j^{(q)}$ has a $K_q^{(r)}$-decomposition $\mathcal{K}_{t+1}$ by (the remark after) $(*)_r$, which is q-disjoint from $\mathcal{K}_1, \dots, \mathcal{K}_t$. □

Note that Corollary 3.13 together with Example 3.8 implies that whenever $1/n \ll 1/q$ and $K_n^{(r)}$ is $K_q^{(r)}$-divisible, then $K_n^{(r)}$ has a (q, r, λ)-design for all $1 \leq \lambda \leq \frac{1}{11 \cdot 7^r q!} n^{q-r}$, which improves the bound $\lambda / n^{q-r} \ll 1$ in [20].

On the other hand, note that $G^{(r)}$ being (q, r, λ)-divisible does not imply that $G^{(r)}$ is $(q, r, 1)$-divisible. Thus, we cannot directly apply our main theorem to a (q, r, λ)-divisible graph to obtain a (q, r, λ)-design. Nevertheless, by applying our main theorem to a number of suitable subgraphs, we can deduce the following theorem (see Section 9 for the proof).

Theorem 3.14. *Let $1/n \ll \varepsilon, \xi, 1/q, 1/\lambda$ and $r \in [q-1]$ such that $2(2\sqrt{\mathrm{e}})^r \varepsilon \leq \xi$. Let G be an (ε, ξ, q, r)-supercomplex on n vertices. If $G^{(r)}$ is (q, r, λ)-divisible then $G^{(r)}$ has a (q, r, λ)-design.*

4. Outline of the methods

Rather than an algebraic approach as in [20], we pursue a combinatorial approach based on 'iterative absorption'. In particular, we do not make use of any nontrivial algebraic techniques and results, but rely only on probabilistic tools.

4.1. **Iterative absorption.** Suppose that we aim to find a $K_q^{(r)}$-decomposition of a suitable complex G. The Rödl nibble (see e.g. [1, 32, 34, 41]) allows us to obtain an approximate $K_q^{(r)}$-decomposition of G, i.e. a set of r-disjoint q-sets covering almost all r-edges of G. However, one has little control over the resulting uncovered leftover set. The basic aim of an absorbing approach is to overcome this issue by removing an absorbing structure A right at the beginning and then applying the Rödl nibble to $G - A$, to obtain an approximate decomposition with a very small uncovered remainder R. Ideally, A was chosen in such a way that $A \cup R$ has a $K_q^{(r)}$-decomposition.

Such an approach was introduced systematically by Rödl, Ruciński and Szemerédi [36] in order to find spanning structures in graphs and hypergraphs (but actually goes back further than this, see e.g. Krivelevich [25]). In the context of decompositions, the first results based on an absorbing approach were obtained in [23, 26]. In contrast to the construction of spanning subgraphs, the decomposition setting gives rise to the additional challenge that the number of and possible shape of uncovered remainder graphs R is comparatively large. So in general it is much less clear how to construct a structure A which can deal with all such possibilities for R (to appreciate this issue, note that $V(R) = V(G)$ in this scenario).

The method developed in [23, 26] consisted of an iterative approach: each iteration consists of an approximate decomposition of the previous leftover, together with a partial absorption (or 'cleaning') step, which further restricts the structure of the current leftover. In our context, we carry out this iteration by considering a 'vortex'. Such a vortex is a nested sequence $V(G) = U_0 \supseteq U_1 \supseteq \cdots \supseteq U_\ell$, where $|U_i|/|U_{i+1}|$ and $|U_\ell|$ are large but bounded. Crucially, after the ith iteration, all r-edges belonging to the current leftover R_i will be induced by U_i. In the $(i+1)$th iteration, we make use of a suitable r-graph H_i on U_i which we set aside at the start. We first apply the Rödl nibble to R_i to obtain a sparse remainder R'_i. We then apply what we refer to as the 'Cover down lemma' to find a $K_q^{(r)}$-packing $\mathcal{K}_i$ of $H_i \cup R'_i$ so that the remainder R_{i+1} consists entirely of r-edges induced by U_{i+1} (see Lemma 7.4). Ultimately, we arrive at a leftover R_ℓ induced by U_ℓ.

Since $|U_\ell|$ is bounded, this means there are only a bounded number of possibilities $S_1, \dots, S_b$ for R_ℓ. This gives a natural approach to the construction of an absorber A for R_ℓ: it suffices to construct an 'exclusive' absorber A_i for each S_i (in the sense that A_i can absorb S_i but nothing else). More precisely, we aim to construct edge-disjoint r-graphs $A_1, \dots, A_b$ so that both A_i and $A_i \cup S_i$ have a $K_q^{(r)}$-decomposition, and then let $A := A_1 \cup \cdots \cup A_b$. Then $A \cup R_\ell$ must also have $K_q^{(r)}$-decomposition.

Iterative absorption based on vortices was introduced in [14], building on a related (but more complicated approach) in [5]. Developing the above approach in the setting of hypergraph decompositions gives rise to two main challenges: constructing the 'exclusive' absorbers and proving the Cover down lemma, which we discuss in the next two subsections, respectively.

One difficulty with the iteration process is that after finishing one iteration, the error terms are too large to carry out the next one. Fortunately, we are able to 'boost' our regularity parameters before each iteration by excluding suitable q-sets from future consideration (see Lemma 6.3). For this, we adopt gadgets introduced in [4]. This 'boosting step' is the reason for introducing the 'density' requirement in the definition of a supercomplex. Moreover, the 'Boost lemma' enables us to obtain explicit bounds e.g. in the minimum degree version (Theorem 1.1).

4.2. **The Cover down lemma.** For simplicity, write U' for U_i and U for U_{i+1}. As indicated above, the goal here is as follows: Given a complex G and a vertex set U' in G, we need to construct H^* in $G[U']^{(r)}$ so that for any sparse leftover R on U', we can find a $K_q^{(r)}$-packing in $G[H^* \cup R]$ where any leftover edges lie in U. (In addition, we need to ensure that the distribution of the leftover edges within U is sufficiently well-behaved so that we can continue with the next iteration, but we do not discuss this aspect here.)

We achieve this goal in several stages: given an edge $e \in H^* \cup R$, we refer to the size of its intersection with U as its type. We first cover all edges of type 0. This can be done using an appropriate greedy approach, i.e. for each edge e of type 0 in turn, we extend e to a copy of $K_q^{(r)}$ using edges of H^* (this works if H^* is a suitable random subgraph of G consisting of edges of nonzero type).

Suppose now that for some $i \in [r-1]$, we have covered all edges of type at most $r-i-1$. To cover the edges of type $r-i$, we consider each i-tuple e of vertices outside U in turn. We now need to find a $K_{q-i}^{(r-i)}$-decomposition $\mathcal{K}_e$ of $H_e := G[H^* \cup R](e)$. (Note that H_e lies in U as we have already covered all edges of type at most $r-i-1$.) Then $\mathcal{K}_e$ corresponds to a $K_q^{(r)}$-packing which covers all leftover edges (of type $r-i$) containing e, see Fact 10.1. (For example, consider the triangle case $q = 3$ and $r = 2$, and suppose $i = 1$. Then e can be viewed as a vertex and $\mathcal{K}_e$ corresponds to a perfect matching on the neighbours of e in U. This yields a triangle packing which covers all edges incident to e.) Inductively, one can use $(*)_{r-i}$ to show that such a decomposition $\mathcal{K}_e$ does exist if we choose H^* appropriately. However, the problem is that we cannot just select these decompositions greedily for successive i-tuples e. Since different H_e overlap, the choice of each $\mathcal{K}_e$ restricts the choices of subsequent $\mathcal{K}_{e'}$ in $H_{e'}$ to such an extent that we cannot apply induction to the (leftover of) subsequent $H_{e'}$ anymore.

Our solution is to split this step of covering the r-edges of type $r-i$ at e into several substeps. We cover a suitable subset of the r-edges of type $r-i$ directly using a probabilistic choice of a suitable $K_{q-i}^{(r-i)}$-decomposition (whose vertex set is some small subset $U_e \subseteq U$). We cover the remaining r-edges of type $r-i$ using an inductive approach (where the induction is on $r-i$). The resulting proof of the Cover down lemma is given in Section 10 (which also includes a more detailed sketch of this part of the argument).

4.3. **Transformers and absorbers.** Recall that our remaining goal is to construct an exclusive absorber A_S for a given 'leftover' r-graph S of bounded size. In other words, both $A_S \cup S$ as well as A_S need to have a $K_q^{(r)}$-decomposition. Clearly, we must (and can) assume that S is $K_q^{(r)}$-divisible.

Based on an idea introduced in [5], we will construct A_S as a concatenation of 'transformers': given S, a transformer T_S can be viewed as transforming S into a new leftover L (which has the same number of edges and is still divisible). Formally, we require that $S \cup T_S$ and $T_S \cup L$ both have a $K_q^{(r)}$-decomposition (and will set aside T_S and L at the beginning of the proof). Since transformers act transitively, the idea is to concatenate them in order to transform S into a vertex-disjoint union of $K_q^{(r)}$, i.e. we gradually transform the given leftover S into a graph which is trivially decomposable.

Roughly speaking, we approach this by choosing L to be a suitable 'canonical' graph (i.e. L only depends on $|S|$). Let S' denote the vertex-disjoint union of copies of $K_q^{(r)}$ such that $|S| = |S'|$, and let $T_{S'}$ be the corresponding transformer from S' into L. Then it is easy to see that we could let $A_S := T_S \cup L \cup T_{S'} \cup S'$. The construction of both the canonical graph L as well as that of the transformer T_S is based on an inductive approach, i.e. we assume that $(*)_1$–$(*)_{r-1}$ hold in Theorem 3.7. Moreover, the construction of the canonical graph L is the point where we need the extendability property in the definition of a supercomplex. The above construction is given in Section 8.

5. Tools

5.1. **Basic tools.** We first state two basic properties of supercomplexes that we will use in Section 8 to construct absorbers.

Proposition 5.1. *Let G be an (ε, ξ, q, r)-supercomplex and let $F \subseteq G^{(i)}$ with $1 \leq |F| \leq 2^i$ for some $i \in [r]_0$. Then $\bigcap_{f \in F} G(f)$ is an $(\varepsilon, \xi, q-i, r-i)$-supercomplex.*

Proof. Let $i' \in [r-i]_0$ and $F' \subseteq (\bigcap_{f\in F} G(f))^{(i')}$ with $1 \leq |F'| \leq 2^{i'}$. Let $F^* := \{f \cup f' : f \in F, f' \in F'\}$. Note that $F^* \subseteq G^{(i+i')}$ and $|F^*| \leq 2^{i+i'}$. Thus,

$$\bigcap_{f' \in F'} (\bigcap_{f \in F} G(f))(f') = \bigcap_{f^* \in F^*} G(f^*)$$

is an $(\varepsilon, \xi, q-i-i', r-i-i')$-complex by Definition 3.3, as required. □

Fact 5.2. *If G is an (ε, ξ, q, r)-supercomplex, then for all distinct $e, e' \in G^{(r)}$, we have $|G^{(q)}(e) \cap G^{(q)}(e')| \geq (\xi - \varepsilon)(n-2r)^{q-r}$.*

In what follows, we gather tools that show that supercomplexes are robust with respect to small perturbations. We first bound the number of q-sets that can affect a given edge e. We provide two bounds, one that we use when optimising our bounds (e.g. in the derivation of Theorem 1.1) and a more convenient one that we use when the precise value of the parameters is irrelevant (e.g. in the proof of Proposition 5.6).

Fact 5.3. *Let L be an r-graph on n vertices with $\Delta(L) \leq \gamma n$. Then for each $i \in [r-1]_0$, we have $\Delta_i(L) \leq \gamma n^{r-i}/(r-i)!$, and for each $S \in \binom{V(L)}{i}$, we have $\Delta(L(S)) \leq \gamma n$.*

Proposition 5.4. *Let $q,r'\in\mathbb{N}$ and $r\in\mathbb{N}_0$ with $q>r$. Let L be an r'-graph on n vertices with $\Delta(L)\le\gamma n$. Then every $e\in\binom{V(L)}{r}$ that does not contain any edge of L is contained in at most $\min\{2^r,\frac{\binom{q}{r'}}{(q-r)!}\}\gamma n^{q-r}$ q-sets of $V(L)$ that contain an edge of L.*

Proof. Consider any $e\in\binom{V(L)}{r}$ that does not contain any edge of L. For a fixed edge $e'\in L$ with $|e\cup e'|\le q$ and $|e\cap e'|=i$, there are at most $\binom{n-|e\cup e'|}{q-|e\cup e'|}\le n^{q-r-r'+i}/(q-r-r'+i)!$ q-sets of $V(L)$ that contain both e and e'. Moreover, since $e'\not\subseteq e$, we have $i<r'$. Hence, by Fact 5.3, there are at most $\binom{r}{i}\Delta_i(L)\le\binom{r}{i}\gamma n^{r'-i}/(r'-i)!$ edges $e'\in L$ with $|e\cap e'|=i$. Let $s:=\max\{r+r'-q,0\}$. Thus, the number of q-sets in $V(L)$ that contain e and an edge of L is at most

$$\sum_{i=s}^{r'-1}\gamma\binom{r}{i}\frac{n^{r'-i}}{(r'-i)!}\frac{n^{q-r-r'+i}}{(q-r-r'+i)!}=\gamma n^{q-r}\sum_{i=s}^{r'-1}\binom{r}{i}\frac{\binom{q-r}{r'-i}}{(q-r)!}.$$

Clearly, $\frac{\binom{q-r}{r'-i}}{(q-r)!}\le 1$, and we can bound $\sum_{i=s}^{r'-1}\binom{r}{i}\le 2^r$. Also, using Vandermonde's convolution, we have $\sum_{i=s}^{r'-1}\binom{r}{i}\frac{\binom{q-r}{r'-i}}{(q-r)!}\le\frac{\binom{q}{r'}}{(q-r)!}$. □

Fact 5.5. *Let $0\le i\le r$. For a complex G, an r-graph H and $F\subseteq G^{(i)}$, we have*

$$\bigcap_{f\in F}(G-H)(f)=\bigcap_{f\in F}G(f)-H-\bigcup_{S\in\bigcup F}H(S)-\bigcup_{S\in\bigcup_{f\in F}\binom{f}{2}}H(S)-\dots-\bigcup_{f\in F}H(f).$$

If $F\not\subseteq(G-H)^{(i)}$, then both sides are empty.

Proposition 5.6. *Let $q,r'\in\mathbb{N}$ and $r\in\mathbb{N}_0$ with $q>r$ and $r'\ge r$. Let G be a complex on $n\ge r2^{r+1}$ vertices and let H be an r'-graph on $V(G)$ with $\Delta(H)\le\gamma n$. Then the following hold:*

(i) *If G is (ε,d,q,r)-regular, then $G-H$ is $(\varepsilon+2^r\gamma,d,q,r)$-regular.*
(ii) *If G is (ξ,q,r)-dense, then $G-H$ is $(\xi-2^r\gamma,q,r)$-dense.*
(iii) *If G is (ξ,q,r)-extendable, then $G-H$ is $(\xi-2^r\gamma,q,r)$-extendable.*
(iv) *If G is an (ε,ξ,q,r)-complex, then $G-H$ is an $(\varepsilon+2^r\gamma,\xi-2^r\gamma,q,r)$-complex.*
(v) *If G is an (ε,ξ,q,r)-supercomplex, then $G-H$ is an $(\varepsilon+2^{2r+1}\gamma,\xi-2^{2r+1}\gamma,q,r)$-supercomplex.*

Proof. (i)–(iii) follow directly from Proposition 5.4. (iv) follows from (i)–(iii). To see (v), suppose that $i\in[r]_0$ and $F\subseteq(G-H)^{(i)}$ with $1\le|F|\le 2^i$. By assumption, $\bigcap_{f\in F}G(f)$ is an $(\varepsilon,\xi,q-i,r-i)$-complex. By Fact 5.5, we can obtain $\bigcap_{f\in F}(G-H)(f)$ from $\bigcap_{f\in F}G(f)$ by repeatedly deleting an $(r'-|S|)$-graph $H(S)$, where $S\subseteq f\in F$. There are at most $|F|2^i\le 2^{2i}$ such graphs. Unless $|S|=r'$, we have $\Delta(H(S))\le\gamma n\le 2\gamma(n-|\bigcup F|)$ by Fact 5.3. Note that if $|S|=r'$, then $S\in F$ and hence $H(S)$ is empty, in which case we can ignore its removal. Thus, a repeated application of (iv) (with $r'-|S|,r-i$ playing the roles of r',r) shows that $\bigcap_{f\in F}(G-H)(f)$ is an $(\varepsilon+2^{r+i+1}\gamma,\xi-2^{r+i+1}\gamma,q-i,r-i)$-complex. □

5.2. **Probabilistic tools.** The following Chernoff-type bounds form the basis of our concentration results that we use for probabilistic arguments.

Lemma 5.7 (see [18, Corollary 2.3, Corollary 2.4, Remark 2.5 and Theorem 2.8])**.** *Let X be the sum of n independent Bernoulli random variables. Then the following hold.*

(i) *For all $t\ge 0$, $\mathbb{P}(|X-\mathbb{E}X|\ge t)\le 2\mathrm{e}^{-2t^2/n}$.*
(ii) *For all $0\le\varepsilon\le 3/2$, $\mathbb{P}(|X-\mathbb{E}X|\ge\varepsilon\mathbb{E}X)\le 2\mathrm{e}^{-\varepsilon^2\mathbb{E}X/3}$.*
(iii) *If $t\ge 7\mathbb{E}X$, then $\mathbb{P}(X\ge t)\le\mathrm{e}^{-t}$.*

We will also use the following simple result.

Proposition 5.8 (Jain, see [33, Lemma 8])**.** *Let $X_1, \dots, X_n$ be Bernoulli random variables such that, for any $i \in [n]$ and any $x_1, \dots, x_{i-1} \in \{0,1\}$,*

$$\mathbb{P}(X_i = 1 \mid X_1 = x_1, \dots, X_{i-1} = x_{i-1}) \leq p.$$

Let $B \sim B(n,p)$ and $X := X_1 + \cdots + X_n$. Then $\mathbb{P}(X \geq a) \leq \mathbb{P}(B \geq a)$ for any $a \geq 0$.

Lemma 5.9. *Let $1/n \ll p, \alpha, 1/a, 1/B$. Let $\mathcal{I}$ be a set of size αn^a and let $(X_i)_{i\in\mathcal{I}}$ be a family of Bernoulli random variables with $\mathbb{P}(X_i = 1) \geq p$. Suppose that $\mathcal{I}$ can be partitioned into at most Bn^{a-1} sets $\mathcal{I}_1, \dots, \mathcal{I}_k$ such that for each $j \in [k]$, the variables $(X_i)_{i\in\mathcal{I}_j}$ are independent. Let $X := \sum_{i\in\mathcal{I}} X_i$. Then we have*

$$\mathbb{P}(|X - \mathbb{E}X| \geq n^{-1/5}\mathbb{E}X) \leq \mathrm{e}^{-n^{1/6}}.$$

Proof. Let $\mathcal{J}_1 := \{j \in [k] : |\mathcal{I}_j| \geq n^{3/5}\}$ and $\mathcal{J}_2 := [k] \setminus \mathcal{J}_1$. Let $Y_j := \sum_{i\in\mathcal{I}_j} X_i$ and $\varepsilon := n^{-1/5}$. Suppose that $|Y_j - \mathbb{E}Y_j| \leq 0.9\varepsilon\mathbb{E}Y_j$ for all $j \in \mathcal{J}_1$. Then

$$|X - \mathbb{E}X| \leq \sum_{j\in[k]} |Y_j - \mathbb{E}Y_j| \leq n^{3/5} \cdot Bn^{a-1} + \sum_{j\in\mathcal{J}_1} 0.9\varepsilon\mathbb{E}Y_j \leq Bn^{a-2/5} + 0.9\varepsilon\mathbb{E}X \leq \varepsilon\mathbb{E}X.$$

Thus,

$$\begin{aligned}\mathbb{P}(|X - \mathbb{E}X| \geq \varepsilon\mathbb{E}X) &\leq \sum_{j\in\mathcal{J}_1} \mathbb{P}(|Y_j - \mathbb{E}Y_j| \geq 0.9\varepsilon\mathbb{E}Y_j) \overset{\text{Lemma 5.7(ii)}}{\leq} \sum_{j\in\mathcal{J}_1} 2\mathrm{e}^{-0.81\varepsilon^2\mathbb{E}Y_j/3} \\ &\leq 2Bn^{a-1}\mathrm{e}^{-0.27n^{-2/5}pn^{3/5}} \leq \mathrm{e}^{-n^{1/6}}.\end{aligned}$$

□

Similarly as in [17], Lemma 5.9 can be conveniently applied in the following situation: We are given an r-graph H on n vertices and H' is a random subgraph of H, where every edge of H survives with some probability $\geq p$. The following folklore observation allows us to apply Lemma 5.9 in order to obtain a concentration result for $|H'|$.

Fact 5.10. *Every r-graph on n vertices can be decomposed into rn^{r-1} matchings.*

Corollary 5.11. *Let $1/n \ll p, 1/r, \alpha$. Let H be an r-graph on n vertices with $|H| \geq \alpha n^r$. Let H' be a random subgraph of H, where each edge of H survives with some probability $\geq p$. Moreover, suppose that for every matching M in H, the edges of M survive independently. Then we have*

$$\mathbb{P}(||H'| - \mathbb{E}|H'|| \geq n^{-1/5}\mathbb{E}|H'|) \leq \mathrm{e}^{-n^{1/6}}.$$

Whenever we apply Corollary 5.11, it will be clear that for every matching M in H, the edges of M survive independently, and we will not discuss this explicitly.

Lemma 5.12. *Let $1/n \ll p, 1/r$. Let H be an r-graph on n vertices. Let H' be a random subgraph of H, where each edge of H survives with some probability $\leq p$. Suppose that for every matching M in H, the edges of M survive independently. Then we have*

$$\mathbb{P}(|H'| \geq 7pn^r) \leq rn^{r-1}\mathrm{e}^{-7pn/r}.$$

Proof. Partition H into at most rn^{r-1} matchings $M_1, \dots, M_k$. For each $i \in [k]$, by Lemma 5.7(iii) we have $\mathbb{P}(|H' \cap M_i| \geq 7pn/r) \leq \mathrm{e}^{-7pn/r}$ since $\mathbb{E}|H' \cap M_i| \leq pn/r$. □

5.3. **Random subsets and subgraphs.** In this subsection, we apply the above tools to obtain basic results about random subcomplexes. The first one deals with taking a random subset of the vertex set, and the second one considers the complex obtained by randomly sparsifying $G^{(r)}$.

Proposition 5.13. *Let $1/n \ll \varepsilon, \xi, 1/q$ and $1/n \ll \gamma \ll \mu, 1/q$ and $r \in [q-1]_0$. Let G be an (ε, ξ, q, r)-complex on n vertices. Suppose that U is a random subset of $V(G)$ obtained by including every vertex from $V(G)$ independently with probability μ. Then with probability at least $1 - \mathrm{e}^{-n^{1/7}}$, the following holds: for any $W \subseteq V(G)$ with $|W| \le \gamma n$, $G[U \triangle W]$ is an $(\varepsilon + 2n^{-1/5} + \tilde{\gamma}^{2/3}, \xi - n^{-1/5} - \tilde{\gamma}^{2/3}, q, r)$-complex, where $\tilde{\gamma} := \max\{|W|/n, n^{-1/3}\}$.*

Proof. If $G^{(r)}$ is empty, there is nothing to prove, so assume the contrary.

By assumption, there exists $Y \subseteq G^{(q)}$ such that $G[Y]$ is (ε, d, q, r)-regular for some $d \ge \xi$, $(\xi, q+r, r)$-dense and (ξ, q, r)-extendable. The latter implies that there exists $X \subseteq V(G)$ with $|X| \ge \xi n$ such that for all $e \in \binom{X}{r}$, we have $|Ext_e| \ge \xi n^{q-r}$, where Ext_e is the set of all $(q-r)$-sets $Q \subseteq V(G) \setminus e$ such that $\binom{Q \cup e}{r} \setminus \{e\} \subseteq G^{(r)}$.

First, by Lemma 5.7(i), with probability at least $1 - 2\mathrm{e}^{-2n^{1/3}}$, we have $|U| = \mu n \pm n^{2/3}$, and with probability at least $1 - 2\mathrm{e}^{-2n^{1/4}}$, $|X \cap U| \ge \mu|X| - |X|^{2/3}$.

Claim 1: For all $e \in G^{(r)}$, with probability at least $1 - \mathrm{e}^{-n^{1/6}}$, $|G[Y]^{(q)}(e)[U]| = (d \pm (\varepsilon + 2n^{-1/5}))(\mu n)^{q-r}$.

Proof of claim: Fix $e \in G^{(r)}$. Note that $\mathbb{E}|G[Y]^{(q)}(e)[U]| = \mu^{q-r}|G[Y]^{(q)}(e)| = (d \pm \varepsilon)(\mu n)^{q-r}$. Viewing $G[Y]^{(q)}(e)$ as a $(q-r)$-graph and $G[Y]^{(q)}(e)[U]$ as a random subgraph, we deduce with Corollary 5.11 that

$$\mathbb{P}(|G[Y]^{(q)}(e)[U]| \neq (1 \pm n^{-1/5})(d \pm \varepsilon)(\mu n)^{q-r}) \le \mathrm{e}^{-n^{1/6}}.$$

–

Claim 2: For all $e \in G^{(r)}$, with probability at least $1 - \mathrm{e}^{-n^{1/6}}$, $|G[Y]^{(q+r)}(e)[U]| \ge (\xi - n^{-1/5})(\mu n)^{q}$.

Proof of claim: Note that $\mathbb{E}|G^{(q+r)}(e)[U]| = \mu^q|G^{(q+r)}(e)| \ge \xi(\mu n)^q$. Viewing $G^{(q+r)}(e)$ as a q-graph and $G^{(q+r)}(e)[U]$ as a random subgraph, we deduce with Corollary 5.11 that

$$\mathbb{P}(|G^{(q+r)}(e)[U]| \le (1 - n^{-1/5})\xi(\mu n)^q) \le \mathrm{e}^{-n^{1/6}}.$$

–

For $e \in \binom{X}{r}$, let Ext'_e be the random subgraph of Ext_e containing all $Q \in Ext_e$ with $Q \subseteq U$.

Claim 3: For all $e \in \binom{X}{r}$, with probability at least $1 - \mathrm{e}^{-n^{1/6}}$, $|Ext'_e| \ge (\xi - n^{-1/5})(\mu n)^{q-r}$.

Proof of claim: Let $e \in \binom{X}{r}$. Note that $\mathbb{E}|Ext'_e| = \mu^{q-r}|Ext_e| \ge \xi(\mu n)^{q-r}$. Again, Corollary 5.11 implies that

$$\mathbb{P}(|Ext'_e| \le (1 - n^{-1/5})\xi(\mu n)^{q-r}) \le \mathrm{e}^{-n^{1/6}}.$$

–

Hence, a union bound yields that with probability at least $1 - \mathrm{e}^{-n^{1/7}}$, we have $|U| = \mu n \pm n^{2/3}$, $|X \cap U| \ge \mu|X| - |X|^{2/3}$ and the above claims hold for all relevant e simultaneously. Assume that this holds for some outcome U. We now deduce the desired result deterministically. Let $W \subseteq V(G)$ with $|W| \le \gamma n$. Define $G' := G[U \triangle W]$ and $n' := |U \triangle W|$. Note that $\mu n = (1 \pm 4\mu^{-1}\tilde{\gamma})n'$. For all $e \in G'^{(r)}$, we have

$$\begin{aligned}|G'[Y]^{(q)}(e)| = |G[Y]^{(q)}(e)[U]| \pm |W|n^{q-r-1} &= (d \pm (\varepsilon + 2n^{-1/5} + \frac{|W|}{\mu^{q-r}n}))(\mu n)^{q-r} \\ &= (d \pm (\varepsilon + 2n^{-1/5} + \mu^{-(q-r)}\tilde{\gamma}))(1 \pm 2^{q-r}4\mu^{-1}\tilde{\gamma})n'^{q-r} \\ &= (d \pm (\varepsilon + 2n^{-1/5} + \tilde{\gamma}^{2/3}))n'^{q-r}\end{aligned}$$

and

$$|G'[Y]^{(q+r)}(e)| \geq |G[Y]^{(q+r)}(e)[U]| - |W|n^{q-1} \geq (\xi - n^{-1/5} - \frac{|W|}{\mu^q n})(\mu n)^q$$
$$\geq (\xi - n^{-1/5} - \mu^{-q}\tilde{\gamma})(1 - 2^q 4\mu^{-1}\tilde{\gamma})n'^{q-r} \geq (\xi - n^{-1/5} - \tilde{\gamma}^{2/3})n'^{q-r},$$

so $G'[Y]$ is $(\varepsilon + 2n^{-1/5} + \tilde{\gamma}^{2/3}, d, q, r)$-regular and $(\xi - n^{-1/5} - \tilde{\gamma}^{2/3}, q+r, r)$-dense.

Finally, let $X' := (X \cap U) \setminus W$. Clearly, $X' \subseteq V(G')$ and $|X'| \geq (\xi - n^{-1/5} - \tilde{\gamma}^{2/3})n'$. Moreover, for every $e \in \binom{X'}{r}$, there are at least

$$|Ext'_e| - |W|n^{q-r-1} \geq (\xi - n^{-1/5} - \tilde{\gamma}^{2/3})n'^{q-r}$$

$(q-r)$-sets $Q \subseteq V(G') \setminus e$ such that $\binom{Q \cup e}{r} \setminus \{e\} \subseteq G'^{(r)}$. Thus, G' (and therefore $G'[Y]$) is $(\xi - n^{-1/5} - \tilde{\gamma}^{2/3}, q, r)$-extendable. □

The next result is a straightforward consequence of Proposition 5.13 and the definition of a supercomplex.

Corollary 5.14. *Let $1/n \ll \gamma \ll \mu \ll \varepsilon \ll \xi, 1/q$ and $r \in [q-1]$. Let G be an (ε, ξ, q, r)-supercomplex on n vertices. Suppose that U is a random subset of $V(G)$ obtained by including every vertex from $V(G)$ independently with probability μ. Then whp for any $W \subseteq V(G)$ with $|W| \leq \gamma n$, $G[U \triangle W]$ is an $(2\varepsilon, \xi - \varepsilon, q, r)$-supercomplex.*

Next, we investigate the effect on G of inducing to a random subgraph H of $G^{(r)}$. For our applications, we need to be able to choose edges with different probabilities. It turns out that under suitable restrictions on these probabilities, the relevant properties of G are inherited by $G[H]$.

Proposition 5.15. *Let $1/n \ll \varepsilon, \gamma, p, \xi, 1/q$ and $r \in [q-1]$, $i \in [r]_0$. Let*

$$\xi' := 0.95\xi p^{2^r \binom{q+r}{r}} \geq 0.95\xi p^{(8^q)} \text{ and } \gamma' := 1.1 \cdot 2^i \frac{\binom{q+r}{r}}{(q-r)!}\gamma.$$

Let G be a complex on n vertices and $F \subseteq G^{(i)}$ with $1 \leq |F| \leq 2^i$. Suppose that

$$G_F := \bigcap_{f \in F} G(f) \text{ is an } (\varepsilon, \xi, q-i, r-i)\text{-complex.}$$

Assume that $\mathcal{P}$ is a partition of $G^{(r)}$ satisfying the following containment conditions:

(I) *For every $f \in F$, there exists a class $\mathcal{E}_f \in \mathcal{P}$ such that $f \cup e \in \mathcal{E}_f$ for all $e \in G_F^{(r-i)}$.*

(II) *For every $\mathcal{E} \in \mathcal{P}$ there exists $D_{\mathcal{E}} \in \mathbb{N}_0$ such that for all $Q \in G_F^{(q-i)}$, we have that $|\{e \in \mathcal{E} : \exists f \in F : e \subseteq f \cup Q\}| = D_{\mathcal{E}}$.*

Let $\beta \colon \mathcal{P} \to [p, 1]$ assign a probability to every class of $\mathcal{P}$. Now, suppose that H is a random subgraph of $G^{(r)}$ obtained by independently including every edge of $X \in \mathcal{P}$ with probability $\beta(X)$ (for all $X \in \mathcal{P}$). Then with probability at least $1 - \mathrm{e}^{-n^{1/8}}$, the following holds: for all $L \subseteq G^{(r)}$ with $\Delta(L) \leq \gamma n$,

$$\bigcap_{f \in F} G[H \triangle L](f) \text{ is a } (3\varepsilon + \gamma', \xi' - \gamma', q-i, r-i)\text{-complex.}$$

Note that (I) and (II) certainly hold if $\mathcal{P} = \{G^{(r)}\}$.

Proof. If $G_F^{(r-i)}$ is empty, then the statement is vacuously true. So let us assume that $G_F^{(r-i)}$ is not empty. Let $n_F := |V(G) \setminus \bigcup F| = |V(G_F)|$. By assumption, there exists $Y \subseteq G_F^{(q-i)}$ such that $G_F[Y]$ is $(\varepsilon, d_F, q-i, r-i)$-regular for some $d_F \geq \xi$, $(\xi, q+r-2i, r-i)$-dense and $(\xi, q-i, r-i)$-extendable. Define

$$p_F := \left(\prod_{f \in F} \beta(\mathcal{E}_f)\right)^{-1} \prod_{\mathcal{E} \in \mathcal{P}} (\beta(\mathcal{E}))^{D_{\mathcal{E}}}.$$

Note that $p_F \geq p^{|F|\binom{q}{r}} \geq p^{2^r\binom{q+r}{r}}$ and thus $p_F d_F \geq \xi'$. For every $e \in G_F^{(r-i)}$, let

$$\mathcal{Q}_e := G_F[Y]^{(q-i)}(e) \quad \text{and} \quad \tilde{\mathcal{Q}}_e := G_F[Y]^{(q+r-2i)}(e).$$

By assumption, we have $|\mathcal{Q}_e| = (d_F \pm \varepsilon)n_F^{q-r}$ and $|\tilde{\mathcal{Q}}_e| \geq \xi n_F^{q-i}$ for all $e \in G_F^{(r-i)}$. Moreover, since $G_F[Y]$ is $(\xi, q-i, r-i)$-extendable, there exists $X \subseteq V(G_F)$ with $|X| \geq \xi n_F$ such that for all $e \in \binom{X}{r-i}$, we have $|Ext_e| \geq \xi n_F^{q-r}$, where Ext_e is the set of all $(q-r)$-sets $Q \subseteq V(G_F) \setminus e$ such that $\binom{Q\cup e}{r-i} \setminus \{e\} \subseteq G_F^{(r-i)} = G_F[Y]^{(r-i)}$.

We consider the following (random) subsets. For every $e \in G_F^{(r-i)}$, let $\mathcal{Q}'_e$ contain all $Q \in \mathcal{Q}_e$ such that for all $f \in F$, $\binom{f\cup Q\cup e}{r} \setminus \{f \cup e\} \subseteq H$ and define $\tilde{\mathcal{Q}}'_e$ analogously with $\tilde{\mathcal{Q}}_e$ playing the role of $\mathcal{Q}_e$. For every $e \in \binom{X}{r-i}$, let Ext'_e contain all $Q \in Ext_e$ such that for all $f \in F$ and $e' \in \binom{Q\cup e}{r-i} \setminus \{e\}$, $f \cup e' \in H$.

Claim 1: For each $e \in G_F^{(r-i)}$, with probability at least $1 - \mathrm{e}^{-n_F^{1/6}}$, $|\mathcal{Q}'_e| = (p_F d_F \pm 3\varepsilon)n_F^{q-r}$.

Proof of claim: We view $\mathcal{Q}_e$ as a $(q-r)$-graph and $\mathcal{Q}'_e$ as a random subgraph. Note that

$$\mathbb{P}(\forall f \in F\colon f \cup e \in H) = \prod_{f\in F} \mathbb{P}(f \cup e \in H) \overset{\text{(I)}}{=} \prod_{f\in F} \beta(\mathcal{E}_f).$$

Hence, we have for every $Q \in \mathcal{Q}_e$ that

$$\begin{aligned}
\mathbb{P}(Q \in \mathcal{Q}'_e) &= \frac{\mathbb{P}(\forall f \in F\colon \binom{f\cup Q\cup e}{r} \subseteq H)}{\mathbb{P}(\forall f \in F\colon f \cup e \in H)} \\
&= \left(\prod_{f\in F} \beta(\mathcal{E}_f)\right)^{-1} \prod_{e'\in G^{(r)}\colon \exists f\in F\colon e'\subseteq f\cup Q\cup e} \mathbb{P}(e' \in H) \\
&= \left(\prod_{f\in F} \beta(\mathcal{E}_f)\right)^{-1} \prod_{\mathcal{E}\in\mathcal{P}} (\beta(\mathcal{E}))^{|\{e'\in\mathcal{E}\colon \exists f\in F\colon e'\subseteq f\cup Q\cup e\}|} \\
&\overset{\text{(II)}}{=} \left(\prod_{f\in F} \beta(\mathcal{E}_f)\right)^{-1} \prod_{\mathcal{E}\in\mathcal{P}} (\beta(\mathcal{E}))^{D_\mathcal{E}} = p_F.
\end{aligned}$$

Thus, $\mathbb{E}|\mathcal{Q}'_e| = p_F|\mathcal{Q}_e|$. Hence, we deduce with Corollary 5.11 that with probability at least $1 - \mathrm{e}^{-n_F^{1/6}}$ we have $|\mathcal{Q}'_e| = (1 \pm \varepsilon)\mathbb{E}|\mathcal{Q}'_e| = (p_F d_F \pm 3\varepsilon)n_F^{q-r}$. –

Claim 2: For each $e \in G_F^{(r-i)}$, with probability at least $1 - \mathrm{e}^{-n_F^{1/6}}$, $|\tilde{\mathcal{Q}}'_e| \geq \xi' n_F^{q-i}$.

Proof of claim: We view $\tilde{\mathcal{Q}}_e$ as a $(q-i)$-graph and $\tilde{\mathcal{Q}}'_e$ as a random subgraph. Observe that for every $Q \in \tilde{\mathcal{Q}}_e$, we have

$$\mathbb{P}(Q \in \tilde{\mathcal{Q}}'_e) \geq p^{|F|(\binom{q+r-i}{r}-1)} \geq p^{2^r\binom{q+r}{r}}$$

and thus $\mathbb{E}|\tilde{\mathcal{Q}}'_e| \geq p^{2^r\binom{q+r}{r}}|\tilde{\mathcal{Q}}_e| \geq \xi p^{2^r\binom{q+r}{r}} n_F^{q-i}$. Thus, we deduce with Corollary 5.11 that with probability at least $1 - \mathrm{e}^{-n_F^{1/6}}$ we have $|\tilde{\mathcal{Q}}'_e| \geq \xi' n_F^{q-i}$. –

Claim 3: For every $e \in \binom{X}{r-i}$, with probability at least $1 - \mathrm{e}^{-n_F^{1/6}}$, $|Ext'_e| \geq \xi' n_F^{q-r}$.

Proof of claim: We view Ext_e as a $(q-r)$-graph and Ext'_e as a random subgraph. Observe that for every $Q \in Ext_e$, we have

$$\mathbb{P}(Q \in Ext'_e) \geq p^{|F|(\binom{q-i}{r-i}-1)} \geq p^{2^r\binom{q+r}{r}}$$

and thus $\mathbb{E}|Ext'_e| \geq p^{2^r\binom{q+r}{r}}|Ext_e| \geq \xi p^{2^r\binom{q+r}{r}} n_F^{q-r}$. Thus, we deduce with Corollary 5.11 that with probability at least $1 - \mathrm{e}^{-n_F^{1/6}}$ we have $|Ext'_e| \geq \xi' n_F^{q-r}$. –

Applying a union bound, we can see that with probability at least $1 - e^{-n^{1/8}}$, H satisfies Claims 1–3 simultaneously for all relevant e.

Assume that this applies. We now deduce the desired result deterministically. Let $L \subseteq G^{(r)}$ be any graph with $\Delta(L) \leq \gamma n$. Let $G' := \bigcap_{f \in F} G[H \triangle L](f)$. First, we claim that $G'[Y]$ is $(3\varepsilon + \gamma', p_F d_F, q-i, r-i)$-regular. Consider $e \in G'[Y]^{(r-i)}$. We have that $|\mathcal{Q}'_e| = (p_F d_F \pm 3\varepsilon) n_F^{q-r}$.

Claim 4: If $Q \in G'[Y]^{(q-i)}(e) \triangle \mathcal{Q}'_e$, then there is some $f \in F$ such that $f \cup Q \cup e$ contains some edge from $L - \{f \cup e\}$.

Proof of claim: Clearly, $Q \in G_F[Y]^{(q-i)}(e)$. First, suppose that $Q \in G'[Y]^{(q-i)}(e) - \mathcal{Q}'_e$. Since $Q \notin \mathcal{Q}'_e$, there exists $f \in F$ such that $\binom{f \cup Q \cup e}{r} \setminus \{f \cup e\} \not\subseteq H$, that is, there is $e' \in \binom{f \cup Q \cup e}{r} \setminus \{f \cup e\}$ with $e' \notin H$. But since $Q \in G'[Y]^{(q-i)}(e)$, we have $e' \in H \triangle L$. Thus, $e' \in L$. Next, suppose that $Q \in \mathcal{Q}'_e - G'[Y]^{(q-i)}(e)$. Since $Q \notin G'[Y]^{(q-i)}(e)$, there exists $f \in F$ such that $f \cup Q \cup e \notin G[H \triangle L]$, that is, there is $e' \in \binom{f \cup Q \cup e}{r}$ with $e' \notin H \triangle L$. Since $e \in G'[Y]^{(r-i)}$, we have that $f \cup e \in H \triangle L$, so $e' \neq f \cup e$. Thus, since $Q \in \mathcal{Q}'_e$, we have that $e' \in H$. Therefore, $e' \in L$. −

For fixed $f \in F$, Proposition 5.4 implies that there are at most $\frac{\binom{q}{r}}{(q-r)!}\gamma n^{q-r}$ q-sets that contain $f \cup e$ and some edge from $L - \{f \cup e\}$. Thus, we conclude with Claim 4 that $|G'[Y]^{(q-i)}(e) \triangle \mathcal{Q}'_e| \leq |F| \frac{\binom{q}{r}}{(q-r)!}\gamma n^{q-r}$. Hence,

$$|G'[Y]^{(q-i)}(e)| = |\mathcal{Q}'_e| \pm \gamma' n_F^{q-r} = (p_F d_F \pm (3\varepsilon + \gamma')) n_F^{q-r},$$

meaning that $G'[Y]$ is indeed $(3\varepsilon + \gamma', p_F d_F, q-i, r-i)$-regular.

Next, we claim that $G'[Y]$ is $(\xi' - \gamma', q+r-2i, r-i)$-dense. Consider $e \in G'[Y]^{(r-i)}$. We have that $|\tilde{\mathcal{Q}}'_e| \geq \xi' n_F^{q-i}$. Similarly to Claim 4, for every $Q \in \tilde{\mathcal{Q}}'_e - G'[Y]^{(q+r-2i)}(e)$ there is some $f \in F$ such that $f \cup Q \cup e$ contains some edge from $L - \{f \cup e\}$. Thus, using Proposition 5.4 again (with $q + r - i$ playing the role of q), we deduce that

$$|\tilde{\mathcal{Q}}'_e - G'[Y]^{(q+r-2i)}(e)| \leq |F| \frac{\binom{q+r-i}{r}}{(q-i)!}\gamma n^{q-i} \leq 2^i \frac{\binom{q+r}{r}}{(q-r)!}\gamma n^{q-i}$$

and thus $|G'[Y]^{(q+r-2i)}(e)| \geq (\xi' - \gamma') n_F^{q-i}$.

Finally, we claim that $G'[Y]$ is $(\xi' - \gamma', q-i, r-i)$-extendable. Let $e \in \binom{X}{r-i}$. We have that $|Ext'_e| \geq \xi' n_F^{q-r}$. Let $Ext_{e,G'}$ contain all $Q \in Ext_e$ such that $\binom{Q \cup e}{r-i} \setminus \{e\} \subseteq G'[Y]^{(r-i)}$. Suppose that $Q \in Ext'_e \setminus Ext_{e,G'}$. Then there is $e' \in \binom{Q \cup e}{r-i} \setminus \{e\}$ and $f \in F$ such that $f \cup e' \notin H \triangle L$. On the other hand, we have $f \cup e' \in H$. Thus, $f \cup e' \in L$. Thus, for all $Q \in Ext'_e \setminus Ext_{e,G'}$, there is some $f \in F$ such that $f \cup Q \cup e$ contains some edge from $L - \{f \cup e\}$. Proposition 5.4 implies that there are at most $|F| \frac{\binom{q}{r}}{(q-r)!}\gamma n^{q-r}$ such Q. Thus,

$$|Ext_{e,G'}| \geq |Ext'_e| - 2^i \frac{\binom{q}{r}}{(q-r)!}\gamma n^{q-r} \geq (\xi' - \gamma') n_F^{q-r}.$$

We conclude that G' is a $(3\varepsilon + \gamma', \xi' - \gamma', q-i, r-i, i')$-complex, as required. □

In particular, the above proposition implies the following.

Corollary 5.16. *Let $1/n \ll \varepsilon, \gamma, \xi, p, 1/q$ and $r \in [q-1]$. Let*

$$\xi' := 0.95 \xi p^{2^r \binom{q+r}{r}} \geq 0.95 \xi p^{(8^q)} \text{ and } \gamma' := 1.1 \cdot 2^r \frac{\binom{q+r}{r}}{(q-r)!}\gamma.$$

Suppose that G is an (ε, ξ, q, r)-supercomplex on n vertices and that $H \subseteq G^{(r)}$ is a random subgraph obtained by including every edge of $G^{(r)}$ independently with probability p. Then whp the following holds: for all $L \subseteq G^{(r)}$ with $\Delta(L) \leq \gamma n$, $G[H \triangle L]$ is a $(3\varepsilon + \gamma', \xi' - \gamma', q, r)$-supercomplex.

6. Nibbles, boosting and greedy covers

6.1. **The nibble.** There are numerous results based on the Rödl nibble which guarantee the existence of an almost perfect matching in a near regular hypergraph with small codegrees. Our application of this is as follows: Let G be a complex. Define the auxiliary $\binom{q}{r}$-graph H with $V(H) = E(G^{(r)})$ and $E(H) = \{\binom{Q}{r} : Q \in G^{(q)}\}$. Note that for every $e \in V(H)$, $|H(e)| = |G^{(q)}(e)|$. Thus, if G is (ε, d, q, r)-regular, then every vertex of H has degree $(d \pm \varepsilon)n^{q-r}$. Moreover, for two vertices $e, e' \in V(H)$, we have $|H(\{e, e'\})| \leq n^{q-r-1}$, thus $\Delta_2(H) \leq n^{q-r-1}$. Standard nibble theorems would in this setting imply the existence of an almost perfect matching in H, which translates into a $K_q^{(r)}$-packing in G that covers all but $o(n^r)$ r-edges. We need a stronger result in the sense that we want the leftover r-edges to induce an r-graph with small maximum degree. Alon and Yuster [2] observed that one can use a result of Pippenger and Spencer [32] (on the chromatic index of uniform hypergraphs) to show that a near regular hypergraph with small codegrees has an almost perfect matching which is 'well-behaved'. The following is an immediate consequence of Theorem 1.2 in [2] (applied to the auxiliary hypergraph H above).

Theorem 6.1 ([2])**.** *Let $1/n \ll \varepsilon \ll \gamma, d, 1/q$ and $r \in [q-1]$. Suppose that G is an (ε, d, q, r)-regular complex on n vertices. Then G contains a $K_q^{(r)}$-packing $\mathcal{K}$ such that $\Delta(G^{(r)} - \mathcal{K}^{(r)}) \leq \gamma n$.*

6.2. **The Boost lemma.** We will now state and prove the 'Boost lemma', which 'boosts' the regularity of a complex by restricting to a suitable set Y of q-sets. It will help us to keep the error terms under control during the iteration process and also helps us to obtain meaningful resilience and minimum degree bounds.

The proof is based on the following 'edge-gadgets', which were used in [4] to obtain fractional $K_q^{(r)}$-decompositions of r-graphs with high minimum degree. These edge-gadgets allow us to locally adjust a given weighting of q-sets so that this changes the total weight at only one r-set.

Proposition 6.2 (see [4, Proposition 3.3])**.** *Let $q > r \geq 1$ and let e and J be disjoint sets with $|e| = r$ and $|J| = q$. Let G be the complete complex on $e \cup J$. There exists a function $\psi\colon G^{(q)} \to \mathbb{R}$ such that*

(i) *for all $e' \in G^{(r)}$, $\sum_{Q \in G^{(q)}(e')} \psi(Q \cup e') = \begin{cases} 1, & e' = e, \\ 0, & e' \neq e; \end{cases}$*

(ii) *for all $Q \in G^{(q)}$, $|\psi(Q)| \leq \frac{2^{r-j}(r-j)!}{\binom{q-r+j}{j}}$, where $j := |e \cap Q|$.*

We use these gadgets as follows. We start off with a complex that is (ε, d, q, r)-regular for some reasonable ε and consider a uniform weighting of all q-sets. We then use the edge-gadgets to shift weights until we have a 'fractional $K_q^{(r)}$-equicovering' in the sense that the weight of each edge is exactly $d'n^{q-r}$ for some suitable d'. We then use this fractional equicovering as an input for a probabilistic argument.

Lemma 6.3 (Boost lemma)**.** *Let $1/n \ll \varepsilon, \xi, 1/q$ and $r \in [q-1]$ such that $2(2\sqrt{\mathrm{e}})^r \varepsilon \leq \xi$. Let $\xi' := 0.9(1/4)^{\binom{q+r}{q}}\xi$. Suppose that G is a complex on n vertices and that G is (ε, d, q, r)-regular for some $d \geq \xi$ and $(\xi, q+r, r)$-dense. Then there exists $Y \subseteq G^{(q)}$ such that $G[Y]$ is $(n^{-(q-r)/2.01}, d/2, q, r)$-regular and $(\xi', q+r, r)$-dense.*

Proof. Let $d' := d/2$. Assume that $\psi\colon G^{(q)} \to [0,1]$ is a function such that for every $e \in G^{(r)}$,

$$\sum_{Q' \in G^{(q)}(e)} \psi(Q' \cup e) = d'n^{q-r},$$

and $1/4 \leq \psi(Q) \leq 1$ for all $Q \in G^{(q)}$. We can then choose $Y \subseteq G^{(q)}$ by including every $Q \in G^{(q)}$ with probability $\psi(Q)$ independently. We then have for every $e \in G^{(r)}$, $\mathbb{E}|G[Y]^{(q)}(e)| = d'n^{q-r}$. By Lemma 5.7(ii), we conclude that

$$\mathbb{P}(|G[Y]^{(q)}(e)| \neq (1 \pm n^{-(q-r)/2.01})d'n^{q-r}) \leq 2\mathrm{e}^{-\frac{n^{-2(q-r)/2.01}d'n^{q-r}}{3}} \leq \mathrm{e}^{-n^{0.004}}.$$

Thus, whp $G[Y]$ is $(n^{-(q-r)/2.01}, d', q, r)$-regular. Moreover, for any $e \in G^{(r)}$ and $Q \in G^{(q+r)}(e)$, we have that

$$\mathbb{P}(Q \in G[Y]^{(q+r)}(e)) = \prod_{Q' \in \binom{Q \cup e}{q}} \psi(Q') \geq (1/4)^{\binom{q+r}{q}}.$$

Therefore, $\mathbb{E}|G[Y]^{(q+r)}(e)| \geq (1/4)^{\binom{q+r}{q}} \xi n^q$, and using Corollary 5.11 we deduce that

$$\mathbb{P}(|G[Y]^{(q+r)}(e)| \leq 0.9(1/4)^{\binom{q+r}{q}} \xi n^q) \leq \mathrm{e}^{-n^{1/6}}.$$

Thus, whp $G[Y]$ is $(0.9(1/4)^{\binom{q+r}{q}}\xi, q+r, r)$-dense.

It remains to show that ψ exists. For every $e \in G^{(r)}$, define

$$c_e := \frac{d' n^{q-r} - 0.5|G^{(q)}(e)|}{|G^{(q+r)}(e)|}.$$

Observe that $|c_e| \leq \frac{\varepsilon n^{q-r}}{2\xi n^q} = \frac{\varepsilon}{2\xi} n^{-r}$ for all $e \in G^{(r)}$.

By Proposition 6.2, for every $e \in G^{(r)}$ and $J \in G^{(q+r)}(e)$, there exists a function $\psi_{e,J} \colon G^{(q)} \to \mathbb{R}$ such that

(i) $\psi_{e,J}(Q) = 0$ for all $Q \not\subseteq e \cup J$;

(ii) for all $e' \in G^{(r)}$, $\sum_{Q' \in G^{(q)}(e')} \psi_{e,J}(Q' \cup e') = \begin{cases} 1, & e' = e, \\ 0, & e' \neq e; \end{cases}$

(iii) for all $Q \in G^{(q)}$, $|\psi_{e,J}(Q)| \leq \frac{2^{r-j}(r-j)!}{\binom{q-r+j}{j}}$, where $j := |e \cap Q|$.

We now define $\psi \colon G^{(q)} \to [0,1]$ as

$$\psi := 1/2 + \sum_{e \in G^{(r)}} c_e \sum_{J \in G^{(q+r)}(e)} \psi_{e,J}.$$

For every $e \in G^{(r)}$, we have

$$\sum_{Q' \in G^{(q)}(e)} \psi(Q' \cup e) = 0.5|G^{(q)}(e)| + \sum_{e' \in G^{(r)}} c_{e'} \sum_{J \in G^{(q+r)}(e')} \sum_{Q' \in G^{(q)}(e)} \psi_{e',J}(Q' \cup e)$$

$$\overset{\text{(ii)}}{=} 0.5|G^{(q)}(e)| + c_e |G^{(q+r)}(e)| = d' n^{q-r},$$

as desired. Moreover, for every $Q \in G^{(q)}$ and $j \in [r]_0$, there are at most $\binom{n}{r}\binom{q}{j}\binom{r}{r-j}$ pairs (e, J) for which $e \in G^{(r)}$, $J \in G^{(q+r)}(e)$, $Q \subseteq e \cup J$ and $|Q \cap e| = j$. Hence,

$$\begin{aligned}
|\psi(Q) - 1/2| &= \left| \sum_{e \in G^{(r)}} c_e \sum_{J \in G^{(q+r)}(e)} \psi_{e,J}(Q) \right| \overset{\text{(i)}}{\leq} \sum_{e \in G^{(r)}, J \in G^{(q+r)}(e) \colon Q \subseteq e \cup J} |c_e| |\psi_{e,J}(Q)| \\
&\overset{\text{(iii)}}{\leq} \sum_{j=0}^{r} \binom{n}{r}\binom{q}{j}\binom{r}{r-j} \cdot \frac{\varepsilon}{2\xi} n^{-r} \cdot \frac{2^{r-j}(r-j)!}{\binom{q-r+j}{j}} \\
&\leq \frac{2^{r-1}\varepsilon}{\xi} \sum_{j=0}^{r} \frac{2^{-j}}{j!} \left(\frac{q}{q-r+1} \right)^j \leq \frac{2^{r-1}\varepsilon}{\xi} \sum_{j=0}^{r} \frac{(r/2)^j}{j!} \leq 1/4,
\end{aligned}$$

implying that $1/4 \leq \psi(Q) \leq 3/4$ for all $Q \in G^{(q)}$, as needed. □

Proof of Lemma 3.4. Let G be an (ε, ξ, q, r)-complex on n vertices. By definition, there exists $Y \subseteq G^{(q)}$ such that $G[Y]$ is (ε, d, q, r)-regular for some $d \geq \xi$, $(\xi, q+r, r)$-dense and (ξ, q, r)-extendable. We can thus apply the Boost lemma (Lemma 6.3) (with $G[Y]$ playing the role of G). This yields $Y' \subseteq Y$ such that $G[Y']$ is $(n^{-1/3}, d/2, q, r)$-regular and $(\xi', q+r, r)$-dense. Since $G[Y']^{(r)} = G[Y]^{(r)}$, $G[Y']$ is also (ξ, q, r)-extendable. Thus, G is an $(n^{-1/3}, \xi', q, r)$-complex.

Suppose now that G is an (ε,ξ,q,r)-supercomplex. Let $i\in[r]_0$ and $F\subseteq G^{(i)}$ with $1\le|F|\le 2^i$. We have that $G_F:=\bigcap_{f\in F}G(f)$ is an $(\varepsilon,\xi,q-i,r-i)$-complex. If $i<r$, we deduce by the above that G_F is an $(n_F^{-1/3},\xi',q-i,r-i)$-complex. If $i=r$, this also holds by Fact 3.2. □

Lemma 6.3 together with Theorem 6.1 immediately implies the following 'Boosted nibble lemma'. Whenever we need an approximate decomposition in the proof of Theorem 3.7, we will obtain it via Lemma 6.4.

Lemma 6.4 (Boosted nibble lemma)**.** *Let $1/n\ll\gamma,\varepsilon\ll\xi,1/q$ and $r\in[q-1]$. Let G be a complex on n vertices such that G is (ε,d,q,r)-regular and $(\xi,q+r,r)$-dense for some $d\ge\xi$. Then G contains a $K_q^{(r)}$-packing $\mathcal{K}$ such that $\Delta(G^{(r)}-\mathcal{K}^{(r)})\le\gamma n$.*

6.3. **Greedy coverings and divisibility.** The following lemma allows us to extend a given collection of r-sets into suitable r-disjoint q-cliques (see Corollary 6.7). The full strength of Lemma 6.5 will only be needed in Section 8. The proof consists of a sequential random greedy algorithm. A probabilistic approach can probably be avoided here, but seems much simpler to analyse.

Lemma 6.5. *Let $1/n\ll\gamma\ll\alpha,1/s,1/q$ and $r\in[q-1]$. Let G be a complex on n vertices and let $L\subseteq G^{(r)}$ satisfy $\Delta(L)\le\gamma n$. Suppose that L decomposes into $L_1,\dots,L_m$ with $1\le|L_j|\le s$. Suppose that for every $j\in[m]$, we are given some candidate set $\mathcal{Q}_j\subseteq\bigcap_{e\in L_j}G^{(q)}(e)$ with $|\mathcal{Q}_j|\ge\alpha n^{q-r}$. Then there exists $Q_j\in\mathcal{Q}_j$ for each $j\in[m]$ such that, writing $K_j:=(Q_j\uplus L_j)^{\le}$, we have that K_j and $K_{j'}$ are r-disjoint for all distinct $j,j'\in[m]$, and $\Delta(\bigcup_{j\in[m]}K_j^{(r)})\le\sqrt{\gamma}n$.*

Proof. Let $t:=0.5\alpha n^{q-r}$ and consider Algorithm 6.6. We claim that with positive probability,

Algorithm 6.6

for j from 1 to m **do**
  define the r-graph $T_j:=\bigcup_{j'=1}^{j-1}K_{j'}^{(r)}$ and let $\mathcal{Q}_j'$ contain all $Q\in\mathcal{Q}_j$ such that $(Q\uplus L_j)^{\le}$ does not contain any edge from T_j or $L-L_j$.
  if $|\mathcal{Q}_j'|\ge t$ **then**
    pick $Q\in\mathcal{Q}_j'$ uniformly at random and let $K_j:=(Q\uplus L_j)^{\le}$
  else
    return 'unsuccessful'
  end if
end for

Algorithm 6.6 outputs $K_1,\dots,K_m$ as desired.

It is enough to ensure that with positive probability, $\Delta(T_j)\le sqr\gamma^{2/3}n$ for all $j\in[m]$. Indeed, note that we have $L_j\cap T_j=\emptyset$ by construction. Hence, if $\Delta(T_j)\le sqr\gamma^{2/3}n$, then Proposition 5.4 implies that every $e\in L_j$ is contained in at most $(\gamma+sqr\gamma^{2/3})2^rn^{q-r}$ q-sets of $V(G)$ that also contain an edge of $T_j\cup(L-L_j)$. Thus, there are at most $s(\gamma+sqr\gamma^{2/3})2^rn^{q-r}\le 0.5\alpha n^{q-r}$ candidates $Q\in\mathcal{Q}_j$ such that $(Q\uplus L_j)^{\le}$ contains some edge from $T_j\cup(L-L_j)$. Hence, $|\mathcal{Q}_j'|\ge|\mathcal{Q}_j|-0.5\alpha n^{q-r}\ge t$, so the algorithm succeeds in round j.

For every $(r-1)$-set $S\subseteq V(G)$ and $j\in[m]$, let Y_j^S be the indicator variable of the event that S is covered by K_j.

For every $(r-1)$-set $S\subseteq V(G)$ and $k\in[r-1]_0$, define $\mathcal{J}_{S,k}:=\{j\in[m]:\max_{e\in L_j}|S\cap e|=k\}$. Observe that if $Y_j^S=1$, then K_j covers at most sq r-edges that contain S. Therefore, we have

$$|T_j(S)|\le sq\sum_{j'=1}^{j-1}Y_{j'}^S=sq\sum_{k=0}^{r-1}\sum_{j'\in\mathcal{J}_{S,k}\cap[j-1]}Y_{j'}^S.$$

The following claim thus implies the lemma.

Claim 1: With positive probability, we have $\sum_{j'\in\mathcal{J}_{S,k}\cap[j-1]} Y^S_{j'} \leq \gamma^{2/3}n$ for all $(r-1)$-sets S, $k\in[r-1]_0$ and $j\in[m]$.

Fix an $(r-1)$-set S, $k\in[r-1]_0$ and $j\in[m]$. For $j'\in\mathcal{J}_{S,k}$, there are at most

$$\sum_{e\in L_{j'}} n^{q-|S\cup e|} \leq sn^{\max_{e\in L_{j'}}(q-|S\cup e|)} = sn^{q-2r+1+k}$$

q-sets that contain S and some edge of $L_{j'}$.

In order to apply Proposition 5.8, let $j_1,\dots,j_b$ be an enumeration of $\mathcal{J}_{S,k}\cap[j-1]$. We then have for all $a\in[b]$ and all $y_1,\dots,y_{a-1}\in\{0,1\}$ that

$$\mathbb{P}(Y^S_{j_a}=1 \mid Y^S_{j_1}=y_1,\dots,Y^S_{j_{a-1}}=y_{a-1}) \leq \frac{sn^{q-2r+1+k}}{t} = 2s\alpha^{-1}n^{-r+k+1}.$$

Let $p:=\min\{2s\alpha^{-1}n^{-r+k+1},1\}$ and let $B\sim Bin(|\mathcal{J}_{S,k}\cap[j-1]|,p)$.

Note that $|\mathcal{J}_{S,k}|\leq\binom{|S|}{k}\Delta_k(L)\leq\binom{r-1}{k}\gamma n^{r-k}$ by Fact 5.3. Thus,

$$7\mathbb{E}B = 7|\mathcal{J}_{S,k}\cap[j-1]|\cdot p \leq 7\cdot\binom{r-1}{k}\gamma n^{r-k}\cdot 2s\alpha^{-1}n^{-r+k+1} \leq \gamma^{2/3}n.$$

Therefore,

$$\mathbb{P}(\sum_{j'\in\mathcal{J}_{S,k}\cap[j-1]} Y^S_{j'} \geq \gamma^{2/3}n) \overset{\text{Proposition 5.8}}{\leq} \mathbb{P}(B\geq\gamma^{2/3}n) \overset{\text{Lemma 5.7(iii)}}{\leq} \mathrm{e}^{-\gamma^{2/3}n}.$$

A union bound now easily proves the claim. □

Corollary 6.7. *Let $1/n\ll\gamma\ll\alpha,1/q$ and $r\in[q-1]$. Let G be a complex on n vertices and let $H\subseteq G^{(r)}$ with $\Delta(H)\leq\gamma n$ and $|G^{(q)}(e)|\geq\alpha n^{q-r}$ for all $e\in H$. Then there is a $K^{(r)}_q$-packing $\mathcal{K}$ in G that covers all edges of H and such that $\Delta(\mathcal{K}^{(r)})\leq\sqrt{\gamma}n$.*

Proof. Let $e_1,\dots,e_m$ be an enumeration of H. For $j\in[m]$, define $L_j:=\{e_j\}$ and $\mathcal{Q}_j:=G^{(q)}(e)$. Apply Lemma 6.5 and let $\mathcal{K}:=\bigcup_{j\in[m]}K_j$. □

Note that Corollary 6.7 and Theorem 6.1 immediately imply the main result of [13], namely the existence of an 'almost' (n,q,r)-Steiner system in the sense that every r-set is covered either once or twice.

We can combine Lemma 6.4 and Corollary 6.7 to deduce the following result. It allows us to make an r-graph divisible by deleting a small fraction of edges (even if we are forbidden to delete a certain set of edges H).

Corollary 6.8. *Let $1/n\ll\gamma,\varepsilon\ll\xi,1/q$ and $r\in[q-1]$. Suppose that G is a complex on n vertices which is (ε,d,q,r)-regular for some $d\geq\xi$ and $(\xi,q+r,r)$-dense. Let $H\subseteq G^{(r)}$ satisfy $\Delta(H)\leq\varepsilon n$. Then there exists $L\subseteq G^{(r)}-H$ such that $\Delta(L)\leq\gamma n$ and $G^{(r)}-L$ is $K^{(r)}_q$-divisible.*

Proof. We clearly have $|G^{(q)}(e)|\geq 0.5\xi n^{q-r}$ for all $e\in H$. Thus, by Corollary 6.7, there exists a $K^{(r)}_q$-packing $\mathcal{K}_0$ in G which covers all edges of H and satisfies $\Delta(\mathcal{K}^{(r)}_0)\leq\sqrt{\varepsilon}n$. By Proposition 5.6(i) and (ii), $G':=G-\mathcal{K}^{(r)}_0$ is still $(2^{r+1}\sqrt{\varepsilon},d,q,r)$-regular and $(\xi/2,q+r,r)$-dense. Thus, by Lemma 6.4, there exists a $K^{(r)}_q$-packing $\mathcal{K}_{nibble}$ in G' such that $\Delta(L)\leq\gamma n$, where $L:=G'^{(r)}-\mathcal{K}^{(r)}_{nibble}=G^{(r)}-\mathcal{K}^{(r)}_0-\mathcal{K}^{(r)}_{nibble}\subseteq G^{(r)}-H$. Clearly, $G^{(r)}-L$ is $K^{(r)}_q$-divisible (in fact, $K^{(r)}_q$-decomposable). □

7. Vortices

7.1. **Statement of the Cover down lemma.** In Section 10, we will prove the Cover down lemma (Lemma 7.4). Roughly speaking, if G is a supercomplex and U a random subset of linear size, we aim to find a $K_q^{(r)}$-packing in G that covers all r-edges that are not inside U by using only few r-edges inside U. The majority of these r-edges will be covered using the Boosted nibble lemma (Lemma 6.4), leaving a very sparse leftover L^*. The Cover down lemma shows the existence of a suitable sparse r-graph H^* which is capable of dealing with any such leftover (i.e. $G[H^* \cup L^*]$ has a $K_q^{(r)}$-packing covering all edges of $H^* \cup L^*$ which are not inside U).

Definition 7.1. Let G be a complex on n vertices. We say that U is $(\varepsilon, \mu, \xi, q, r)$*-random in* G if there exists a q-graph Y on $V(G)$ such that the following hold:

(R1) $U \subseteq V(G)$ with $|U| = \mu n \pm n^{2/3}$;
(R2) there exists $d \geq \xi$ such that for all $x \in [q-r]_0$ and all $e \in G^{(r)}$, we have that
$$|\{Q \in G[Y]^{(q)}(e) \, : \, |Q \cap U| = x\}| = (1 \pm \varepsilon) bin(q-r, \mu, x) dn^{q-r};$$
(R3) for all $e \in G^{(r)}$ we have $|G[Y]^{(q+r)}(e)[U]| \geq \xi (\mu n)^q$;
(R4) for all $h \in [r]_0$ and all $F \subseteq G^{(h)}$ with $1 \leq |F| \leq 2^h$ we have that $\bigcap_{f \in F} G(f)[U]$ is an $(\varepsilon, \xi, q-h, r-h)$-complex.

We record the following easy consequences for later use.

Fact 7.2. *The following hold.*

(i) *If G is an (ε, ξ, q, r)-supercomplex, then $V(G)$ is $(\varepsilon/\xi, 1, \xi, q, r)$-random in G.*
(ii) *If U is $(\varepsilon, \mu, \xi, q, r)$-random in G, then $G[U]$ is an (ε, ξ, q, r)-supercomplex.*

Here, (ii) follows immediately from (R4). Note that (R4) is stronger in the sense that F is not restricted to U.

Definition 7.3. Let G be a complex on n vertices and $H \subseteq G^{(r)}$. We say that G is (ξ, q, r)-*dense with respect to* H if for all $e \in G^{(r)}$, we have $|G[H \cup \{e\}]^{(q)}(e)| \geq \xi n^{q-r}$.

Recall that $(*)_r$ is the statement of our main theorem (Theorem 3.7), which we intend to prove by induction.

Lemma 7.4 (Cover down lemma)**.** *Let $1/n \ll \gamma \ll \varepsilon \ll \nu \ll \mu, \xi, 1/q$ and $r \in [q-1]$ with $\mu \leq 1/2$. Assume that $(*)_i$ is true for all $i \in [r-1]$. Let G be a complex on n vertices and suppose that U is $(\varepsilon, \mu, \xi, q, r)$-random in G. Let $\tilde{G}$ be a complex on $V(G)$ with $G \subseteq \tilde{G}$ such that $\tilde{G}$ is (ε, q, r)-dense with respect to $G^{(r)} - G^{(r)}[\bar{U}]$, where $\bar{U} := V(G) \setminus U$.*

Then there exists a subgraph $H^* \subseteq G^{(r)} - G^{(r)}[\bar{U}]$ with $\Delta(H^*) \leq \nu n$ such that for any $L \subseteq \tilde{G}^{(r)}$ with $\Delta(L) \leq \gamma n$ and $H^* \cup L$ being $K_q^{(r)}$-divisible, there exists a $K_q^{(r)}$-packing in $\tilde{G}[H^* \cup L]$ which covers all edges of $H^* \cup L$ except possibly some inside U.

7.2. **Existence of vortices.** A vortex consists of a suitable nested sequence of vertex sets. It provides the framework in which we can iteratively apply the Boosted nibble lemma (Lemma 6.4) and the Cover down lemma.

Definition 7.5 (Vortex)**.** Let G be a complex. An $(\varepsilon, \mu, \xi, q, r, m)$*-vortex in* G is a sequence $U_0 \supseteq U_1 \supseteq \cdots \supseteq U_\ell$ such that

(V1) $U_0 = V(G)$;
(V2) $|U_i| = \lfloor \mu |U_{i-1}| \rfloor$ for all $i \in [\ell]$;
(V3) $|U_\ell| = m$;
(V4) for all $i \in [\ell]$, U_i is $(\varepsilon, \mu, \xi, q, r)$-random in $G[U_{i-1}]$;
(V5) for all $i \in [\ell - 1]$, $U_i \setminus U_{i+1}$ is $(\varepsilon, \mu(1-\mu), \xi, q, r)$-random in $G[U_{i-1}]$.

The goal of this subsection is to prove the following lemma, which guarantees the existence of a vortex in a supercomplex.

Lemma 7.6. *Let $1/m' \ll \varepsilon \ll \mu, \xi, 1/q$ such that $\mu \leq 1/2$ and $r \in [q-1]$. Let G be an (ε, ξ, q, r)-supercomplex on $n \geq m'$ vertices. Then there exists a $(2\sqrt{\varepsilon}, \mu, \xi - \varepsilon, q, r, m)$-vortex in G for some $\mu m' \leq m \leq m'$.*

Fact 7.7. *For all $p_1, p_2 \in [0,1]$ and $i, n \in \mathbb{N}_0$, we have*

$$\sum_{j=i}^{n} bin(n, p_1, j) bin(j, p_2, i) = bin(n, p_1 p_2, i). \tag{7.1}$$

Proposition 7.8. *Let $1/n \ll \varepsilon \ll \mu_1, \mu_2, 1-\mu_2, \xi, 1/q$ and $r \in [q-1]$. Let G be a complex on n vertices and suppose that U is $(\varepsilon, \mu_1, \xi, q, r)$-random in G. Let U' be a random subset of U obtained by including every vertex from U independently with probability μ_2. Then whp for all $W \subseteq U$ of size $|W| \leq |U|^{3/5}$, $U' \triangle W$ is $(\varepsilon + 0.5|U|^{-1/6}, \mu_1\mu_2, \xi - 0.5|U|^{-1/6}, q, r)$-random in G.*

Proof. Let $Y \subseteq G^{(q)}$ and $d \geq \xi$ be such that (R1)–(R4) hold for U. By Lemma 5.7(i) we have that whp $|U'| = \mu_2|U| \pm |U|^{3/5}$. So for any admissible W, we have that $|U' \triangle W| = \mu_2|U| \pm 2|U|^{3/5} = \mu_1\mu_2 n \pm (\mu_2 n^{2/3} + 2n^{3/5}) = \mu_1\mu_2 n \pm n^{2/3}$, implying (R1).

We next check (R2). For all $x \in [q-r]_0$ and $e \in G^{(r)}$, we have that $|\mathcal{Q}_{e,x}| = (1 \pm \varepsilon) bin(q-r, \mu_1, x) d n^{q-r}$, where $\mathcal{Q}_{e,x} := \{Q \in G[Y]^{(q)}(e) \,:\, |Q \cap U| = x\}$. Consider $e \in G^{(r)}$ and $x, y \in [q-r]_0$. We view $\mathcal{Q}_{e,x}$ as a $(q-r)$-graph and consider the random subgraph $\mathcal{Q}_{e,x,y}$ containing all $Q \in \mathcal{Q}_{e,x}$ such that $|Q \cap U'| = y$.

By the random choice of U', for all $e \in G^{(r)}$ and $x, y \in [q-r]_0$, we have

$$\mathbb{E}|\mathcal{Q}_{e,x,y}| = bin(x, \mu_2, y)|\mathcal{Q}_{e,x}|.$$

Thus, by Corollary 5.11 whp we have for all $e \in G^{(r)}$ and $x, y \in [q-r]_0$ that

$$\begin{aligned} |\mathcal{Q}_{e,x,y}| &= (1 \pm n^{-1/5}) bin(x, \mu_2, y)|\mathcal{Q}_{e,x}| \\ &= (1 \pm n^{-1/5}) bin(x, \mu_2, y)(1 \pm \varepsilon) bin(q-r, \mu_1, x) d n^{q-r} \\ &= (1 \pm (\varepsilon + 2n^{-1/5})) bin(q-r, \mu_1, x) bin(x, \mu_2, y) d n^{q-r}. \end{aligned}$$

Assuming that the above holds for U', we have for all $y \in [q-r]_0$, $e \in G^{(r)}$ and $W \subseteq U$ of size $|W| \leq |U|^{3/5}$ that

$$\begin{aligned} |\{Q \in G[Y]^{(q)}(e) \,:\, |Q \cap (U' \triangle W)| = y\}| &= \sum_{x=y}^{q-r} |\mathcal{Q}_{e,x,y}| \pm |W| n^{q-r-1} \\ = \quad &\sum_{x=y}^{q-r} (1 \pm (\varepsilon + 2n^{-1/5})) bin(q-r, \mu_1, x) bin(x, \mu_2, y) d n^{q-r} \pm n^{-2/5} n^{q-r} \\ \overset{(7.1)}{=} \quad &(1 \pm (\varepsilon + 3n^{-1/5})) bin(q-r, \mu_1\mu_2, y) d n^{q-r}. \end{aligned}$$

We now check (R3). Consider $e \in G^{(r)}$ and let $\tilde{\mathcal{Q}}_e := G[Y]^{(q+r)}(e)[U]$. We have $|\tilde{\mathcal{Q}}_e| \geq \xi(\mu_1 n)^q$. Consider the random subgraph of $\tilde{\mathcal{Q}}_e'$ consisting of all q-sets $Q \in \tilde{\mathcal{Q}}_e$ satisfying $Q \subseteq U'$. For every $Q \in \tilde{\mathcal{Q}}_e$, we have $\mathbb{P}(Q \subseteq U') = \mu_2^q$. Hence, $\mathbb{E}|\tilde{\mathcal{Q}}_e'| = \mu_2^q |\tilde{\mathcal{Q}}_e| \geq \xi(\mu_1\mu_2 n)^q$. Thus, using Corollary 5.11 and a union bound, we deduce that whp for all $e \in G^{(r)}$, we have $|G[Y]^{(q+r)}(e)[U']| \geq (1 - |U|^{-1/5})\xi(\mu_1\mu_2 n)^q$. Assuming that this holds for U', it is easy to see that for all $W \subseteq U$ of size $|W| \leq |U|^{3/5}$, we have $|G[Y]^{(q+r)}(e)[U' \triangle W]| \geq (1 - |U|^{-1/5})\xi(\mu_1\mu_2 n)^q - |W| n^{q-1} \geq (\xi - 2|U|^{-1/5})(\mu_1\mu_2 n)^q$.

Finally, we check (R4). Let $h \in [r]_0$ and $F \subseteq G^{(h)}$ with $1 \leq |F| \leq 2^h$. Since U is $(\varepsilon, \mu_1, \xi, q, r)$-random in G, we have that $\bigcap_{f \in F} G(f)[U]$ is an $(\varepsilon, \xi, q-h, r-h)$-complex. Then, by Proposition 5.13, with probability at least $1 - \mathrm{e}^{-|U|/8}$, $\bigcap_{f \in F} G(f)[U' \triangle W]$ is an $(\varepsilon + 4|U|^{-1/5}, \xi - 3|U|^{-1/5}, q-h, r-h)$-complex for all $W \subseteq U$ of size $|W| \leq |U|^{3/5}$. Thus, a union bound yields the desired result. □

Proposition 7.9. *Let $1/n \ll \varepsilon \ll \mu_1, \mu_2, 1-\mu_2, \xi, 1/q$ and $r \in [q-1]$. Let G be a complex on n vertices and let $U \subseteq V(G)$ be of size $\lfloor \mu_1 n \rfloor$ and $(\varepsilon, \mu_1, \xi, q, r)$-random in G. Then there exists $\tilde{U} \subseteq U$ of size $\lfloor \mu_2 |U| \rfloor$ such that*

(i) *$\tilde{U}$ is $(\varepsilon + |U|^{-1/6}, \mu_2, \xi - |U|^{1/6}, q, r)$-random in $G[U]$ and*
(ii) *$U \setminus \tilde{U}$ is $(\varepsilon + |U|^{-1/6}, \mu_1(1-\mu_2), \xi - |U|^{1/6}, q, r)$-random in G.*

Proof. Pick $U' \subseteq U$ randomly by including every vertex from U independently with probability μ_2. Clearly, by Lemma 5.7(i), we have with probability at least $1 - 2\mathrm{e}^{-2|U|^{1/7}}$ that $|U'| = \mu_2 |U| \pm |U|^{4/7}$.

It is easy to see that U is $(\varepsilon + 0.5|U|^{-1/6}, 1, \xi - 0.5|U|^{-1/6}, q, r)$-random in $G[U]$. Hence, by Proposition 7.8, whp $U' \triangle W$ is $(\varepsilon + |U|^{-1/6}, \mu_2, \xi - |U|^{1/6}, q, r)$-random in $G[U]$ for all $W \subseteq U$ of size $|W| \le |U|^{3/5}$. Moreover, since $U'' := U \setminus U'$ is a random subset obtained by including every vertex from U independently with probability $1 - \mu_2$, Proposition 7.8 implies that whp $U'' \triangle W$ is $(\varepsilon + 0.5|U|^{-1/6}, \mu_1(1-\mu_2), \xi - 0.5|U|^{1/6}, q, r)$-random in G for all $W \subseteq U$ of size $|W| \le |U|^{3/5}$.

Let U' be a set that has the above properties. Let $W \subseteq V(G)$ be a set with $|W| \le |U|^{3/5}$ such that $|U' \triangle W| = \lfloor \mu_2 |U| \rfloor$ and let $\tilde{U} := U' \triangle W$. By the above, $\tilde{U}$ satisfies (i) and (ii). □

We can now obtain a vortex by inductively applying Proposition 7.9.

Proof of Lemma 7.6. Recursively define $n_0 := n$ and $n_i := \lfloor \mu n_{i-1} \rfloor$. Observe that $\mu^i n \ge n_i \ge \mu^i n - 1/(1-\mu)$. Further, for $i \in \mathbb{N}$, let $a_i := 2n^{-1/6} \sum_{j \in [i]} \mu^{-(j-1)/6}$. Let $\ell := 1 + \max\{i \ge 0 : n_i \ge m'\}$ and let $m := n_\ell$. Note that $\lfloor \mu m' \rfloor \le m \le m'$. Moreover, we have that

$$a_\ell = 2n^{-1/6} \frac{\mu^{-\ell/6} - 1}{\mu^{-1/6} - 1} \le 2 \frac{(\mu^{\ell-1} n)^{-1/6}}{1 - \mu^{1/6}} \le 2 \frac{m'^{-1/6}}{1 - \mu^{1/6}} \le \varepsilon$$

since $\mu^{\ell-1} n \ge n_{\ell-1} \ge m'$.

By Fact 7.2, $U_0 := V(G)$ is $(\varepsilon/\xi, 1, \xi, q, r)$-random in G. Hence, by Proposition 7.9, there exists a set $U_1 \subseteq U_0$ of size n_1 such that U_1 is $(\sqrt{\varepsilon} + a_1, \mu, \xi - a_1, q, r)$-random in $G[U_0]$. If $\ell = 1$, this completes the proof, so assume that $\ell \ge 2$.

Now, suppose that for some $i \in [\ell - 1]$, we have already found a $(\sqrt{\varepsilon} + a_i, \mu, \xi - a_i, q, r, n_i)$-vortex $U_0, \dots, U_i$ in G. Note that this is true for $i = 1$. In particular, U_i is $(\sqrt{\varepsilon} + a_i, \mu, \xi - a_i, q, r)$-random in $G[U_{i-1}]$ by (V4). By Proposition 7.9, there exists a subset U_{i+1} of U_i of size n_{i+1} such that U_{i+1} is $(\sqrt{\varepsilon} + a_i + n_i^{-1/6}, \mu, \xi - a_i - n_i^{-1/6}, q, r)$-random in $G[U_i]$ and $U_i \setminus U_{i+1}$ is $(\sqrt{\varepsilon} + a_i + n_i^{-1/6}, \mu(1-\mu), \xi - a_i - n_i^{-1/6}, q, r)$-random in $G[U_{i-1}]$. Thus, $U_0, \dots, U_{i+1}$ is a $(\sqrt{\varepsilon} + a_{i+1}, \mu, \xi - a_{i+1}, q, r, n_{i+1})$-vortex in G.

Finally, $U_0, \dots, U_\ell$ is an $(\sqrt{\varepsilon} + a_\ell, \mu, \xi - a_\ell, q, r, m)$-vortex in G. □

Proposition 7.10. *Let $1/n \ll \varepsilon \ll \mu, \xi, 1/q$ such that $\mu \le 1/2$ and $r \in [q-1]$. Suppose that G is a complex on n vertices and U is $(\varepsilon, \mu, \xi, q, r)$-random in G. Suppose that $L \subseteq G^{(r)}$ satisfies $\Delta(L) \le \varepsilon n$. Then U is still $(\sqrt{\varepsilon}, \mu, \xi - \sqrt{\varepsilon}, q, r)$-random in $G - L$.*

Proof. Clearly, (R1) still holds. Moreover, using Proposition 5.4 it is easy to see that (R2) and (R3) are preserved. To see (R4), let $h \in [r]_0$ and $F \subseteq (G-L)^{(h)}$ with $1 \le |F| \le 2^h$. By assumption, we have that $\bigcap_{f \in F} G(f)[U]$ is an $(\varepsilon, \xi, q-h, r-h)$-complex. By Fact 5.5, we can obtain $\bigcap_{f \in F} (G-L)(f)[U]$ from $\bigcap_{f \in F} G(f)[U]$ by repeatedly deleting an $(r - |S|)$-graph $L(S)$, where $S \subseteq f \in F$. There are at most $|F| 2^h \le 2^{2h}$ such graphs, and we have $\Delta(L(S)) \le \varepsilon n \le \varepsilon^{2/3} |U - \bigcup F|$ by Fact 5.3 if $|S| < r$. If $|S| = r$, we have $S \in F$ and thus $L(S)$ is empty, in which case we can ignore its removal. Thus, a repeated application of Proposition 5.6(iv) (with $r - |S|, r - h$ playing the roles of r', r) shows that $\bigcap_{f \in F} (G-L)(f)[U]$ is an $(\varepsilon + 2^{2r+1} \varepsilon^{2/3}, \xi - 2^{2r+1} \varepsilon^{2/3}, q-h, r-h)$-complex. □

7.3. **Existence of cleaners.** The aim of this subsection is to apply the Cover down lemma to each 'level' i of the vortex to obtain a 'cleaning graph' H_i (playing the role of H^*) for each $i \in [\ell]$ (see Lemma 7.12). Let G be a complex and $U_0 \supseteq U_1 \supseteq \cdots \supseteq U_\ell$ a vortex in G. We say that $H_1, \dots, H_\ell$ is a (γ, ν, q, r)-*cleaner* if the following hold for all $i \in [\ell]$:

(C1) $H_i \subseteq G^{(r)}[U_{i-1}] - G^{(r)}[U_{i+1}]$, where $U_{\ell+1} := \emptyset$;
(C2) $\Delta(H_i) \le \nu |U_{i-1}|$;
(C3) H_i and H_{i+1} are edge-disjoint, where $H_{\ell+1} := \emptyset$;
(C4) whenever $L \subseteq G^{(r)}[U_{i-1}]$ is such that $\Delta(L) \le \gamma |U_{i-1}|$ and $H_i \cup L$ is $K_q^{(r)}$-divisible, there exists a $K_q^{(r)}$-packing $\mathcal{K}$ in $G[H_i \cup L]$ which covers all edges of $H_i \cup L$ except possibly some inside U_i.

Note that (C1) and (C3) together imply that $H_1, \dots, H_\ell$ are edge-disjoint. The following proposition will be used to ensure (C3).

Proposition 7.11. *Let $1/n \ll \varepsilon \ll \mu, \xi, 1/q$ and $r \in [q-1]$. Let $\xi' := \xi(1/2)^{(8^q+1)}$. Let G be a complex on n vertices and let $U \subseteq V(G)$ of size μn and $(\varepsilon, \mu, \xi, q, r)$-random in G. Suppose that H is a random subgraph of $G^{(r)}$ obtained by including every edge of $G^{(r)}$ independently with probability $1/2$. Then with probability at least $1 - \mathrm{e}^{-n^{1/10}}$,*

(i) *U is $(\sqrt{\varepsilon}, \mu, \xi', q, r)$-random in $G[H]$ and*
(ii) *G is $(\sqrt{\varepsilon}, q, r)$-dense with respect to $H - G^{(r)}[\bar{U}]$, where $\bar{U} := V(G) \setminus U$.*

Proof. Let $Y \subseteq G^{(q)}$ and $d \ge \xi$ be such that (R1)–(R4) hold for U and G. We first consider (i). Clearly, (R1) holds. We next check (R2). For $e \in G^{(r)}$ and $x \in [q-r]_0$, let $\mathcal{Q}_{e,x} := \{Q \in G[Y]^{(q)}(e) : |Q \cap U| = x\}$. Thus, $|\mathcal{Q}_{e,x}| = (1 \pm \varepsilon) bin(q-r, \mu, x) d n^{q-r}$.

Consider $e \in G^{(r)}$ and $x \in [q-r]_0$. We view $\mathcal{Q}_{e,x}$ as a $(q-r)$-graph and consider the random subgraph $\mathcal{Q}'_{e,x}$ containing all $Q \in \mathcal{Q}_{e,x}$ such that $\binom{Q \cup e}{r} \setminus \{e\} \subseteq H$. For each $Q \in \mathcal{Q}_{e,x}$, we have $\mathbb{P}(Q \in \mathcal{Q}'_{e,x}) = (1/2)^{\binom{q}{r}-1}$. Thus, using Corollary 5.11 we deduce that with probability at least $1 - \mathrm{e}^{-n^{1/6}}$ we have

$$\begin{aligned} |\mathcal{Q}'_{e,x}| &= (1 \pm \varepsilon) \mathbb{E} |\mathcal{Q}'_{e,x}| = (1 \pm \varepsilon)(1/2)^{\binom{q}{r}-1} (1 \pm \varepsilon) bin(q-r, \mu, x) d n^{q-r} \\ &= (1 \pm \sqrt{\varepsilon}) d' bin(q-r, \mu, x) d n^{q-r}, \end{aligned}$$

where $d' := d(1/2)^{\binom{q}{r}-1} \ge \xi'$. Thus, a union bound yields that with probability at least $1 - \mathrm{e}^{-n^{1/7}}$, (R2) holds.

Next, we check (R3). By assumption, we have $|G[Y]^{(q+r)}(e)[U]| \ge \xi (\mu n)^q$ for all $e \in G^{(r)}$. Let $Q_e := G[Y]^{(q+r)}(e)[U]$ and consider the random subgraph $\mathcal{Q}'_e$ containing all $Q \in \mathcal{Q}_e$ such that $\binom{Q \cup e}{r} \setminus \{e\} \subseteq H$. For each $Q \in \mathcal{Q}_e$, we have $\mathbb{P}(Q \in \mathcal{Q}'_e) = (1/2)^{\binom{q+r}{r}-1}$. Thus, using Corollary 5.11 we deduce that with probability at least $1 - \mathrm{e}^{-n^{1/6}}$ we have

$$|\mathcal{Q}'_e| = (1 \pm \varepsilon) \mathbb{E} |\mathcal{Q}'_e| \ge (1 - \varepsilon)(1/2)^{\binom{q+r}{r}-1} \xi (\mu n)^q \ge \xi' (\mu n)^q,$$

and a union bound implies that this is true for all $e \in G^{(r)}$ with probability at least $1 - \mathrm{e}^{-n^{1/7}}$.

Next, we check (R4). Let $h \in [r]_0$ and $F \subseteq G^{(h)}$ with $1 \le |F| \le 2^h$. We know that $\bigcap_{f \in F} G(f)[U]$ is an $(\varepsilon, \xi, q-h, r-h)$-complex. By Proposition 5.15 (applied with $G[U \cup \bigcup F], \{G[U \cup \bigcup F]^{(r)}\}$ playing the roles of $G, \mathcal{P}$), with probability at least $1 - \mathrm{e}^{-|U|^{1/8}}$, $\bigcap_{f \in F} G[H](f)[U]$ is a $(\sqrt{\varepsilon}, \xi', q-h, r-h)$-complex. Thus, a union bound over all $h \in [r]_0$ and $F \subseteq G^{(h)}$ with $1 \le |F| \le 2^h$ yields that with probability at least $1 - \mathrm{e}^{-n^{1/9}}$, (R4) holds.

Finally, we check (ii). Consider $e \in G^{(r)}$ and let $Q_e := G[(G^{(r)} - G^{(r)}[\bar{U}]) \cup e]^{(q)}(e)$. Note by (R2), we have $|G[Y]^{(q)}(e)[U]| = (1 \pm \varepsilon) bin(q-r, \mu, q-r) d n^{q-r}$, so $|\mathcal{Q}_e| \ge |G[Y]^{(q)}(e)[U]| \ge (1 - \varepsilon) \xi \mu^{q-r} n^{q-r}$. We view Q_e as a $(q-r)$-graph and consider the random subgraph Q'_e containing all $Q \in \mathcal{Q}_e$ such that $\binom{Q \cup e}{r} \setminus \{e\} \subseteq H$. For each $Q \in \mathcal{Q}_e$, we have $\mathbb{P}(Q \in Q'_e) = (1/2)^{\binom{q}{r}-1}$.

Thus, using Corollary 5.11 we deduce that with probability at least $1-\mathrm{e}^{-n^{1/6}}$ we have

$$|\mathcal{Q}'_e| \geq 0.9\mathbb{E}|\mathcal{Q}'_e| \geq 0.9(1/2)^{\binom{q}{r}-1}(1-\varepsilon)\xi\mu^{q-r}n^{q-r} \geq \sqrt{\varepsilon}n^{q-r}.$$

A union bound easily implies that with probability at least $1-\mathrm{e}^{-n^{1/7}}$, this holds for all $e \in G^{(r)}$. □

Lemma 7.12. *Let $1/m \ll \gamma \ll \varepsilon \ll \nu \ll \mu, \xi, 1/q$ be such that $\mu \leq 1/2$ and $r \in [q-1]$. Assume that $(*)_i$ is true for all $i \in [r-1]$. Let G be a complex and $U_0 \supseteq U_1 \supseteq \dots \supseteq U_\ell$ an $(\varepsilon, \mu, \xi, q, r, m)$-vortex in G. Then there exists a (γ, ν, q, r)-cleaner.*

Proof. For $i \in [\ell]$, define $U'_i := U_i \setminus U_{i+1}$, where $U_{\ell+1} := \emptyset$. For $i \in [\ell-1]$, let $\mu_i := \mu(1-\mu)$, and let $\mu_\ell := \mu$. By (V4) and (V5), we have for all $i \in [\ell]$ that U'_i is $(\varepsilon, \mu_i, \xi, q, r)$-random in $G[U_{i-1}]$.

Split $G^{(r)}$ randomly into G_0 and G_1, that is, independently for every edge $e \in G^{(r)}$, put e into G_0 with probability $1/2$ and into G_1 otherwise. We claim that with positive probability, the following hold for every $i \in [\ell]$:

(i) U'_i is $(\sqrt{\varepsilon}, \mu_i, \xi(1/2)^{(8^q+1)}, q, r)$-random in $G[G_{i \bmod 2}][U_{i-1}]$;
(ii) $G[U_{i-1}]$ is $(\sqrt{\varepsilon}, q, r)$-dense with respect to $G_{i \bmod 2}[U_{i-1}] - G^{(r)}[U_{i-1} \setminus U'_i]$.

By Proposition 7.11, the probability that (i) or (ii) do not hold for $i \in [\ell]$ is at most $\mathrm{e}^{-|U_{i-1}|^{1/10}} \leq |U_{i-1}|^{-2}$. Since $\sum_{i=1}^{\ell} |U_{i-1}|^{-2} < 1$, we deduce that with positive probability, (i) and (ii) hold for all $i \in [\ell]$.

Therefore, there exist G_0, G_1 satisfying the above properties. For every $i \in [\ell]$, we will find H_i using the Cover down lemma (Lemma 7.4). Let $i \in [\ell]$. Apply Lemma 7.4 with the following objects/parameters:

object/parameter	$G[G_{i \bmod 2}][U_{i-1}]$	U'_i	$G[U_{i-1}]$	$\lvert U_{i-1}\rvert$	γ	$\sqrt{\varepsilon}$	ν	μ_i	$\xi(1/2)^{(8^q+1)}$
playing the role of	G	U	$\tilde{G}$	n	γ	ε	ν	μ	ξ

Hence, there exists

$$H_i \subseteq G_{i \bmod 2}[U_{i-1}] - G_{i \bmod 2}[U_{i-1} \setminus U'_i] \subseteq G_{i \bmod 2}[U_{i-1}] - G^{(r)}[U_{i+1}]$$

with $\Delta(H_i) \leq \nu|U_{i-1}|$ such that whenever $L \subseteq G^{(r)}[U_{i-1}]$ is such that $\Delta(L) \leq \gamma|U_{i-1}|$ and $H_i \cup L$ is $K_q^{(r)}$-divisible, there exists a $K_q^{(r)}$-packing $\mathcal{K}$ in $G[H_i \cup L]$ which covers all edges of $H_i \cup L$ except possibly some inside $U'_i \subseteq U_i$. Thus, (C1), (C2) and (C4) hold.

Since G_0 and G_1 are edge-disjoint, (C3) holds as well. Thus, $H_1, \dots, H_\ell$ is a (γ, ν, q, r)-cleaner. □

7.4. **Obtaining a near-optimal packing.**

We can now carry out the actual iteration to obtain a near optimal packing, i.e. a $K_q^{(r)}$-packing which covers all but a bounded number of edges.

Lemma 7.13. *Let $1/m \ll \varepsilon \ll \mu \ll \xi, 1/q$ and $r \in [q-1]$. Assume that $(*)_k$ is true for all $k \in [r-1]$. Let G be a $K_q^{(r)}$-divisible (ε, ξ, q, r)-supercomplex and $U_0 \supseteq U_1 \supseteq \dots \supseteq U_\ell$ an $(\varepsilon, \mu, \xi, q, r, m)$-vortex in G. Then there exists a $K_q^{(r)}$-packing $\mathcal{K}$ in G which covers all edges of $G^{(r)}$ except possibly some inside U_ℓ.*

Proof. Choose new constants $\gamma, \nu > 0$ such that $1/m \ll \gamma \ll \varepsilon \ll \nu \ll \mu \ll \xi, 1/q$.

Apply Lemma 7.12 to obtain a (γ, ν, q, r)-cleaner $H_1, \dots, H_\ell$. Note that by (V4) and Fact 7.2(ii), $G[U_i]$ is an (ε, ξ, q, r)-supercomplex for all $i \in [\ell]$, and the same holds for $i = 0$ by assumption.

Suppose that for some $i \in [\ell]$, we have found a $K_q^{(r)}$-packing $\mathcal{K}^*_{i-1}$ in G such that:

(i) $\mathcal{K}^*_{i-1}$ covers all edges of $G^{(r)}$ that are not inside U_{i-1};
(ii) $\mathcal{K}^*_{i-1}$ does not cover any edges from $H_i \cup G^{(r)}[U_i]$;

(iii) $\Delta(\mathcal{K}^{*(r)}_{i-1}[U_{i-1}]) \leq \mu|U_{i-1}|$.

Note that this is trivially true for $i = 1$ with $\mathcal{K}^*_0 := \emptyset$. Let $H_{\ell+1} := \emptyset$ and $U_{\ell+1} := \emptyset$.

We now intend to find a $K_q^{(r)}$-packing $\mathcal{K}_i$ in $G - \mathcal{K}^{*(r)}_{i-1}$ which covers all edges from $G^{(r)}[U_{i-1}] - \mathcal{K}^{*(r)}_{i-1}$ that are not inside U_i, does not cover any edges from $H_{i+1} \cup G^{(r)}[U_{i+1}]$, and satisfies $\Delta(\mathcal{K}^{(r)}_i[U_i]) \leq \mu|U_i|$. We will obtain $\mathcal{K}_i$ as the union of two packings, one obtained from the Boosted nibble lemma (Lemma 6.4) and one using (C4).

Let $G_{i,nibble} := G^{(r)}[U_{i-1}] - \mathcal{K}^{*(r)}_{i-1} - H_i - G^{(r)}[U_i]$. Note

$$\Delta(\mathcal{K}^{*(r)}_{i-1}[U_{i-1}] \cup H_i \cup G^{(r)}[U_i]) \leq \mu|U_{i-1}| + \nu|U_{i-1}| + \mu|U_{i-1}| \leq 3\mu|U_{i-1}|.$$

So Proposition 5.6(i) and (ii) imply that $G[U_{i-1}][G_{i,nibble}]$ is still $(2^{r+2}\mu, d, q, r)$-regular and $(\xi/2, q + r, r)$-dense for some $d \geq \xi$. Since $\mu \ll \xi$, we can apply the boosted nibble lemma (Lemma 6.4) to obtain a $K_q^{(r)}$-packing $\mathcal{K}_{i,nibble}$ in $G[U_{i-1}][G_{i,nibble}]$ such that $\Delta(L_{i,nibble}) \leq \frac{1}{2}\gamma|U_{i-1}|$, where $L_{i,nibble} := G_{i,nibble} - \mathcal{K}^{(r)}_{i,nibble}$.

Since

$$G^{(r)} - \mathcal{K}^{*(r)}_{i-1} - \mathcal{K}^{(r)}_{i,nibble} = G^{(r)}[U_{i-1}] - \mathcal{K}^{*(r)}_{i-1} - \mathcal{K}^{(r)}_{i,nibble} = H_i \cup G^{(r)}[U_i] \cup L_{i,nibble},$$

we know that $H_i \cup G^{(r)}[U_i] \cup L_{i,nibble}$ is $K_q^{(r)}$-divisible. By (C1) and (C3), we know that $H_{i+1} \cup G^{(r)}[U_{i+1}] \subseteq G^{(r)}[U_i] - H_i$. We can thus apply Corollary 6.8 (with $G[U_i] - H_i$ playing the role of G) to find a $K_q^{(r)}$-divisible subgraph R_i of $G^{(r)}[U_i] - H_i$ containing $H_{i+1} \cup G^{(r)}[U_{i+1}]$ such that $\Delta(L_{i,res}) \leq \frac{1}{2}\gamma|U_i|$, where $L_{i,res} := G^{(r)}[U_i] - H_i - R_i$.

Let $L_i := L_{i,nibble} \cup L_{i,res}$. Clearly, $L_i \subseteq G^{(r)}[U_{i-1}]$ and $\Delta(L_i) \leq \gamma|U_{i-1}|$. Moreover, note that

$$H_i \cup L_i = (H_i \mathbin{\dot\cup} (G^{(r)}[U_i] - H_i) \mathbin{\dot\cup} L_{i,nibble}) - R_i$$

is $K_q^{(r)}$-divisible. Thus, by (C4) there exists a $K_q^{(r)}$-packing $\mathcal{K}_{i,clean}$ in $G[H_i \cup L_i]$ which covers all edges of $H_i \cup L_i$ except possibly some inside U_i. We claim that $\mathcal{K}_i := \mathcal{K}_{i,nibble} \cup \mathcal{K}_{i,clean}$ is the desired packing.

Clearly, $\mathcal{K}_i$ covers all edges of $G^{(r)}[U_{i-1}] - \mathcal{K}^{*(r)}_{i-1}$ that are not inside U_i. On the other hand, the choice of R_i ensures that $\mathcal{K}_i$ does not cover any edges from $H_{i+1} \cup G^{(r)}[U_{i+1}]$. Moreover, $\Delta(\mathcal{K}^{(r)}_i[U_i]) \leq \Delta(H_i \cup L_i) \leq \nu|U_{i-1}| + \gamma|U_{i-1}| \leq \mu|U_i|$.

Let $\mathcal{K}^*_i := \mathcal{K}^*_{i-1} \cup \mathcal{K}_i$. Then (i)–(iii) hold with i being replaced by $i + 1$. Finally, $\mathcal{K}^*_\ell$ will be the desired packing. □

8. Absorbers

In this section we show that for any (divisible) r-graph H in a supercomplex G, we can find an 'exclusive' absorber r-graph A. The following definition makes this precise and the main result of this section is Lemma 8.2.

Definition 8.1 (Absorber)**.** Let G be a complex and $H \subseteq G^{(r)}$. A subgraph $A \subseteq G^{(r)}$ is a $K_q^{(r)}$*-absorber for H in G* if A and H are edge-disjoint and both $G[A]$ and $G[A \cup H]$ have a $K_q^{(r)}$-decomposition.

Lemma 8.2 (Absorbing lemma)**.** *Let $1/n \ll \gamma, 1/h, \varepsilon \ll \xi, 1/q$ and $r \in [q-1]$. Assume that $(*)_i$ is true for all $i \in [r-1]$. Let G be an (ε, ξ, q, r)-supercomplex on n vertices and let H be a $K_q^{(r)}$-divisible subgraph of $G^{(r)}$ with $|H| \leq h$. Then there exists a $K_q^{(r)}$-absorber A for H in G with $\Delta(A) \leq \gamma n$.*

Building on [5], we will construct absorbers as a concatenation of 'transformers' and special 'canonical graphs'. The goal is to transform an arbitrary divisible r-graph H into a canonical graph. In the following subsection, we will construct transformers. In Section 8.2, we will prove the existence of suitable canonical graphs. We will prove Lemma 8.2 in Section 8.3.

We now briefly discuss the case $r = 1$. Recall that a $K_q^{(1)}$-decomposition of a complex G corresponds to a perfect matching of $G^{(q)}$ (ignoring isolated vertices). Assume first that $H = \{e_1, \dots, e_q\}$. Choose any q-set $Q_0 \in G^{(q)}$ with $Q_0 = \{v_1, \dots, v_q\}$. Now, for every $i \in [q]$, choose a $Q_i \in G^{(q)}(e_i) \cap G^{(q)}(\{v_i\})$ (cf. Fact 5.2). Choose these sets such that $\bigcup H, Q_0, \dots, Q_q$ are pairwise disjoint. Let $A := \bigcup_{i \in [q]_0} \binom{Q_i}{1}$. It is then easy to see that A is a $K_q^{(1)}$-absorber for H in G. More generally, if H is any $K_q^{(1)}$-divisible 1-graph, then $q \mid |H|$, so we can partition the edges of H into $|H|/q$ subgraphs of equal size and then find an absorber for each of these subgraphs (successively so that they are edge-disjoint.) Thus, for the remainder of this section, we will assume that $r \geq 2$.

8.1. **Transformers.** Roughly speaking, a transformer T can be viewed as transforming a leftover graph H into a new leftover H' (where we set aside T and H' earlier).

Definition 8.3 (Transformer)**.** Let G be a complex and assume that $H, H' \subseteq G^{(r)}$. A subgraph $T \subseteq G^{(r)}$ is an *$(H, H'; K_q^{(r)})$-transformer in G* if T is edge-disjoint from both H and H', and both $G[T \cup H]$ and $G[T \cup H']$ have a $K_q^{(r)}$-decomposition.

Definition 8.4. Let H, H' be r-graphs. A *homomorphism from H to H'* is a map $\phi\colon V(H) \to V(H')$ such that $\phi(e) \in H'$ for all $e \in H$. We let $\phi(H)$ denote the subgraph of H' with vertex set $\phi(V(H))$ and edge set $\{\phi(e) : e \in H\}$. We say that ϕ is *edge-bijective* if $|H| = |\phi(H)| = |H'|$. For two r-graphs H and H', we write $H \rightsquigarrow H'$ if there exists an edge-bijective homomorphism from H to H'.

Note that if $H \rightsquigarrow H'$ and H is $K_q^{(r)}$-divisible, then so is H'. The main lemma of this subsection guarantees a transformer from H to H' if $H \rightsquigarrow H'$.

Lemma 8.5. *Let $1/n \ll \gamma, 1/h, \varepsilon \ll \xi, 1/q$ and $2 \leq r < q$. Assume that $(*)_i$ is true for all $i \in [r-1]$. Let G be an (ε, ξ, q, r)-supercomplex on n vertices and H, H' be vertex-disjoint $K_q^{(r)}$-divisible subgraphs of $G^{(r)}$ of order at most h and such that $H \rightsquigarrow H'$. Then there exists an $(H, H'; K_q^{(r)})$-transformer T in G with $\Delta(T) \leq \gamma n$.*

Suppose that $H \rightsquigarrow H'$ are as in Lemma 8.5, and we aim to find an $(H, H'; K_q^{(r)})$-transformer T. A first attempt would be to pair off each $e \in H$ with $e' = \phi(e) \in H'$ (and view e' as the 'mirror image' of e). Note that $|e \cap e'| = 0$. For each $e \in H$ we now pick a $(q-r)$-set $Q_e^* \in G^{(q)}(e) \cap G^{(q)}(e')$ such that all the Q_e^* are vertex-disjoint from each other. Thus Q_e^* extends e into a q-set $e \cup Q_e^* =: Q_e$ and e' into a q-set $e' \cup Q_e^* =: Q_{e'}$. (Note this is similar to the above argument for $r = 1$ and also part of the transformer construction in [5]. It is also our first step in the proof of Lemma 8.5.) Let $T_0^\diamond := \bigcup_{e \in H} (Q_e^{\leq} \cup Q_{e'}^{\leq})^{(r)} - H - H'$. Note that there exists a $K_q^{(r)}$-packing $\mathcal{K} := \bigcup_{e \in H} Q_e^{\leq}$ in $G[T_0^\diamond \cup H]$ covering all edges of H and a $K_q^{(r)}$-packing $\mathcal{K}' := \bigcup_{e' \in H'} Q_{e'}^{\leq}$ in $G[T_0^\diamond \cup H']$ covering all edges of H'. We next define 'remainders' $R_1' := T_0^\diamond - \mathcal{K}^{(r)}$ and $R_1 := T_0^\diamond - \mathcal{K}'^{(r)}$. Note that R_1' can be viewed as a 'mirror image' of R_1 (in particular, $R_1 \rightsquigarrow R_1'$). Also, the existence of a $(R_1, R_1'; K_q^{(r)})$-transformer T_1 would yield the desired T by defining $T := T_0^\diamond \cup T_1$. The crucial difference to the task we faced originally is that now the mirror image e' of each $e \in R_1$ satisfies $|e \cap e'| \geq 1$.

The idea is now to proceed inductively. We view the construction of $T_0^\diamond$ as step 0. In the ith step we would ideally seek a $(R_i, R_i'; K_q^{(r)})$-transformer T_i, where for each $e \in R_i$ and its mirror image $e' \in R_i'$, we have $|e \cap e'| \geq i$. (We refer to the latter property of R_i as being '$(r-i)$-projectable'.) Instead of constructing T_i explicitly, we construct a 'partial transformer' $T_i^\diamond$ with the property that $T_i^\diamond \cup R_i$ and $T_i^\diamond \cup R_i'$ both have a $K_q^{(r)}$-packing such that the remainders R_{i+1} and R_{i+1}' uncovered by these packings form mirror images of each other and such that R_{i+1} is $(r-i-1)$-projectable. Continuing in this way, we arrive at a pair R_r, R_r' of remainders which are mirror images of each other and such that R_r is 0-projectable, at which point we can terminate the process.

For $i \in [r-1]$, the 'partial transformers' $T_i^\diamond$ are constructed in Lemma 8.8. For this, we consider the 'ith level L_i of the overlap' between R_i and R_i' (so L_i can be viewed as an i-graph). We then add an 'absorber' A to the overlap, i.e. both A and $A \cup L_i$ have a $K_{q-r+i}^{(i)}$-decomposition (which are $(i+1)$-disjoint). Since $i \in [r-1]$, we can simply apply the inductive assertion $(*)_i$ to find these decompositions. This has the effect of absorbing the part of R_i (and its mirror image) that is not $(r-i-1)$-projectable, resulting in the desired partial transformer $T_i^\diamond$. We can finally concatenate these partial transformers $T_0^\diamond, \dots, T_{r-1}^\diamond$ in Lemma 8.5 to form the desired transformer T.

We now formalise the notion of being projectable.

Definition 8.6. Let V be a set and let V_1, V_2 be disjoint subsets of V, and let $\phi\colon V_1 \to V_2$ be a map. Let $\bar{\phi}$ be the extension of ϕ to $V \setminus V_2$, where $\bar{\phi}(x) := x$ for all $x \in V \setminus (V_1 \cup V_2)$. Let $r \in \mathbb{N}$ and suppose that R is an r-graph with $V(R) \subseteq V$ and $i \in [r]_0$. We say that R is (ϕ, V, V_1, V_2, i)*-projectable* if the following hold:

- (Y1) for every $e \in R$, we have that $e \cap V_2 = \emptyset$ and $|e \cap V_1| \in [i]$ (so if $i = 0$, then R must be empty since $[0] = \emptyset$);
- (Y2) for every $e \in R$, we have $|\bar{\phi}(e)| = r$;
- (Y3) for every two distinct edges $e, e' \in R$, we have $\bar{\phi}(e) \neq \bar{\phi}(e')$.

Note that if ϕ is injective and $e \cap V_2 = \emptyset$ for all $e \in R$, then (Y2) and (Y3) always hold. If R is (ϕ, V, V_1, V_2, i)-projectable, then let $\phi(R)$ be the r-graph on $\bar{\phi}(V(R) \setminus V_2)$ with edge set $\{\bar{\phi}(e) : e \in R\}$. For an r-graph P with $V(P) \subseteq V \setminus V_2$ that satisfies (Y2), let P^ϕ the r-graph on $V(P) \cup V_1$ that consists of all $e \in \binom{V \setminus V_2}{r}$ such that $\bar{\phi}(e) = \bar{\phi}(e')$ for some $e' \in P$.

The following properties are straightforward to check.

Proposition 8.7. *Let $V, V_1, V_2, \phi, R, P, r, i$ be as above and assume that R is (ϕ, V, V_1, V_2, i)-projectable. Then the following hold:*

- (i) $R \rightsquigarrow \phi(R)$;
- (ii) *for all $e' \in \phi(R)$, we have $e' \cap V_1 = \emptyset$ and $|e' \cap V_2| \in [i]$;*
- (iii) *assume that for all $e \in R$, we have $|e \cap V_1| = i$, and let $\mathcal{S}$ contain all $S \in \binom{V_1}{i}$ such that S is contained in some edge of R, then*

$$R = \dot{\bigcup_{S \in \mathcal{S}}} (S \uplus R(S)) \quad \text{and} \quad \phi(R) = \dot{\bigcup_{S \in \mathcal{S}}} (\phi(S) \uplus R(S)).$$

Lemma 8.8. *Let $1/n \ll \gamma' \ll \gamma, \varepsilon \ll \xi, 1/q$ and $1 \le i < r < q$. Assume that $(*)_{r-i}$ is true. Let G be an (ε, ξ, q, r)-supercomplex on n vertices and let $S_1, S_2 \in G^{(i)}$ with $S_1 \cap S_2 = \emptyset$. Let $\phi\colon S_1 \to S_2$ be a bijection and let $\bar{\phi}$ be as in Definition 8.6 with S_1, S_2 playing the roles of V_1, V_2, respectively. Moreover, suppose that L is a $K_{q-i}^{(r-i)}$-divisible subgraph of $G(S_1)^{(r-i)} \cap G(S_2)^{(r-i)}$ with $|V(L)| \le \gamma' n$.*

Then there exist $T, R \subseteq G^{(r)}$ such that the following hold:

- (TR1) *R is $(\phi, V(G), S_1, S_2, i-1)$-projectable;*
- (TR2) *T is an $((S_1 \uplus L) \cup \phi(R), (S_2 \uplus L) \cup R; K_q^{(r)})$-transformer in G;*
- (TR3) $|V(T \cup R)| \le \gamma n$.

Proof. We may assume that $\gamma \ll \varepsilon$. Choose $\mu > 0$ with $\gamma' \ll \mu \ll \gamma \ll \varepsilon$. We split the argument into two parts. First, we will establish the following claim, which is the essential part and relies on $(*)_{r-i}$.

Claim 1: There exist $\hat{T}, R_{1,A}, R_{1,A\cup L} \subseteq G^{(r)}$ such that the following hold:

- (tr1) *$R_{1,A}$ and $R_{1,A\cup L}$ are $(\phi, V(G), S_1, S_2, i-1)$-projectable;*
- (tr2) *$\hat{T}$, $S_1 \uplus L$, $S_2 \uplus L$, $R_{1,A}$, $\phi(R_{1,A})$, $R_{1,A\cup L}$, $\phi(R_{1,A\cup L})$ are pairwise edge-disjoint r-graphs;*
- (tr3) *$\hat{T}$ is an $((S_1 \uplus L) \cup R_{1,A\cup L} \cup \phi(R_{1,A}), (S_2 \uplus L) \cup R_{1,A} \cup \phi(R_{1,A\cup L}); K_q^{(r)})$-transformer in G;*

(tr4) $|V(\hat{T} \cup R_{1,A} \cup R_{1,A\cup L})| \leq 2\mu n$.

Proof of claim: By Corollary 5.14 and Lemma 5.7(i), there exists a subset $U \subseteq V(G)$ with $0.9\mu n \leq |U| \leq 1.1\mu n$ such that $G' := G[U \cup S_1 \cup S_2 \cup V(L)]$ is a $(2\varepsilon, \xi - \varepsilon, q, r)$-supercomplex. By Proposition 5.1, $G'' := G'(S_1) \cap G'(S_2)$ is a $(2\varepsilon, \xi-\varepsilon, q-i, r-i)$-supercomplex. Clearly, $L \subseteq G''^{(r-i)}$ and $\Delta(L) \leq \gamma' n \leq \sqrt{\gamma'}|U|$. Thus, by Proposition 5.6(v), $G''-L$ is a $(3\varepsilon, \xi-2\varepsilon, q-i, r-i)$-supercomplex. By Corollary 6.8, there exists $H \subseteq G''^{(r-i)} - L$ such that $A := G''^{(r-i)} - L - H$ is $K_{q-i}^{(r-i)}$-divisible and $\Delta(H) \leq \gamma' n$. In particular, by Proposition 5.6(v) we have that

(i) $G''[A]$ is a $K_{q-i}^{(r-i)}$-divisible $(3\varepsilon, \xi/2, q-i, r-i)$-supercomplex;
(ii) $G''[A \cup L]$ is a $K_{q-i}^{(r-i)}$-divisible $(3\varepsilon, \xi/2, q-i, r-i)$-supercomplex.

By (i) and $(*)_{r-i}$, there exists a $K_{q-i}^{(r-i)}$-decomposition $\mathcal{K}_A$ of $G''[A]$. Clearly, $\Delta(\mathcal{K}_A^{(r-i+1)}) \leq q$. Thus, by (ii), Proposition 5.6(v) and $(*)_{r-i}$, there also exists a $K_{q-i}^{(r-i)}$-decomposition $\mathcal{K}_{A\cup L}$ of $G''[A \cup L] - \mathcal{K}_A^{(r-i+1)}$. For $j \in [2]$, define

$$\begin{aligned} \mathcal{K}_{j,A} &:= (S_j \uplus \mathcal{K}_A^{(q-i)})^{\leq}, \\ \mathcal{K}_{j,A\cup L} &:= (S_j \uplus \mathcal{K}_{A\cup L}^{(q-i)})^{\leq}, \\ R_{j,A} &:= \{e \in \mathcal{K}_{j,A}^{(r)} : |e \cap S_j| \in [i-1]\}, \\ R_{j,A\cup L} &:= \{e \in \mathcal{K}_{j,A\cup L}^{(r)} : |e \cap S_j| \in [i-1]\}. \end{aligned}$$

Note that $R_{1,A}, R_{2,A}, R_{1,A\cup L}, R_{2,A\cup L}$ are empty if $i = 1$. Crucially, since $\mathcal{K}_A$ and $\mathcal{K}_{A\cup L}$ are $(r-i+1)$-disjoint and since A and L are edge-disjoint, we have that

(†) $S_1 \uplus L$, $S_2 \uplus L$, $S_1 \uplus A$, $S_2 \uplus A$, $\mathcal{K}_A^{(r)}$, $\mathcal{K}_{A\cup L}^{(r)}$, $R_{1,A}$, $R_{2,A}$, $R_{1,A\cup L}$, $R_{2,A\cup L}$ are pairwise edge-disjoint r-graphs.

Observe that for $j \in [2]$, we have

$$\mathcal{K}_{j,A}^{(r)} = (S_j \uplus A) \mathbin{\dot\cup} R_{j,A} \mathbin{\dot\cup} \mathcal{K}_A^{(r)}, \tag{8.1}$$

$$\mathcal{K}_{j,A\cup L}^{(r)} = (S_j \uplus (A \cup L)) \mathbin{\dot\cup} R_{j,A\cup L} \mathbin{\dot\cup} \mathcal{K}_{A\cup L}^{(r)}. \tag{8.2}$$

Define

$$\hat{T} := (S_1 \uplus A) \cup (S_2 \uplus A) \cup \mathcal{K}_A^{(r)} \cup \mathcal{K}_{A\cup L}^{(r)}.$$

We now check that (tr1)–(tr4) hold. First observe that $R_{1,A}$ and $R_{1,A\cup L}$ are $(\phi, V(G), S_1, S_2, i)$-projectable since ϕ is injective and for all $e \in R_{1,A} \cup R_{1,A\cup L}$, we have $|e \cap S_1| \in [i-1]$ and $e \cap S_2 = \emptyset$, so (tr1) holds. Note that $R_{2,A} = \phi(R_{1,A})$ and $R_{2,A\cup L} = \phi(R_{1,A\cup L})$. Hence, (†) implies (tr2). Moreover, by (8.1), (8.2) we have that

$$\begin{aligned} \hat{T} \cup (S_1 \uplus L) \cup R_{1,A\cup L} \cup \phi(R_{1,A}) &= \mathcal{K}_{1,A\cup L}^{(r)} \mathbin{\dot\cup} \mathcal{K}_{2,A}^{(r)}, \\ \hat{T} \cup (S_2 \uplus L) \cup R_{1,A} \cup \phi(R_{1,A\cup L}) &= \mathcal{K}_{1,A}^{(r)} \mathbin{\dot\cup} \mathcal{K}_{2,A\cup L}^{(r)}, \end{aligned}$$

so (tr3) holds. Finally, $|V(\hat{T} \cup R_{1,A} \cup R_{1,A\cup L})| \leq |V(G')| \leq 2\mu n$, proving the claim. –

The transformer $\hat{T}$ almost has the required properties, except that to satisfy (TR2) we would have needed $R_{1,A\cup L}$ and $\phi(R_{1,A\cup L})$ to be on the 'other side' of the transformation. In order to resolve this, we carry out an additional transformation step. (Since $R_{1,A}$ and $R_{1,A\cup L}$ are empty if $i = 1$, this additional step is vacuous in this case.)

Claim 2: There exist $T', R' \subseteq G^{(r)}$ such that the following hold:

(tr1′) *R' is $(\phi, V(G), S_1, S_2, i-1)$-projectable;*
(tr2′) *T', R', $\phi(R')$, $\hat{T}$, $S_1 \uplus L$, $S_2 \uplus L$, $R_{1,A}$, $\phi(R_{1,A})$, $R_{1,A\cup L}$, $\phi(R_{1,A\cup L})$ are pairwise edge-disjoint r-graphs;*
(tr3′) *T' is an $(R_{1,A\cup L} \cup R', \phi(R_{1,A\cup L}) \cup \phi(R'); K_q^{(r)})$-transformer in G;*
(tr4′) *$|V(T' \cup R')| \leq 0.7\gamma n$.*

Proof of claim: Let $H' := \hat{T} \cup R_{1,A} \cup \phi(R_{1,A}) \cup (S_1 \uplus L) \cup (S_2 \uplus L)$. Clearly, $\Delta(H') \leq 5\mu n$.

Let $W := V(R_{1,A\cup L}) \cup V(\phi(R_{1,A\cup L}))$. By (tr4), we have that $|W| \leq 4\mu n$. Similarly to the above, by Corollary 5.14 and Lemma 5.7(i), there exists a subset $U' \subseteq V(G)$ with $0.4\gamma n \leq |U'| \leq 0.6\gamma n$ such that $G''' := G[U' \cup W]$ is a $(2\varepsilon, \xi - \varepsilon, q, r)$-supercomplex. Let $\tilde{n} := |U' \cup W|$. Note that $\Delta(H') \leq 5\mu n \leq \sqrt{\mu}\tilde{n}$. Thus, by Proposition 5.6(v), $\tilde{G} := G''' - H'$ is still a $(3\varepsilon, \xi - 2\varepsilon, q, r)$-supercomplex. For every $e \in R_{1,A\cup L}$, let

$$\mathcal{Q}_e := \{Q \in \tilde{G}^{(q)}(e) \cap \tilde{G}^{(q)}(\bar{\phi}(e)) \,:\, Q \cap (S_1 \cup S_2) = \emptyset\}.$$

By Fact 5.2, for every $e \in R_{1,A\cup L} \subseteq \tilde{G}^{(r)}$, we have that $|\tilde{G}^{(q)}(e) \cap \tilde{G}^{(q)}(\bar{\phi}(e))| \geq 0.5\xi\tilde{n}^{q-r}$. Thus, we have that $|\mathcal{Q}_e| \geq 0.4\xi\tilde{n}^{q-r}$. Since $\Delta(R_{1,A\cup L} \cup \bar{\phi}(R_{1,A\cup L})) \leq 4\mu n \leq \sqrt{\mu}\tilde{n}$, we can apply Lemma 6.5 (with $|R_{1,A\cup L}|, 2, \{e, \bar{\phi}(e)\}, \mathcal{Q}_e$ playing the roles of $m, s, L_j, \mathcal{Q}_j$) to find for every $e \in R_{1,A\cup L}$ some $Q_e \in \mathcal{Q}_e$ such that $\binom{e\cup Q_e}{r} \cup \binom{\bar{\phi}(e)\cup Q_e}{r}$ and $\binom{e'\cup Q_{e'}}{r} \cup \binom{\bar{\phi}(e')\cup Q_{e'}}{r}$ are disjoint for distinct $e, e' \in R_{1,A\cup L}$. For each $e \in R_{1,A\cup L}$, let $\tilde{Q}_e := (e \cup Q_e)^{\leq}$ and $\tilde{Q}_{\bar{\phi}(e)} := (\bar{\phi}(e) \cup Q_e)^{\leq}$. Let

$$T' := \bigcup_{e\in R_{1,A\cup L}} (\tilde{Q}_e^{(r)} \cap \tilde{Q}_{\bar{\phi}(e)}^{(r)}) = \bigcup_{e\in R_{1,A\cup L}} \binom{(e \setminus S_1) \cup Q_e}{r}, \tag{8.3}$$

$$R' := \left(\bigcup_{e\in R_{1,A\cup L}} \tilde{Q}_e^{(r)}\right) - T' - R_{1,A\cup L}. \tag{8.4}$$

We clearly have $T', R' \subseteq G^{(r)}$. We now check (tr1′)–(tr4′). Let $e' \in R'$. Thus, there exists $e \in R_{1,A\cup L}$ with $e' \subseteq e \cup Q_e$. If $e' \cap S_1 = \emptyset$, then $e' \subseteq (e \setminus S_1) \cup Q_e$, thus $e' \in T'$, so this cannot happen. Moreover, by (tr1) we have $|e' \cap S_1| \leq |(e \cup Q_e) \cap S_1| = |e \cap S_1| \leq i - 1$ and $|e' \cap S_2| \leq |(e \cup Q_e) \cap S_2| = 0$. Therefore, R' is $(\phi, V(G), S_1, S_2, i-1)$-projectable, so (tr1′) holds. Observe that

$$\phi(R') = \left(\bigcup_{e\in R_{1,A\cup L}} \tilde{Q}_{\bar{\phi}(e)}^{(r)}\right) - T' - \phi(R_{1,A\cup L}). \tag{8.5}$$

In order to check (tr2′), note first that $T', R', \phi(R') \subseteq \tilde{G}^{(r)} \subseteq G^{(r)} - H'$. Thus, by (tr2), it is enough to check that $T', R', \phi(R'), R_{1,A\cup L}, \phi(R_{1,A\cup L})$ are pairwise edge-disjoint. Note that no edge of T' intersects $S_1 \cup S_2$. Thus, (tr1), (8.4), (8.5), (Y1) and Proposition 8.7(ii) imply that $T', R', \phi(R'), R_{1,A\cup L}, \phi(R_{1,A\cup L})$ are indeed pairwise edge-disjoint, proving (tr2′).

By (8.4) and (8.5), we have $T' \cup R_{1,A\cup L} \cup R' = \bigcup_{e\in R_{1,A\cup L}} \tilde{Q}_e^{(r)}$ and $T' \cup \phi(R_{1,A\cup L}) \cup \phi(R') = \bigcup_{e\in R_{1,A\cup L}} \tilde{Q}_{\bar{\phi}(e)}^{(r)}$. Hence, T' satisfies (tr3′).

Finally, we can easily check that $|V(T' \cup R')| \leq \tilde{n} \leq 0.7\gamma n$. –

Let $T := \hat{T} \cup R_{1,A\cup L} \cup \phi(R_{1,A\cup L}) \cup T'$ and $R := R_{1,A} \cup R'$. Clearly, (tr1) and (tr1′) imply that (TR1) holds. Moreover, (tr2′) implies that T is edge-disjoint from both $(S_1 \uplus L) \cup \phi(R)$ and $(S_2 \uplus L) \cup R$. Observe that

$$\begin{aligned} T \cup (S_1 \uplus L) \cup \phi(R) &= \hat{T} \cup R_{1,A\cup L} \cup \phi(R_{1,A\cup L}) \cup T' \cup (S_1 \uplus L) \cup \phi(R_{1,A}) \cup \phi(R') \\ &= \hat{T} \cup (S_1 \uplus L) \cup R_{1,A\cup L} \cup \phi(R_{1,A}) \cup T' \cup \phi(R_{1,A\cup L}) \cup \phi(R'). \end{aligned}$$

Using (tr3) and (tr3′) we can thus see that $G[T \cup (S_1 \uplus L) \cup \phi(R)]$ has a $K_q^{(r)}$-decomposition. Similarly, $G[T \cup (S_2 \uplus L) \cup R]$ has a $K_q^{(r)}$-decomposition, so (TR2) holds. Finally, we have $|V(T \cup R)| \leq 4\mu n + 0.7\gamma n \leq \gamma n$ by (tr4) and (tr4′). □

Proof of Lemma 8.5. We can assume that $\gamma \ll 1/h, \varepsilon$. Choose new constants $\gamma_2, \ldots, \gamma_r, \gamma_2' \ldots, \gamma_r' > 0$ such that

$$1/n \ll \gamma_r \ll \gamma_r' \ll \gamma_{r-1} \ll \gamma_{r-1}' \ll \cdots \ll \gamma_2 \ll \gamma_2' \ll \gamma \ll 1/h, \varepsilon \ll \xi, 1/q.$$

Let $\phi\colon V(H) \to V(H')$ be an edge-bijective homomorphism from H to H'. Let $\bar{\phi}$ be as in Definition 8.6 with $V(H), V(H')$ playing the roles of V_1, V_2. Since ϕ is edge-bijective, we have that

$$\phi\restriction_S \text{ is injective whenever } S \subseteq e \text{ for some } e \in H. \tag{8.6}$$

For every $e \in H$, we have $|G^{(q)}(e) \cap G^{(q)}(\bar{\phi}(e))| \geq 0.5\xi n^{q-r}$ by Fact 5.2. It is thus easy to find for each $e \in H$ some $Q_e \in G^{(q)}(e) \cap G^{(q)}(\bar{\phi}(e))$ with $Q_e \cap (V(H) \cup V(H')) = \emptyset$ such that $Q_e \cap Q_{e'} = \emptyset$ for all distinct $e, e' \in H$. For each $e \in H$, let $\tilde{Q}_e := (e \cup Q_e)^{\leq}$ and $\tilde{Q}_{\bar{\phi}(e)} := (\bar{\phi}(e) \cup Q_e)^{\leq}$. Define

$$\begin{aligned} T^*_r &:= \bigcup_{e\in H} \binom{Q_e}{r} = \bigcup_{e \in H} (\tilde{Q}_e^{(r)} \cap \tilde{Q}_{\bar{\phi}(e)}^{(r)}), \\ R^*_r &:= \bigcup_{e \in H} \{e' \in \tilde{Q}_e^{(r)} : |e' \cap V(H)| \in [r-1]\} = \bigcup_{e \in H} \tilde{Q}_e^{(r)} - T^*_r - H. \end{aligned} \tag{8.7}$$

Let $\gamma_1 := \gamma$. Given $i \in [r-1]$ and $T^*_{i+1}, R^*_{i+1} \subseteq G^{(r)}$, we define the following conditions:

(TR1*)$_i$ R^*_{i+1} is $(\phi, V(G), V(H), V(H'), i)$-projectable;

(TR2*)$_i$ T^*_{i+1} is an $(H \cup R^*_{i+1}, H' \cup \phi(R^*_{i+1}); K_q^{(r)})$-transformer in G;

(TR3*)$_i$ $|V(T^*_{i+1} \cup R^*_{i+1})| \leq \gamma_{i+1} n$.

Note that (TR2*)$_i$ implies that T^*_{i+1} is edge-disjoint from both R^*_{i+1} and $\phi(R^*_{i+1})$, and that all three are subgraphs of $G^{(r)}$. Note also that R^*_{i+1} and $\phi(R^*_{i+1})$ are edge-disjoint by (TR1*)$_i$ and Proposition 8.7(ii).

Claim 1: T^*_r and R^*_r as defined in (8.7) *satisfy* (TR1*)$_{r-1}$–(TR3*)$_{r-1}$.

Proof of claim: (TR3*)$_{r-1}$ clearly holds. To see (TR1*)$_{r-1}$, let $e' \in R^*_r$. There exists $e \in H$ such that $e' \subseteq e \cup Q_e$ and $|e' \cap V(H)| \in [r-1]$. Clearly, $e' \cap V(H') \subseteq (e \cup Q_e) \cap V(H') = \emptyset$, so (Y1) holds. Moreover, $e' \cap V(H) \subseteq e$, so $\phi\restriction_{e' \cap V(H)}$ is injective by (8.6), and (Y2) holds. Let $e', e'' \in R^*_r$ and suppose that $\bar{\phi}(e') = \bar{\phi}(e'')$. We thus have $e' \setminus V(H) = e'' \setminus V(H)$. Since the Q_e's were chosen to be vertex-disjoint, we must have $e', e'' \subseteq e \cup Q_e$ for some $e \in H$. Hence, $(e' \cup e'') \cap V(H) \subseteq e$ and so $\phi\restriction_{(e' \cup e'') \cap V(H)}$ is injective by (8.6). Since $\phi(e' \cap V(H)) = \phi(e'' \cap V(H))$ by assumption, we have $e' \cap V(H) = e'' \cap V(H)$, and thus $e' = e''$. Altogether, (Y3) holds, so (TR1*)$_{r-1}$ is satisfied.

T^*_r is clearly edge-disjoint from both $H \cup R^*_r$ and $H' \cup \phi(R^*_r)$. Moreover, note that $\bigcup_{e\in H} \tilde{Q}_e$ is a $K_q^{(r)}$-decomposition of $T^*_r \cup H \cup R^*_r$ and $\bigcup_{e \in H} \tilde{Q}_{\bar{\phi}(e)}$ is a $K_q^{(r)}$-decomposition of $T^*_r \cup H' \cup \phi(R^*_r)$, so T^*_r satisfies (TR2*)$_{r-1}$. –

Suppose that for some $i \in [r-1]$, we have already found $T^*_{i+1}, R^*_{i+1} \subseteq G^{(r)}$ such that (TR1*)$_i$–(TR3*)$_i$ hold. We will now find T^*_i and R^*_i such that (TR1*)$_{i-1}$–(TR3*)$_{i-1}$ hold. To this end, let

$$R_i := \{e \in R^*_{i+1} : |e \cap V(H)| = i\}.$$

Let $\mathcal{S}_i$ be the set of all $S \in \binom{V(H)}{i}$ such that S is contained in some edge of R_i. For each $S \in \mathcal{S}_i$, let $L_S := R_i(S)$. By Proposition 8.7(iii), we have that

$$R_i = \dot{\bigcup_{S \in \mathcal{S}_i}} (S \uplus L_S) \quad \text{and} \quad \phi(R_i) = \dot{\bigcup_{S\in\mathcal{S}_i}} (\phi(S) \uplus L_S). \tag{8.8}$$

We intend to apply Lemma 8.8 to each pair $S, \phi(S)$ with $S \in \mathcal{S}_i$ individually. For each $S \in \mathcal{S}_i$, define

$$V_S := (V(G) \setminus (V(H) \cup V(H'))) \cup S \cup \phi(S).$$

Claim 2: For every $S \in \mathcal{S}_i$, $L_S \subseteq G[V_S](S)^{(r-i)} \cap G[V_S](\phi(S))^{(r-i)}$ and $|V(L_S)| \leq 1.1\gamma_{i+1}|V_S|$.

Proof of claim: The second assertion clearly holds by (TR3*)$_i$. To see the first one, let $e' \in L_S = R_i(S)$. Since $R_i \subseteq R^*_{i+1} \subseteq G^{(r)}$, we have $e' \in G(S)^{(r-i)}$. Moreover, $\phi(S) \cup e' \in \phi(R_i) \subseteq \phi(R^*_{i+1}) \subseteq G^{(r)}$ by (8.8). Since R^*_{i+1} is $(\phi, V(G), V(H), V(H'), i)$-projectable, we have that $e' \cap (V(H) \cup V(H')) = \emptyset$. Thus, $S \cup e' \subseteq V_S$ and $\phi(S) \cup e' \subseteq V_S$. –

Claim 3: For every $S \in \mathcal{S}_i$, L_S is $K_{q-i}^{(r-i)}$-divisible.

Proof of claim: Let $f \subseteq V(L_S)$ with $|f| \le r-i-1$. We have to check that $\binom{q-i-|f|}{r-i-|f|} \mid |L_S(f)|$. By (TR2*)$_i$, we have that $T^*_{i+1} \cup H \cup R^*_{i+1}$ and $T^*_{i+1} \cup H' \cup \phi(R^*_{i+1})$ are $K_q^{(r)}$-divisible. Clearly, H' does not contain an edge that contains S. Note that by (TR1*)$_i$ and Proposition 8.7(ii), $\phi(R^*_{i+1})$ does not contain an edge that contains S either, hence $|T^*_{i+1}(S \cup f)| = |(T^*_{i+1} \cup H' \cup \phi(R^*_{i+1}))(S \cup f)| \equiv 0 \mod \binom{q-|S\cup f|}{r-|S\cup f|}$. Moreover, since H is $K_q^{(r)}$-divisible, we have $|(T^*_{i+1} \cup R^*_{i+1})(S \cup f)| \equiv |(T^*_{i+1} \cup H \cup R^*_{i+1})(S \cup f)| \equiv 0 \mod \binom{q-|S\cup f|}{r-|S\cup f|}$. Thus, we have $\binom{q-|S\cup f|}{r-|S\cup f|} \mid |R^*_{i+1}(S \cup f)|$. Moreover, $|R^*_{i+1}(S \cup f)| = |R_i(S \cup f)| = |L_S(f)|$, proving the claim. –

We now intend to apply Lemma 8.8 for every $S \in \mathcal{S}_i$ in order to define $T_S, R_S \subseteq G^{(r)}$ such that the following hold:

(TR1′) R_S is $(\phi, V(G), V(H), V(H'), i-1)$-projectable;
(TR2′) T_S is an $((S \uplus L_S) \cup \phi(R_S), (\phi(S) \uplus L_S) \cup R_S; K_q^{(r)})$-transformer in G;
(TR3′) $|V(T_S \cup R_S)| \le \gamma'_{i+1} n$.

In order to ensure that these graphs are all edge-disjoint, we find them successively. Recall that P^ϕ (for a given r-graph P) was defined in Definition 8.6. Let $\mathcal{S}' \subseteq \mathcal{S}_i$ be the set of all $S' \in \mathcal{S}_i$ for which $T_{S'}$ and $R_{S'}$ have already been defined such that (TR1′)–(TR3′) hold. Suppose that next we want to find T_S and R_S. Let

$$P_S := R^*_{i+1} \cup \bigcup_{S' \in \mathcal{S}'} R_{S'},$$
$$M_S := T^*_{i+1} \cup R^*_{i+1} \cup \phi(R^*_{i+1}) \cup \bigcup_{S' \in \mathcal{S}'} (T_{S'} \cup R_{S'} \cup \phi(R_{S'})),$$
$$G_S := G[V_S] - ((M_S \cup P_S^\phi) - ((S \uplus L_S) \cup (\phi(S) \uplus L_S))).$$

Observe that (TR3*)$_i$ and (TR3′) imply that

$$\begin{aligned} |V(M_S \cup P_S)| &\le |V(T^*_{i+1} \cup R^*_{i+1} \cup \phi(R^*_{i+1}))| + \sum_{S' \in \mathcal{S}'} |V(T_{S'} \cup R_{S'} \cup \phi(R_{S'}))| \\ &\le 2\gamma_{i+1} n + 2\binom{h}{i}\gamma'_{i+1} n \le \gamma_i n. \end{aligned}$$

In particular, $|V(P_S^\phi)| \le |V(P_S) \cup V(H)| \le \gamma_i n + h$. Thus, by Proposition 5.6(v) G_S is still a $(2\varepsilon, \xi/2, q, r)$-supercomplex. Moreover, note that $L_S \subseteq G_S(S)^{(r-i)} \cap G_S(\phi(S))^{(r-i)}$ by Claim 2 and L_S is $K_{q-i}^{(r-i)}$-divisible by Claim 3.

Finally, by definition of $\mathcal{S}_i$, S is contained in some $e \in R_i$. Since R_i satisfies (Y2) by (TR1*)$_i$, we know that $\phi\restriction_e$ is injective. Thus, $\phi\restriction_S \colon S \to \phi(S)$ is a bijection. We can thus apply Lemma 8.8 with the following objects/parameters:

object/parameter	G_S	i	S	$\phi(S)$	$\phi\restriction_S$	L_S	$1.1\gamma_{i+1}$	γ'_{i+1}	2ε	$\lvert V_S\rvert$	$\xi/2$
playing the role of	G	i	S_1	S_2	ϕ	L	γ'	γ	ε	n	ξ

This yields T_S and R_S such that (TR2′) and (TR3′) hold and R_S is $(\phi\restriction_S, V(G_S), S, \phi(S), i-1)$-projectable. Note that the latter implies that R_S is $(\phi, V(G), V(H), V(H'), i-1)$-projectable as $V(H) \cap V(G_S) = S$ and $V(H') \cap V(G_S) = \phi(S)$, so (TR1′) holds as well. Moreover, by construction we have that

(a) $H, H', T^*_{i+1}, R^*_{i+1}, \phi(R^*_{i+1}), (T_S)_{S \in \mathcal{S}_i}, (R_S)_{S \in \mathcal{S}_i}, (\phi(R_S))_{S \in \mathcal{S}_i}$ are pairwise edge-disjoint;
(b) for all distinct $S, S' \in \mathcal{S}_i$ and all $e \in R_S$, $e' \in R_{S'}$, $e'' \in R^*_{i+1} - R_i$ we have that $\bar\phi(e)$, $\bar\phi(e')$ and $\bar\phi(e'')$ are pairwise distinct.

Here, (a) holds by the choice of M_S and (b) holds by the definition of P_S^ϕ. Let

$$T_i^* := T_{i+1}^* \cup R_i \cup \phi(R_i) \cup \bigcup_{S\in\mathcal{S}_i} T_S,$$
$$R_i^* := (R_{i+1}^* - R_i) \cup \bigcup_{S\in\mathcal{S}_i} R_S.$$

We check that (TR1*)$_{i-1}$–(TR3*)$_{i-1}$ hold. Using (TR3*)$_i$ and (TR3′), we can confirm that

$$|V(T_i^*\cup R_i^*)| \leq |V(T_{i+1}^*\cup R_{i+1}^* \cup \phi(R_{i+1}^*))| + \sum_{S\in\mathcal{S}_i}|V(T_S\cup R_S)|$$
$$\leq 2\gamma_{i+1}n + \binom{h}{i}\gamma'_{i+1}n \leq \gamma_i n.$$

In order to check that R_i^* is $(\phi, V(G), V(H), V(H'), i-1)$-projectable, note that (Y1) and (Y2) hold by (TR1*)$_i$, the definition of R_i and (TR1′). Moreover, (Y3) is implied by (TR1*)$_i$, (TR1′) and (b).

Finally, we check (TR2*)$_{i-1}$. Observe that

$$\begin{aligned}
T_i^*\cup H\cup R_i^* &= T_{i+1}^*\cup R_i\cup\phi(R_i)\cup\bigcup_{S\in\mathcal{S}_i}T_S\cup H\cup(R_{i+1}^*-R_i)\cup\bigcup_{S\in\mathcal{S}_i}R_S\\
&\overset{(8.8)}{=} (T_{i+1}^*\cup H\cup R_{i+1}^*)\cup\bigcup_{S\in\mathcal{S}_i}(T_S\cup(\phi(S)\uplus L_S)\cup R_S),\\
T_i^*\cup H'\cup\phi(R_i^*) &= T_{i+1}^*\cup R_i\cup\phi(R_i)\cup\bigcup_{S\in\mathcal{S}_i}T_S\cup H'\cup(\phi(R_{i+1}^*)-\phi(R_i))\cup\bigcup_{S\in\mathcal{S}_i}\phi(R_S)\\
&\overset{(8.8)}{=} (T_{i+1}^*\cup H'\cup\phi(R_{i+1}^*))\cup\bigcup_{S\in\mathcal{S}_i}(T_S\cup(S\uplus L_S)\cup\phi(R_S)).
\end{aligned}$$

Thus, by (TR2*)$_i$ and (TR2′), both $G[T_i^*\cup H\cup R_i^*]$ and $G[T_i^*\cup H'\cup\phi(R_i^*)]$ have a $K_q^{(r)}$-decomposition.

Finally, T_1^* is an $(H, H'; K_q^{(r)})$-transformer in G with $\Delta(T_1^*)\leq\gamma_1 n$ by (TR2*)$_0$ and (TR3*)$_0$ since R_1^* is empty by (TR1*)$_0$ and (Y1). □

8.2. **Canonical colourings.** Ideally, for each $K_q^{(r)}$-divisible H, we would like to find H' with $H \rightsquigarrow H'$ so that H' is (trivially) $K_q^{(r)}$-decomposable. Together with the corresponding transformer T guaranteed by Lemma 8.5, this would give us an absorber $A := T\cup H'$ for H. But it is far from clear why such an H' should exist. A strategy to overcome this issue is to search for a 'canonical' L, so that $H\rightsquigarrow L$ for any H with $|H| = |L|$. In particular, together with Lemma 8.5 this would imply the existence of an $(H, L; K_q^{(r)})$-transformer T_H and an $(H_0, L; K_q^{(r)})$-transformer T_0, where H_0 is the union of $|H|/\binom{q}{r}$ disjoint $K_q^{(r)}$'s. This in turn immediately yields an absorber $A := T_H\cup L\cup T_0\cup H_0$ for H. Unfortunately, this strategy still has the problem that any natural construction of L seems to lead to the occurrence of multiple as well as degenerate edges in L.

We are able to overcome this by considering the above strategy with the 'clique complement' ∇H playing the role of H: here ∇H is obtained from H by first extending each edge of H into a copy of $K_q^{(r)}$ and then removing the original edges of H. We can then find the required L which is canonical with respect to clique complements of arbitrary H. The actual construction of L is quite simple to describe. Perhaps surprisingly, the proof that it is indeed canonical is based on the inductive assertion $(*)_{r-1}$.

A *multi-r-graph* G consists of a set of vertices $V(G)$ and a multiset of edges $E(G)$, where each $e\in E(G)$ is a subset of $V(G)$ of size r. We will often identify a multi-r-graph with its edge set. For $S\subseteq V(G)$, let $|G(S)|$ denote the number of edges that contain S (counted with multiplicities). If $|S| = r$, then $|G(S)|$ is called the *multiplicity of S in G*. We say that G is $K_q^{(r)}$*-divisible* if $\binom{q-|S|}{r-|S|}$ divides $|G(S)|$ for all $S \subseteq V(G)$ with $|S| \leq r-1$.

Definition 8.9. We introduce the following operators $\tilde{\nabla}_{q,r}, \nabla_{q,r}$. Given a (multi-)$r$-graph H, let $\tilde{\nabla}_{q,r}H$ be obtained from H by extending every edge of H into a copy of $K_q^{(r)}$. More precisely, for every $e \in H$, let $Z_e := \{z_{e,1}, \dots, z_{e,q-r}\}$ be a vertex set of size $q-r$, such that $Z_e \cap Z_{e'} = \emptyset$ for all distinct (but possibly parallel) $e, e' \in H$ and $V(H) \cap Z_e = \emptyset$ for all $e \in H$. Then $\tilde{\nabla}_{q,r}H$ is the (multi-)r-graph on $V(H) \cup \bigcup_{e \in H} Z_e$ with edge set $\bigcup_{e \in H} \binom{e \cup Z_e}{r}$, that is, $(\tilde{\nabla}_{q,r}H)[e \cup Z_e]$ is a copy of $K_q^{(r)}$ for every $e \in H$. Let $\nabla_{q,r}H := \tilde{\nabla}_{q,r}H - H$. If q and r are clear from the context, we omit the subscripts. Note that if H is $K_q^{(r)}$-divisible, then so is $\nabla_{q,r}H$.

For $r \in \mathbb{N}$, let $\mathcal{M}_r$ contain all pairs $(k, m) \in \mathbb{N}_0^2$ such that $\frac{m}{r-i}\binom{k-i}{r-1-i}$ is an integer for all $i \in [r-1]_0$.

Definition 8.10. Given $q > r \geq 2$ and $(k, m) \in \mathcal{M}_r$, define the multi-r-graph $L_{k,m}^{q,r}$ as follows: Let B be a set of size $q-r$ such that $[k] \cap B = \emptyset$. Define $L_{k,m}^{q,r}$ on vertex set $[k] \cup B$ such that for every $e \in \binom{[k] \cup B}{r}$, the multiplicity of e is

$$|L_{k,m}^{q,r}(e)| = \begin{cases} 0 & \text{if } e \subseteq [k]; \\ \frac{m}{r-|e \cap [k]|}\binom{k-|e \cap [k]|}{r-1-|e \cap [k]|} & \text{otherwise.} \end{cases}$$

Finally, for an r-graph H, we let $H^{+t \cdot K_q^{(r)}}$ denote the vertex-disjoint union of H and t copies of $K_q^{(r)}$. We write H^{+t} if q is clear from the context.

The main lemma of this subsection is the following, which guarantees the existence of the desired canonical graph.

Lemma 8.11. *Let $q > r \geq 2$ and assume that $(*)_{r-1}$ holds. Then for all $h \in \mathbb{N}$, there exists $(k, m) \in \mathcal{M}_r$ such that for any $K_q^{(r)}$-divisible r-graph H on at most h vertices, there exists $t \in \mathbb{N}$ such that $\nabla(\nabla(H^{+t})) \rightsquigarrow \nabla L_{k,m}^{q,r}$.*

To prove Lemma 8.11 we introduce so called strong colourings. Let H be an r-graph and C a set. A map $c\colon V(H) \to C$ is a *strong C-colouring of H* if for all distinct $x, y \in V(H)$ with $|H(\{x, y\})| > 0$, we have $c(x) \neq c(y)$, that is, no colour appears twice in one edge. For $\alpha \in C$, we let $c^{-1}(\alpha)$ denote the set of all vertices coloured α. For a set $C' \subseteq C$, we let $c^{\subseteq}(C') := \{e \in H : C' \subseteq c(e)\}$. We say that c is *m-regular* if $|c^{\subseteq}(C')| = m$ for all $C' \in \binom{C}{r-1}$.

Given an r-graph H and a strong C-colouring c of H, let $id(H, c)$ denote the multi-r-graph obtained from H by identifying $c^{-1}(\alpha)$ to a new vertex for all $\alpha \in C$, that is, $id(H, c)$ has vertex set C and its edge set is the multiset $\{c(e) : e \in H\}$.

Proposition 8.12. *Let $q > r \geq 2$. Let H be an r-graph and c a strong C-colouring of H. Then $\nabla_{q,r}(H) \rightsquigarrow \nabla_{q,r}(id(H, c))$.*

Proof. Let $V(H) \cup \bigcup_{e \in H} Z_e$ be the vertex set of $\nabla_{q,r}(H)$ as in Definition 8.9. Similarly, for every $e \in H$, let $Z'_e = \{z'_{e,1}, \dots, z'_{e,q-r}\}$ be such that $C \cup \bigcup_{e \in H} Z'_e$ can be taken as the vertex set of $\nabla_{q,r}(id(H, c))$ as in Definition 8.9. Define $\phi\colon V(\nabla_{q,r}(H)) \to V(\nabla_{q,r}(id(H, c)))$ as follows: for all $x \in V(H)$, let $\phi(x) := c(x)$, and for all $e \in H$ and $i \in [q-r]$, let $\phi(z_{e,i}) := z'_{e,i}$. Then ϕ is an edge-bijective homomorphism from $\nabla_{q,r}(H)$ to $\nabla_{q,r}(id(H, c))$. □

Fact 8.13. *Let H be an r-graph and let c be a strong m-regular $[k]$-colouring of H. Then $|c^{\subseteq}(C')| = \frac{m}{r-i}\binom{k-i}{r-1-i}$ for all $i \in [r-1]_0$ and all $C' \in \binom{[k]}{i}$.*

Let H be an r-graph and assume that c is a strong C-colouring of H. There is a natural way to derive a strong colouring $\nabla_{q,r}c$ of $\nabla_{q,r}H$. Let $V(H) \cup \bigcup_{e \in H} Z_e$ be the vertex set of $\nabla_{q,r}H$ as in Definition 8.9. Let $B = \{b_1, \dots, b_{q-r}\}$ be a set of size $q-r$ such that $C \cap B = \emptyset$. Define $\nabla_{q,r}c$ as follows: for all $x \in V(H)$, let $\nabla_{q,r}c(x) := c(x)$, and for every $e \in H$ and $i \in [q-r]$, let $\nabla_{q,r}c(z_{e,i}) := b_i$. Clearly, $\nabla_{q,r}c$ is a strong $(C \cup B)$-colouring of $\nabla_{q,r}H$.

Proposition 8.14. *Let $q > r \geq 2$. Let H be an r-graph and suppose that c is a strong m-regular $[k]$-colouring of H. Then $(k,m) \in \mathcal{M}_r$ and $id(\nabla_{q,r}H, \nabla_{q,r}c) \cong L^{q,r}_{k,m}$.*

Proof. By Fact 8.13, $(k,m) \in \mathcal{M}_r$, so $L^{q,r}_{k,m}$ is defined. Let B be a set as above such that $[k] \cup B$ is the colour set of ∇c. Thus, $[k] \cup B$ is the vertex set of $id(\nabla H, \nabla c)$. We may also assume that $L^{q,r}_{k,m}$ has vertex set $[k] \cup B$ as well. For a set $e \in \binom{[k] \cup B}{r}$, define

$$S(e) := \{e' \in \nabla H \,:\, \nabla c(e') = e\}.$$

It remains to show that for all $e \in \binom{[k]\cup B}{r}$, we have $|S(e)| = |L^{q,r}_{k,m}(e)|$. So let $e \in \binom{[k]\cup B}{r}$. Clearly, if $e \subseteq [k]$, then $S(e) = \emptyset$ since no edge of H is contained in ∇H, and $|L^{q,r}_{k,m}(e)| = 0$, so we can assume that $e \not\subseteq [k]$. We claim that $|S(e)| = |c^{\subseteq}(e \cap [k])|$. Indeed, for every $e'' \in H$ with $e \cap [k] \subseteq c(e'')$, we have that $e' := (e'' \cap c^{-1}(e \cap [k])) \cup \{z_{e'',i} \,:\, b_i \in e \cap B\}$ is in ∇H and $\nabla c(e') = e$, and this assignment is bijective. Thus, since $|e \cap [k]| \leq r-1$, Fact 8.13 implies that

$$|S(e)| = |c^{\subseteq}(e \cap [k])| = \frac{m}{r - |e \cap [k]|}\binom{k - |e\cap[k]|}{r-1-|e \cap [k]|} = |L^{q,r}_{k,m}(e)|.$$

□

The following lemma guarantees the existence of a suitable strong colouring. Together, these tools allow us to deduce Lemma 8.11.

Lemma 8.15. *Let $q > r \geq 2$ and assume that $(*)_{r-1}$ holds. Then for all $h \in \mathbb{N}$, there exist $k, m \in \mathbb{N}$ such that for any $K^{(r)}_q$-divisible r-graph H on at most h vertices, there exists $t \in \mathbb{N}$ such that H^{+t} has a strong m-regular $[k]$-colouring.*

Proof of Lemma 8.11. Let $h \in \mathbb{N}$ and let $k, m \in \mathbb{N}$ be as in Lemma 8.15. Now, let H be any $K^{(r)}_q$-divisible r-graph H on at most h vertices. By Lemma 8.15, there exists $t \in \mathbb{N}$ such that H^{+t} has a strong m-regular $[k]$-colouring c. In particular, ∇c is a strong colouring of $\nabla(H^{+t})$. Thus, by Proposition 8.12, we have $\nabla(\nabla(H^{+t})) \rightsquigarrow \nabla(id(\nabla(H^{+t}), \nabla c))$. By Proposition 8.14, we have $(k,m) \in \mathcal{M}_r$ and $id(\nabla(H^{+t}), \nabla c) \cong L^{q,r}_{k,m}$, completing the proof. □

It remains to prove Lemma 8.15. We need the following result about decompositions of multi-r-graphs (which we will apply with $r-1$ playing the role of r).

Corollary 8.16. *Let $r \in \mathbb{N}$ and assume that $(*)_r$ is true. Let $1/n \ll 1/h, 1/q$ with $q > r$ be such that $K^{(r)}_n$ is $K^{(r)}_q$-divisible. Let $m \in \mathbb{N}$. Suppose that H is a $K^{(r)}_q$-divisible multi-r-graph on $[h]$ with multiplicity at most $m-1$ and let K be the complete multi-r-graph on $[n]$ with multiplicity m. Then $K - H$ has a $K^{(r)}_q$-decomposition.*

Proof. Choose $\varepsilon > 0$ such that $1/n \ll \varepsilon \ll 1/h, 1/q$. We may assume that $\tilde{H} := \tilde{\nabla}_{q,r}H$ is a multi-r-graph on $[n]$. We may also assume that $\hat{H} := \tilde{\nabla}_{q,r}(\tilde{H} - H)$ is an r-graph on $[n]$. Observe that the following are true:

(a) $\tilde{H}$ can be decomposed into $m-1$ (possibly empty) $K^{(r)}_q$-decomposable (simple) r-graphs $H'_1, \dots, H'_{m-1}$;
(b) $\hat{H}$ is a $K^{(r)}_q$-decomposable (simple) r-graph;
(c) $H \cup \hat{H} = \tilde{H} \cup \nabla(\nabla H)$.

By (c), we have that

$$K - H = (K - H - \hat{H}) \cup \hat{H} = \hat{H} \cup (K - \tilde{H} - \nabla(\nabla H)).$$

Let K' be the complete (simple) r-graph on $[n]$. For each $i \in [m-1]$, define $H_i := K' - H'_i$, and let $H_m := K' - \nabla(\nabla H)$. We thus have $K - \tilde{H} - \nabla(\nabla H) = \bigcup_{i \in [m]} H_i$ by (a).

Recall that $K'^{\leftrightarrow}$ is a $(0, 0.99/q!, q, r)$-supercomplex (cf. Example 3.8). We conclude with Proposition 5.6(v) that $H_i^{\leftrightarrow}$ is an $(\varepsilon, 0.5/q!, q, r)$-supercomplex for every $i \in [m]$. Thus, by $(*)_r$, H_i is $K_q^{(r)}$-decomposable for every $i \in [m]$. Thus,

$$K - H = \hat{H} \cup (K - \tilde{H} - \nabla(\nabla H)) = \hat{H} \cup \bigcup_{i \in [m]} H_i$$

has a $K_q^{(r)}$-decomposition. □

Proof of Lemma 8.15. Choose $k \in \mathbb{N}$ such that $1/k \ll 1/h, 1/q$ and such that $K_k^{(r-1)}$ is $K_q^{(r-1)}$-divisible. (Note that for every $a \in \mathbb{N}$, $K_{a(q!)+q}^{(r-1)}$ is $K_q^{(r-1)}$-divisible.) Let G be the complete multi-$(r-1)$-graph on $[k]$ with multiplicity $m' := h + 1$ and let $m := (q - r + 1)m'$.

Let H be any $K_q^{(r)}$-divisible r-graph on at most h vertices. We may assume that $V(H) = [h]$. We first define a multi-$(r-1)$-graph H' on $[h]$ as follows: For each $S \in \binom{[h]}{r-1}$, let the multiplicity of S in H' be $|H'(S)| := |H(S)|$. Clearly, H' has multiplicity at most h. Observe that for each $S \subseteq [h]$ with $|S| \leq r - 1$, we have

$$|H'(S)| = (r - |S|)|H(S)|. \tag{8.9}$$

Note that since H is $K_q^{(r)}$-divisible, we have that $\binom{q-(r-1)}{1} \mid |H(S)|$ for all $S \in \binom{[h]}{r-1}$. Thus, the multiplicity of each $S \in \binom{[h]}{r-1}$ in H' is divisible by $q - r + 1$. Let H'' be the multi-$(r-1)$-graph on $[h]$ obtained from H' by dividing the multiplicity of each $S \in \binom{[h]}{r-1}$ by $q - r + 1$. Hence, by (8.9), for all $S \subseteq [h]$ with $|S| \leq r - 2$, we have

$$|H''(S)| = \frac{|H'(S)|}{q - r + 1} = \frac{r - |S|}{q - r + 1}|H(S)| = \frac{|H(S)|}{\binom{q-|S|}{r-|S|}}\binom{q - |S|}{r - 1 - |S|} \equiv 0 \mod \binom{q - |S|}{r - 1 - |S|}$$

since H is $K_q^{(r)}$-divisible. Thus, H'' is $K_q^{(r-1)}$-divisible. Therefore, by Corollary 8.16 (with $k, m', r-1$ playing the roles of n, m, r) and our choice of k, $G - H''$ has a decomposition into t edge-disjoint copies $K'_1, \dots, K'_t$ of $K_q^{(r-1)}$.

We will show that t is as required in Lemma 8.15. To do this, let $K_1, \dots, K_t$ be vertex-disjoint copies of $K_q^{(r)}$ and vertex-disjoint from H. We will now define a strong m-regular $[k]$-colouring c of $H^{+t} = H \cup \bigcup_{j \in [t]} K_j$. To this end, for every $j \in [t]$, let $x_{j,1}, \dots, x_{j,q}$ be an enumeration of $V(K_j)$ and let $y_{j,1}, \dots, y_{j,q}$ be an enumeration of $V(K'_j)$. For all $x \in V(H)$, let

$$c(x) := x. \tag{8.10}$$

For all $j \in [t]$ and $i \in [q]$, let

$$c(x_{j,i}) := y_{j,i}. \tag{8.11}$$

Clearly, c is a strong $[k]$-colouring of H^{+t}. It remains to check that it is m-regular. Let $C \in \binom{[k]}{r-1}$. We set $|H(C)| = |H''(C)| := 0$ if $C \not\subseteq [h]$. By (8.10), we have that $|H(C)|$ edges e of H satisfy $C \subseteq c(e)$. Let $J(C) := \{j \in [t] : C \subseteq \{y_{j,1}, \dots, y_{j,q}\}\}$. Clearly, if $j \in [t] \setminus J(C)$, then K_j does not contain any edges e with $C \subseteq c(e)$ by (8.11). Moreover, if $j \in J(C)$, then K_j contains $q - (r - 1)$ edges e with $C \subseteq c(e)$, also by (8.11). Thus, $|c^{\subseteq}(C)| = |H(C)| + (q - r + 1)|J(C)|$. Note that C has multiplicity $m' - |H''(C)|$ in $G - H''$, and hence we have $|J(C)| = m' - |H''(C)|$. Since $|H''(C)| = |H(C)|/(q - r + 1)$, we conclude that $|c^{\subseteq}(C)| = (q - r + 1)(|H''(C)| + J(C)) = (q - r + 1)m' = m$. □

8.3. **Proof of the Absorbing lemma.** We can now use Lemma 8.5 and Lemma 8.11 to construct the desired absorber as a concatenation of transformers.

Proof of Lemma 8.2. If H is empty, then we can take A to be empty, so let us assume that H is not empty. In particular, $G^{(r)}$ is not empty. Recall also that we assume $r \geq 2$. By Lemma 8.11, there exist $k, m, t_1, t_2 \in \mathbb{N}$ such that

$$\nabla(\nabla(H^{+t_1})) \rightsquigarrow \nabla L^{q,r}_{k,m} \text{ and } \nabla(\nabla(\emptyset^{+t_2})) \rightsquigarrow \nabla L^{q,r}_{k,m}. \tag{8.12}$$

We can assume that $1/n \ll 1/k, 1/m, 1/t_1, 1/t_2$.

Clearly, there exist disjoint $Q_1, \dots, Q_{t_1}, Q'_1, \dots, Q'_{t_2} \in G^{(q)}$ which are also disjoint from $V(H)$. Let $H_1 := H \cup \bigcup_{j \in [t_1]} G^{(r)}[Q_j]$ and $H_2 := \bigcup_{j \in [t_2]} G^{(r)}[Q'_j]$. So H_1 is a copy of H^{+t_1} and H_2 is a copy of $\emptyset^{+t_2}$. Moreover, both $G[H_1 - H]$ and $G[H_2]$ have a $K^{(r)}_q$-decomposition.

For each $e \in H_1 \cup H_2$, choose $Q_e \in G^{(q-r)}(e)$. We can assume that all Q_e are pairwise disjoint and disjoint from $V(H_1) \cup V(H_2)$. Let $H'_1 := \bigcup_{e \in H_1} (G^{(r)}(Q_e \cup e) - \{e\})$ and $H'_2 := \bigcup_{e \in H_2} (G^{(r)}(Q_e \cup e) - \{e\})$. Thus, H'_1 is a copy of $\nabla(H^{+t_1})$ and H'_2 is a copy of $\nabla(\emptyset^{+t_2})$. Moreover, both $G[H_1 \cup H'_1]$ and $G[H_2 \cup H'_2]$ have a $K^{(r)}_q$-decomposition.

For each $e \in H'_1 \cup H'_2$, choose $Q_e \in G^{(q-r)}(e)$. We can assume that all Q_e are pairwise disjoint and disjoint from $V(H'_1) \cup V(H'_2)$. Let $H''_1 := \bigcup_{e \in H'_1} (G^{(r)}(Q_e \cup e) - \{e\})$ and $H''_2 := \bigcup_{e \in H'_2} (G^{(r)}(Q_e \cup e) - \{e\})$. Thus, H''_1 is a copy of $\nabla(\nabla(H^{+t_1}))$ and H''_2 is a copy of $\nabla(\nabla(\emptyset^{+t_2}))$. Moreover, both $G[H'_1 \cup H''_1]$ and $G[H'_2 \cup H''_2]$ have a $K^{(r)}_q$-decomposition.

Since G is (ξ, q, r)-extendable, it is straightforward to find a copy L' of $\nabla L^{q,r}_{k,m}$ in $G^{(r)}$ which is vertex-disjoint from H''_1 and H''_2. (This step is the reason why the definition of a supercomplex includes the notion of extendability.)

By (8.12), we have $H''_1 \rightsquigarrow L'$ and $H''_2 \rightsquigarrow L'$. Clearly, both H''_1 and H''_2 are $K^{(r)}_q$-divisible, and (again by (8.12)) so is L'. By Proposition 5.6(v) and Lemma 8.5, there exists an $(H''_1, L'; K^{(r)}_q)$-transformer T_1 in $G - (H_1 \cup H'_1 \cup H_2 \cup H'_2 \cup H''_2)$ with $\Delta(T_1) \leq \gamma n/3$. Similarly (using Proposition 5.6(v) and Lemma 8.5 again), we can find an $(H''_2, L'; K^{(r)}_q)$-transformer T_2 in $G - (H_1 \cup H'_1 \cup H''_1 \cup H_2 \cup H'_2 \cup T_1)$ with $\Delta(T_2) \leq \gamma n/3$.

Let

$$A := (H_1 - H) \cup (H'_1 \cup H''_1) \cup (T_1 \cup L') \cup (T_2 \cup H''_2) \cup (H'_2 \cup H_2).$$

Clearly, $A \subseteq G^{(r)}$, and $\Delta(A) \leq \gamma n$. Moreover, A and H are edge-disjoint. It remains to show that both $G[A]$ and $G[A \cup H]$ have a $K^{(r)}_q$-decomposition.

By construction, $G[H_1 - H]$, $G[H'_1 \cup H''_1]$, $G[T_1 \cup L']$, $G[T_2 \cup H''_2]$ and $G[H'_2 \cup H_2]$ are r-disjoint and have a $K^{(r)}_q$-decomposition each. Thus, $G[A]$ has a $K^{(r)}_q$-decomposition. Moreover, we have

$$A \cup H = (H_1 \cup H'_1) \cup (H''_1 \cup T_1) \cup (L' \cup T_2) \cup (H''_2 \cup H'_2) \cup H_2.$$

By construction, $G[H_1 \cup H'_1]$, $G[H''_1 \cup T_1]$, $G[L' \cup T_2]$, $G[H''_2 \cup H'_2]$ and $G[H_2]$ are r-disjoint and have a $K^{(r)}_q$-decomposition each. Thus, $G[A \cup H]$ has a $K^{(r)}_q$-decomposition. So A is indeed a $K^{(r)}_q$-absorber for H in G. □

9. Proof of the main theorems

We can now deduce our main results (modulo the proof of the Cover down lemma).

Proof of Theorem 3.7. We proceed by induction on r. The case $r = 1$ forms the base case of the induction and in this case we do not rely on any inductive assumption. Suppose that $r \in \mathbb{N}$ and that $(*)_i$ is true for all $i \in [r-1]$.

Choose new constants $m' \in \mathbb{N}$, $\gamma, \mu > 0$ such that $1/n \ll \gamma \ll 1/m' \ll \varepsilon \ll \mu \ll \xi, 1/q$.

Let G be a $K^{(r)}_q$-divisible (ε, ξ, q, r)-supercomplex on n vertices. By Lemma 7.6, there exists a $(2\sqrt{\varepsilon}, \mu, \xi - \varepsilon, q, r, m)$-vortex $U_0, U_1, \dots, U_\ell$ in G for some $\mu m' \leq m \leq m'$. Let $H_1, \dots, H_s$ be an enumeration of all spanning $K_q^{(r)}$-divisible subgraphs of $G[U_\ell]^{(r)}$. Clearly, $s \leq 2^{\binom{m}{r}}$. We will now find edge-disjoint subgraphs $A_1, \dots, A_s$ of $G^{(r)}$ such that for all $i \in [s]$ we have that

- (A1) A_i is a $K_q^{(r)}$-absorber for H_i in G;
- (A2) $\Delta(A_i) \leq \gamma n$;
- (A3) $A_i[U_1]$ is empty.

Suppose that for some $t \in [s]$, we have already found edge-disjoint $A_1, \dots, A_{t-1}$ that satisfy (A1)–(A3). Let $T_t := (G^{(r)}[U_1] - H_t) \cup \bigcup_{i \in [t-1]} A_i$. Clearly, $\Delta(T_t) \leq \mu n + s\gamma n \leq 2\mu n$. Thus, $G_{abs,t} := G - T_t$ is still a $(\sqrt{\mu}, \xi/2, q, r)$-supercomplex by Proposition 5.6(v). Using Lemma 8.2, we can find a $K_q^{(r)}$-absorber A_t for H_t in $G_{abs,t}$ with $\Delta(A_t) \leq \gamma n$. Clearly, A_t is edge-disjoint from $A_1, \dots, A_{t-1}$. Moreover, (A3) holds since $G_{abs,t}^{(r)}[U_1] = H_t$.

Let $A^* := A_1 \cup \dots \cup A_s$. It is easy to see that (A1)–(A3) imply the following:

- (A1′) for every $K_q^{(r)}$-divisible subgraph H^* of $G[U_\ell]^{(r)}$, $G[A^* \cup H^*]$ has a $K_q^{(r)}$-decomposition;
- (A2′) $\Delta(A^*) \leq \varepsilon n$;
- (A3′) $A^*[U_1]$ is empty.

Let $G_{almost} := G - A^*$. By (A2′) and Proposition 5.6(v), G_{almost} is an $(\sqrt{\varepsilon}, \xi/2, q, r)$-supercomplex. Moreover, since A^* must be $K_q^{(r)}$-divisible, we have that G_{almost} is $K_q^{(r)}$-divisible. By (A3′), $U_1, \dots, U_\ell$ clearly is a $(2\sqrt{\varepsilon}, \mu, \xi - \varepsilon, q, r, m)$-vortex in $G_{almost}[U_1]$. Moreover, (A2′) and Proposition 7.10 imply that U_1 is $(\varepsilon^{1/5}, \mu, \xi/2, q, r)$-random in G_{almost} and $U_1 \setminus U_2$ is $(\varepsilon^{1/5}, \mu(1-\mu), \xi/2, q, r)$-random in G_{almost}. Hence, $U_0, U_1, \dots, U_\ell$ is still an $(\varepsilon^{1/5}, \mu, \xi/2, q, r, m)$-vortex in G_{almost}. Thus, by Lemma 7.13, there exists a $K_q^{(r)}$-packing $\mathcal{K}_{almost}$ in G_{almost} which covers all edges of $G_{almost}^{(r)}$ except possibly some inside U_ℓ. Let $H^* := (G^{(r)} - \mathcal{K}_{almost}^{(r)})[U_\ell]$. Since H^* is $K_q^{(r)}$-divisible, $G[A^* \cup H^*]$ has a $K_q^{(r)}$-decomposition $\mathcal{K}_{absorb}$ by (A1′). Then, $\mathcal{K}_{almost} \cup \mathcal{K}_{absorb}$ is the desired $K_q^{(r)}$-decomposition of G. □

Proof of Theorem 3.14. Choose new constants $\gamma, \varepsilon', \xi'$ such that $1/n \ll \gamma \ll \varepsilon' \ll \xi' \ll \xi, 1/q, 1/\lambda$. By Lemma 3.4, G is an $(\varepsilon', \sqrt{\xi'}, q, r)$-supercomplex. Split $G^{(r)}$ into two subgraphs G_1 and G_2 such that for $i \in [2]$ and all $L \subseteq G^{(r)}$ with $\Delta(L) \leq \varepsilon' n$,

$$G[G_i \bigtriangleup L] \text{ is a } (\sqrt{\varepsilon'}, \xi', q, r)\text{-supercomplex.} \tag{9.1}$$

That such a splitting exists can be seen by a probabilistic argument: For each edge $e \in G^{(r)}$ independently, put e into G_1 with probability 1/2, and into G_2 otherwise. Then by Corollary 5.16, whp the desired property holds.

By Corollary 6.8, there exists a subgraph $L^* \subseteq G_2$ with $\Delta(L^*) \leq \gamma n$ such that $G_2' := G_2 - L^*$ is $K_q^{(r)}$-divisible. Let $G_1' := G_1 \cup L^* = G^{(r)} - G_2'$. Clearly, G_1' is still (q, r, λ)-divisible, and both $G[G_1']$ and $G[G_2']$ are $(\sqrt{\varepsilon'}, \xi', q, r)$-supercomplexes by (9.1). By repeated applications of Corollary 6.8, we can find edge-disjoint subgraphs $L_1, \dots, L_\lambda$ of G_1' such that $R_i := G_1' - L_i$ is $K_q^{(r)}$-divisible and $\Delta(L_i) \leq \gamma n$ for all $i \in [\lambda]$. Indeed, suppose that we have already found $L_1, \dots, L_{i-1}$. Then $\Delta(L_1 \cup \dots \cup L_{i-1}) \leq \lambda\gamma n \leq \sqrt{\varepsilon'} n$. Thus, by Corollary 6.8, there exists a subgraph $L_i \subseteq G_1' - (L_1 \cup \dots \cup L_{i-1})$ with $\Delta(L_i) \leq \gamma n$ such that $G_1' - L_i$ is $K_q^{(r)}$-divisible.

Let $G_2'' := G_2' \cup L_1 \cup \dots \cup L_\lambda$. We claim that G_2'' is $K_q^{(r)}$-divisible. Let $S \subseteq V(G)$ with $|S| \leq r - 1$. We then have that

$$\begin{aligned} |G_2''(S)| &= |G_2'(S)| + \sum_{i \in [\lambda]} |L_i(S)| = |G_2'(S)| + \sum_{i \in [\lambda]} |(G_1' - R_i)(S)| \\ &= |G_2'(S)| + \lambda |G_1'(S)| - \sum_{i \in [\lambda]} |R_i(S)| \equiv 0 \mod \binom{q - |S|}{r - |S|}. \end{aligned}$$

Thus, G_2'' is $K_q^{(r)}$-divisible. Clearly, $\Delta(L^* \cup L_1 \cup \dots \cup L_\lambda) \leq (\lambda + 1)\gamma n \leq \varepsilon' n$. By (9.1), we can see that each of $G[G_2']$, $G[G_2'']$, $G[R_1], \dots, G[R_\lambda]$ is a $(\sqrt{\varepsilon'}, \xi', q, r)$-supercomplex.

Using Theorem 3.7, we can thus find $K_q^{(r)}$-decompositions $\mathcal{K}_1, \dots, \mathcal{K}_{\lambda-1}$ of $G[G_2']$, a $K_q^{(r)}$-decomposition $\mathcal{K}^*$ of $G[G_2'']$, and for each $i \in [\lambda]$, a $K_q^{(r)}$-decomposition $\mathcal{K}_i'$ of $G[R_i]$. Moreover, we can assume that all these decompositions are pairwise q-disjoint. Indeed, this can be achieved by choosing them successively, deleting the corresponding q-sets from the subsequent complexes and then applying Proposition 3.12 (where $t \le 2\lambda \le \sqrt{\varepsilon'} n^{q-r}$) to verify that the parameters of the subsequent supercomplexes are not affected significantly. We claim that $\mathcal{K} := \mathcal{K}^* \cup \bigcup_{i \in [\lambda-1]} \mathcal{K}_i \cup \bigcup_{i \in [\lambda]} \mathcal{K}_i'$ (viewed as a collection of copies of $K_q^{(r)}$) is the desired (q,r,λ)-design. Indeed, every edge of $G_1' - (L_1 \cup \dots \cup L_\lambda)$ is covered by each of $\mathcal{K}_1', \dots, \mathcal{K}_\lambda'$. For each $i \in [\lambda]$, every edge of L_i is covered by $\mathcal{K}^*$ and each of $\mathcal{K}_1', \dots, \mathcal{K}_{i-1}', \mathcal{K}_{i+1}', \dots, \mathcal{K}_\lambda'$. Finally, every edge of G_2' is covered by each of $\mathcal{K}_1, \dots, \mathcal{K}_{\lambda-1}$ and $\mathcal{K}^*$. □

We note that the same proof can be used to obtain (q,r,λ)-designs of a supercomplex G, where λ is allowed to grow with n, if we utilize an approximate decomposition result which achieves a sublinear maximum degree for the leftover. More precisely, an analogous statement of Theorem 6.1, where we might assume that G is $(n^{-(q-r)/2.01}, d, q, r)$-regular, say, and γn is replaced by $g(n) = o(n^{1-\alpha})$, would imply that Theorem 3.14 holds for all $\lambda \le n^\alpha$.

Proof of Theorem 1.3. Choose $\varepsilon > 0$ such that $1/n \ll \varepsilon \ll p, 1/q, 1/\lambda$ and let

$$c' := \frac{1.1 \cdot 2^r \binom{q+r}{r}}{(q-r)!} c(q,r,p), \quad \xi := 0.99/q!, \quad \xi' := 0.95 \xi p^{2^r \binom{q+r}{r}}, \quad \xi'' := 0.9 (1/4)^{\binom{q+r}{q}} (\xi' - c').$$

Recall that the complete complex K_n is an (ε, ξ, q, r)-supercomplex (cf. Example 3.8).

Let $H \sim \mathcal{H}_r(n,p)$. We can view H as a random subgraph of $K_n^{(r)}$. By Corollary 5.16, the following holds whp for all $L \subseteq K_n^{(r)}$ with $\Delta(L) \le c(q,r,p)n$:

$$K_n[H \triangle L] \text{ is a } (3\varepsilon + c', \xi' - c', q, r)\text{-supercomplex.}$$

Note that $c' \le \frac{p^{2^r \binom{q+r}{r}}}{2.7(2\sqrt{e})^r q!}$. Thus, $2(2\sqrt{e})^r \cdot (3\varepsilon + c') \le \xi' - c'$. Hence, if $H \triangle L$ is (q,r,λ)-divisible, it has a (q,r,λ)-design by Theorem 3.14. □

Proof of Theorem 1.4. By Example 3.9, we have that $H^{\leftrightarrow}$ is an (ε, ξ, q, r)-supercomplex, where $\varepsilon := 2^{q-r+1} c/(q-r)!$ and $\xi := (1 - 2^{q+1} c) p^{2^r \binom{q+r}{r}} / q!$. It is easy to see that (1.1) implies that $2(2\sqrt{\mathrm{e}})^r \varepsilon \le \xi$. Hence, an application of Theorem 3.14 completes the proof. □

Proof of Theorem 1.5. By Example 3.11, we have that $G^{\leftrightarrow}$ is an $(0.01\xi, 0.99\xi, q, 1)$-supercomplex. Moreover, since $q \mid n$, $G^{\leftrightarrow}$ is $K_q^{(1)}$-divisible. Thus, by Corollary 3.13, $G^{\leftrightarrow}$ has $0.01 \xi n^{q-1}$ q-disjoint $K_q^{(1)}$-decompositions, i.e. G has $0.01\xi n^{q-1}$ edge-disjoint perfect matchings. □

10. Covering down

It remains to prove the Cover down lemma (Lemma 7.4), which we do in this section. Suppose that G is a supercomplex, U is a random subset of $V(G)$ and L^* is a very sparse 'leftover' graph. Recall that the lemma guarantees a $K_q^{(r)}$-packing that covers all edges that are not inside U by using only few edges inside U. More precisely, the Cover down lemma shows the existence of a suitable sparse graph H^* so that $G[H^* \cup L^*]$ has a $K_q^{(r)}$-packing covering all edges of $H^* \cup L^*$ except possibly some inside U.

We now briefly sketch how one can attempt to construct such a graph H^*. For the moment, suppose that H^* and L^* are given. For an edge e of $H^* \cup L^*$, we refer to $|e \cap U|$ as its type. A natural way (for divisibility reasons) to try to cover all edges of $H^* \cup L^*$ which are not inside U is to first cover all type-0-edges, then all type-1-edges, etc. and finally all type-$(r-1)$-edges. It is comparatively easy to cover all type-0-edges. The reason for this is that a type-0-edge can be covered by a q-clique that contains no other type-0-edge. Thus, if H^* is a random subgraph of

$G^{(r)} - G^{(r)}[V(G) \setminus U]$, then every type-0-edge (from L^*) is contained in many q-cliques. Since $\Delta(L^*)$ is very small, this allows us to apply Corollary 6.7 in order to cover all type-0-edges with edge-disjoint q-cliques. The situation is very different for edges of higher types. Here we would like to apply the following observation to cover edges of type $r - i$.

Fact 10.1. *Let G be a complex and $S \in \binom{V(G)}{i}$ with $1 \le i < r < q$. Suppose that $\mathcal{K}$ is a $K_{q-i}^{(r-i)}$-decomposition of $G(S)$. Then $(S \uplus \mathcal{K}^{(q-i)})^{\le}$ is a $K_q^{(r)}$-packing in G covering all r-edges that contain S.*

Proof. For each $Q \in \mathcal{K}^{(q-i)}$, we have $S \cup Q \in G^{(q)}$ since $Q \in G(S)^{(q-i)}$. Moreover, for distinct $Q, Q' \in \mathcal{K}^{(q-i)}$, we have $|(S \cup Q) \cap (S \cup Q')| = i + |Q \cap Q'| < i + (r - i) = r$. Thus, $(S \uplus \mathcal{K}^{(q-i)})^{\le}$ is a $K_q^{(r)}$-packing in G. Let $e \in G^{(r)}$ with $S \subseteq e$. Then $e \setminus S \in G(S)^{(r-i)}$, so there exists some $Q \in \mathcal{K}^{(q-i)}$ with $e \setminus S \subseteq Q$. Hence, e is covered by $S \cup Q \in ((S \uplus \mathcal{K}^{(q-i)})^{\le})^{(q)}$. □

A natural approach is then to reserve for every $S \in \binom{V(G) \setminus U}{i}$ a random subgraph H_S of $G(S)[U]^{(r-i)}$ and to protect all the H_S's when applying the Boosted nibble lemma. Let L be the leftover resulting from this application and suppose for simplicity that there are no more leftover edges of types $0, \dots, r - i - 1$. Let $L_S := L(S)$. Thus $L_S \subseteq G(S)[U]^{(r-i)}$. The hope is that the H_S's do not intersect too much, so that it is possible to find a $K_{q-i}^{(r-i)}$-decomposition $\mathcal{K}_S$ for each $H_S \cup L_S$ such that the $K_q^{(r)}$-packings $(S \uplus \mathcal{K}_S^{(q-i)})^{\le}$ are r-disjoint. This would then yield a $K_q^{(r)}$-packing covering all leftover edges of type $r - i$. There are two natural candidates for selecting H_S:

(A) Choose H_S by including every edge of $G(S)[U]^{(r-i)}$ with probability ν.
(B) Choose a random subset U_S of U of size $\rho|U|$ and let $H_S := G(S)^{(r-i)}[U_S]$.

The advantage of Strategy (A) is that $H_S \cup L_S$ is close to being quasirandom. This is not the case for (B): even if the maximum degree of L_S is sublinear, its edges might be spread out over the whole of U. Unfortunately, when pursuing Strategy (A), the H_S intersect too much, so it is not clear how to find the desired decompositions due to the interference between different H_S. However, it turns out that under the additional assumption that $V(L_S) \subseteq U_S$, Strategy (B) does work. We call the corresponding result the 'Localised cover down lemma' (Lemma 10.7).

We will combine both strategies as follows: For each S, we will choose H_S as in (A) and U_S as in (B) and let $J_S := G(S)^{(r-i)}[U_S]$. In a first step we use H_S to find a $K_{q-i}^{(r-i)}$-packing covering all edges of $e \in H_S \cup L_S$ with $e \not\subseteq U_S$, and then afterwards we apply the Localised cover down lemma to cover all remaining edges. Note that the first step resembles the original problem: We are given a graph $H_S \cup L_S$ on U and want to cover all edges that are not inside $U_S \subseteq U$. But the resulting types are now more restricted. This enables us to prove a more general Cover down lemma, the 'Cover down lemma for setups' (Lemma 10.22), by induction on $r - |S|$, which will allow us to perform the first step in the above combined strategy for all S simultaneously.

10.1. **Systems and focuses.** In this subsection, we prove the Localised cover down lemma, which shows that Strategy (B) works under the assumption that each L_S is 'localised'.

Definition 10.2. Given $i \in \mathbb{N}_0$, an *i-system in a set V* is a collection $\mathcal{S}$ of distinct subsets of V of size i. A subset of V is called *$\mathcal{S}$-important* if it contains some $S \in \mathcal{S}$, otherwise we call it *$\mathcal{S}$-unimportant*. We say that $\mathcal{U} = (U_S)_{S \in \mathcal{S}}$ is a *focus for $\mathcal{S}$* if for each $S \in \mathcal{S}$, U_S is a subset of $V \setminus S$.

Definition 10.3. Let G be a complex and $\mathcal{S}$ an i-system in $V(G)$. We call G *r-exclusive with respect to $\mathcal{S}$* if every $f \in G$ with $|f| \ge r$ contains at most one element of $\mathcal{S}$. Let $\mathcal{U}$ be a focus for $\mathcal{S}$. If G is r-exclusive with respect to $\mathcal{S}$, the following functions are well-defined: For $r' \ge r$, let $\mathcal{E}_{r'}$ denote the set of $\mathcal{S}$-important r'-sets in G. Define $\tau_{r'} \colon \mathcal{E}_{r'} \to [r' - i]_0$ as $\tau_{r'}(e) := |e \cap U_S|$, where S is the unique $S \in \mathcal{S}$ contained in e. We call $\tau_{r'}$ the *type function of $G^{(r')}$, $\mathcal{S}$, $\mathcal{U}$*.

Fact 10.4. *Let $r \in \mathbb{N}$ and $i \in [r-1]_0$. Let G be a complex and $\mathcal{S}$ an i-system in $V(G)$ such that G is r-exclusive with respect to $\mathcal{S}$. Let $f \in G$ with $|f| \geq r$ be $\mathcal{S}$-important and let $\mathcal{E}' := \mathcal{E}_r \cap \binom{f}{r}$. Then we have*

(i) $\max_{e \in \mathcal{E}'} \tau_r(e) \leq \tau_{|f|}(f) \leq |f| - r + \min_{e \in \mathcal{E}'} \tau_r(e)$,
(ii) $\min_{e \in \mathcal{E}'} \tau_r(e) = \max\{r + \tau_{|f|}(f) - |f|, 0\}$.

Proof. Let $S \subseteq f$ with $S \in \mathcal{S}$. Clearly, for every $\mathcal{S}$-important r-subset e of f, S is the unique element from $\mathcal{S}$ that e contains. For any such e, we have $\tau_{|f|}(f) = |f \cap U_S| \geq |e \cap U_S| = \tau_r(e)$, implying the first inequality of (i). Also, $|f| - \tau_{|f|}(f) = |f \setminus U_S| \geq |e \setminus U_S| = r - \tau_r(e)$, implying the second inequality of (i).

This also implies that $\min_{e \in \mathcal{E}'} \tau_r(e) \geq \max\{r + \tau_{|f|}(f) - |f|, 0\}$. To see the converse, note that $|f \setminus U_S| = |f| - \tau_{|f|}(f)$. Hence, we can choose an r-set $e \subseteq f$ with $S \subseteq e$ and $|e \setminus U_S| = \min\{|f| - \tau_{|f|}(f), r\}$. Note that $e \in \mathcal{E}'$ and $\tau_r(e) = r - |e \setminus U_S| = r - \min\{|f| - \tau_{|f|}(f), r\} = \max\{r + \tau_{|f|}(f) - |f|, 0\}$. This completes the proof of (ii). □

In order to make Strategy (B) work, it is essential that the U_S do not interfere too much which each other. The following definition makes this more precise.

Let $\mathcal{Z}_{r,i}$ be the set of all quadruples $(z_0, z_1, z_2, z_3) \in \mathbb{N}_0^4$ such that $z_0 + z_1 < i$, $z_0 + z_3 < i$ and $z_0 + z_1 + z_2 + z_3 = r$. Clearly, $|\mathcal{Z}_{r,i}| \leq (r+1)^3$, and $\mathcal{Z}_{r,i} = \emptyset$ if $i = 0$.

Definition 10.5. Let V be a set of size n, let $\mathcal{S}$ be an i-system in V and let $\mathcal{U}$ be a focus for $\mathcal{S}$. We say that $\mathcal{U}$ is a *μ-focus for $\mathcal{S}$* if each $U_S \in \mathcal{U}$ has size $\mu n \pm n^{2/3}$. For all $S \in \mathcal{S}$, $z = (z_0, z_1, z_2, z_3) \in \mathcal{Z}_{r,i}$ and all $(z_1 + z_2 - 1)$-sets $f \subseteq V \setminus S$, define

$$\begin{aligned} \mathcal{J}^f_{S,z} &:= \{S' \in \mathcal{S} \,:\, |S \cap S'| = z_0, f \subseteq S' \cup U_{S'}, |U_{S'} \cap S| \geq z_3\}, \\ \mathcal{J}^f_{S,z,1} &:= \{S' \in \mathcal{J}_{S,z} \,:\, |f \cap S'| = z_1\}, \\ \mathcal{J}^f_{S,z,2} &:= \{S' \in \mathcal{J}_{S,z} \,:\, |f \cap S'| = z_1 - 1, |U_S \cap (S' \setminus f)| \geq 1\}. \end{aligned}$$

We say that $\mathcal{U}$ is a *(ρ_{size}, ρ, r)-focus for $\mathcal{S}$* if

(F1) each U_S has size $\rho_{size}\rho n \pm n^{2/3}$;
(F2) $|U_S \cap U_{S'}| \leq 2\rho^2 n$ for distinct $S, S' \in \mathcal{S}$;
(F3) for all $S \in \mathcal{S}$, $z = (z_0, z_1, z_2, z_3) \in \mathcal{Z}_{r,i}$ and $(z_1 + z_2 - 1)$-sets $f \subseteq V \setminus S$, we have

$$\begin{aligned} |\mathcal{J}^f_{S,z,1}| &\leq 2^{6r} \rho^{z_2 + z_3 - 1} n^{i - z_0 - z_1}, \\ |\mathcal{J}^f_{S,z,2}| &\leq 2^{9r} \rho^{z_2 + z_3 + 1} n^{i - z_0 - z_1 + 1}. \end{aligned}$$

The sets S' in $\mathcal{J}^f_{S,z,1}$ and $\mathcal{J}^f_{S,z,2}$ are those which may give rise to interference when covering the edges containing S. (F3) ensures that there are not too many of them. We now show that a random choice of the U_S satisfies this.

Lemma 10.6. *Let $1/n \ll \rho \ll \rho_{size}, 1/r$ and $i \in [r-1]$. Let V be a set of size n, let $\mathcal{S}$ be an i-system in V and let $\mathcal{U}' = (U'_S)_{S \in \mathcal{S}}$ be a ρ_{size}-focus for $\mathcal{S}$. Let $\mathcal{U} = (U_S)_{S \in \mathcal{S}}$ be a random focus obtained as follows: independently for all pairs $S \in \mathcal{S}$ and $x \in U'_S$, retain x in U_S with probability ρ. Then whp $\mathcal{U}$ is a (ρ_{size}, ρ, r)-focus for $\mathcal{S}$.*

Proof. Clearly, $U_S \subseteq V \setminus S$ for all $S \in \mathcal{S}$.

Step 1: Probability estimates for (F1) and (F2)

For $S \in \mathcal{S}$, Lemma 5.7(i) implies that with probability at least $1 - 2e^{-0.5|U'_S|^{1/3}}$, we have $|U_S| = \mathbb{E}(|U_S|) \pm 0.5|U'_S|^{2/3} = \rho\rho_{size} n \pm (\rho n^{2/3} + 0.5|U'_S|^{2/3})$. Thus, with probability at least $1 - e^{-n^{1/4}}$, (F1) holds.

Let $S, S' \in \mathcal{S}$ be distinct. If $|U'_S \cap U'_{S'}| \leq \rho^2 n$, then we surely have $|U_S \cap U_{S'}| \leq \rho^2 n$, so assume that $|U'_S \cap U'_{S'}| \geq \rho^2 n$. Lemma 5.7(i) implies that with probability at least $1 - 2e^{-2\rho^4 |U'_S \cap U'_{S'}|}$, we have $|U_S \cap U_{S'}| \leq \mathbb{E}(|U_S \cap U_{S'}|) + \rho^2 |U'_S \cap U'_{S'}| \leq 2\rho^2 n$. Thus, with probability at least $1 - \mathrm{e}^{-n^{1/2}}$, (F2) holds.

Step 2: Probability estimates for (F3)

Now, fix $S \in \mathcal{S}$, $z = (z_0, z_1, z_2, z_3) \in \mathcal{Z}_{r,i}$ and an $(z_1 + z_2 - 1)$-set $f \subseteq V \setminus S$. In order to estimate $|\mathcal{J}^f_{S,z,1}|$ and $|\mathcal{J}^f_{S,z,2}|$, define

$$\mathcal{J}' := \{S' \in \mathcal{S} \,:\, |S \cap S'| = z_0, |f \cap S'| = z_1\},$$
$$\mathcal{J}'' := \{S' \in \mathcal{S} \,:\, |S \cap S'| = z_0, |f \cap S'| = z_1 - 1\}.$$

Clearly, $\mathcal{J}^f_{S,z,1} \subseteq \mathcal{J}'$ and $\mathcal{J}^f_{S,z,2} \subseteq \mathcal{J}''$. Moreover, since $f \cap S = \emptyset$, we have that

$$|\mathcal{J}'| \leq \binom{i}{z_0}\binom{z_1+z_2-1}{z_1} n^{i-z_0-z_1} \leq 2^{2r} n^{i-z_0-z_1},$$
$$|\mathcal{J}''| \leq \binom{i}{z_0}\binom{z_1+z_2-1}{z_1-1} n^{i-z_0-z_1+1} \leq 2^{2r} n^{i-z_0-z_1+1}.$$

Consider $S' \in \mathcal{J}'$. By the random choice of $U_{S'}$ and since $f \cap S = \emptyset$, we have that

$$\mathbb{P}(S' \in \mathcal{J}^f_{S,z,1}) = \mathbb{P}(f \setminus S' \subseteq U_{S'}, |U_{S'} \cap S| \geq z_3) = \mathbb{P}(f \setminus S' \subseteq U_{S'}) \cdot \mathbb{P}(|U_{S'} \cap S| \geq z_3).$$

Note that $\mathbb{P}(f \setminus S' \subseteq U_{S'}) \leq \rho^{z_2-1}$ since $|f \setminus S'| = z_2 - 1$. Moreover, $\mathbb{P}(|U_{S'} \cap S| \geq z_3) \leq \binom{i}{z_3}\rho^{z_3} \leq 2^i \rho^{z_3}$.

Hence, $7\mathbb{E}|\mathcal{J}^f_{S,z,1}| \leq 2^3 2^i \rho^{z_2+z_3-1} 2^{2r} n^{i-z_0-z_1}$. Since $i - z_0 - z_1 \geq 1$ and $U_{S'}$ and $U_{S''}$ are chosen independently for any two distinct $S', S'' \in \mathcal{J}'$, Lemma 5.7(iii) implies that

$$\mathbb{P}(|\mathcal{J}^f_{S,z,1}| \geq 2^{6r} \rho^{z_2+z_3-1} n^{i-z_0-z_1}) \leq \mathrm{e}^{-2^{6r}\rho^{z_2+z_3-1} n^{i-z_0-z_1}} \leq \mathrm{e}^{-\sqrt{n}}. \tag{10.1}$$

Now, consider $S' \in \mathcal{J}''$. By the random choice of U_S and $U_{S'}$, we have that

$$\begin{aligned}\mathbb{P}(S' \in \mathcal{J}^f_{S,z,2}) &= \mathbb{P}(f \setminus S' \subseteq U_{S'}, |U_{S'} \cap S| \geq z_3, |U_S \cap (S' \setminus f)| \geq 1)\\ &= \mathbb{P}(f \setminus S' \subseteq U_{S'}) \cdot \mathbb{P}(|U_{S'} \cap S| \geq z_3) \cdot \mathbb{P}(|U_S \cap (S' \setminus f)| \geq 1)\\ &\leq \rho^{z_2} \cdot \binom{i}{z_3}\rho^{z_3} \cdot (i - z_1 + 1)\rho \leq r 2^r \rho^{z_2+z_3+1}.\end{aligned}$$

However, note that the events $S' \in \mathcal{J}^f_{S,z,2}$ and $S'' \in \mathcal{J}^f_{S,z,2}$ are not necessarily independent. To deal with this, define the auxiliary $(i - z_0 - z_1 + 1)$-graph A on V with edge set $\{S' \setminus (S \cup f) : S' \in \mathcal{J}''\}$ and let A' be the (random) subgraph with edge set $\{S' \setminus (S \cup f) \,:\, S' \in \mathcal{J}^f_{S,z,2}\}$. Note that for every edge $e \in A$, there are at most $\binom{i}{z_0}\binom{z_1+z_2-1}{z_1-1} \leq 2^{2r}$ elements $S' \in \mathcal{J}''$ with $e = S' \setminus (S \cup f)$. Hence, $|\mathcal{J}^f_{S,z,2}| \leq 2^{2r}|A'|$. Moreover, every edge of A survives (i.e. lies in A') with probability at most $2^{2r} \cdot r2^r \rho^{z_2+z_3+1}$, and for every matching M in A, the edges of M survive independently. Thus, by Lemma 5.12, we have that

$$\mathbb{P}(|A'| \geq 7r2^{3r}\rho^{z_2+z_3+1} n^{i-z_0-z_1+1}) \leq (i - z_0 - z_1 + 1) n^{i-z_0-z_1} \mathrm{e}^{-7\cdot 2^{3r}\rho^{z_2+z_3+1} n}$$

and thus

$$\mathbb{P}(|\mathcal{J}^f_{S,z,2}| \geq 7r2^{5r}\rho^{z_2+z_3+1} n^{i-z_0-z_1+1}) \leq rn^r \mathrm{e}^{-7\cdot 2^{3r}\rho^{z_2+z_3+1} n} \leq \mathrm{e}^{-\sqrt{n}}. \tag{10.2}$$

Since $|\mathcal{S}| \leq n^i$, a union bound applied to (10.1) and (10.2) shows that with probability at least $1 - \mathrm{e}^{-n^{1/3}}$, (F3) holds. □

Lemma 10.7 (Localised cover down lemma)**.** *Let $1/n \ll \rho \ll \rho_{size}, \xi, 1/q$ and $1 \le i < r < q$. Assume that $(*)_{r-i}$ is true. Let G be a complex on n vertices and let $\mathcal{S} = \{S_1, \dots, S_p\}$ be an i-system in G such that G is r-exclusive with respect to $\mathcal{S}$. Let $\mathcal{U} = \{U_1, \dots, U_p\}$ be a (ρ_{size}, ρ, r)-focus for $\mathcal{S}$. Suppose further that whenever $S_j \subseteq e \in G^{(r)}$, we have $e \setminus S_j \subseteq U_j$. Finally, assume that $G(S_j)[U_j]$ is a $K_{q-i}^{(r-i)}$-divisible $(\rho, \xi, q-i, r-i)$-supercomplex for all $j \in [p]$.*

Then there exists a $K_q^{(r)}$-packing $\mathcal{K}$ in G covering all $\mathcal{S}$-important r-edges.

To prove Lemma 10.7, we will use the fact that by Corollary 3.13, there are many $(q-i)$-disjoint candidates $\mathcal{K}_j$ for a $K_{q-i}^{(r-i)}$-decomposition of $G(S_j)[U_j]$. We will proceed sequentially, choosing one candidate $\mathcal{K}_j$ uniformly at random from a large set of $(q-i)$-disjoint ones such that $\mathcal{K}_j$ does not interfere with any previous choices. Together this translates into the desired packing $\mathcal{K}$. A similar idea was introduced in [5], but the hypergraph analysis is far more intricate.

Proof. Let $t := \rho^{1/6}(0.5\rho\rho_{size}n)^{q-r}$. For all $j \in [p]$, define $H_j := G(S_j)[U_j]$. Consider Algorithm 10.8 which, if successful, outputs a $K_{q-i}^{(r-i)}$-decomposition $\mathcal{K}_j$ of H_j for every $j \in [p]$.

Algorithm 10.8

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for j from 1 to p **do**

for all $z = (z_0, z_1, z_2, z_3) \in \mathcal{Z}_{r,i}$, define T_z^j as the $(z_1 + z_2)$-graph on U_j containing all $Z_1 \mathbin{\dot\cup} Z_2 \subseteq U_j$ with $|Z_1| = z_1, |Z_2| = z_2$ such that for some $j' \in [j-1]$ with $|S_j \cap S_{j'}| = z_0$ and some $K' \in \mathcal{K}_{j'}^{(q-i)}$, we have $Z_1 \subseteq S_{j'}$, $Z_2 \subseteq K'$ and $|K' \cap S_j| = z_3$

if there exist t $K_{q-i}^{(r-i)}$-decompositions $\mathcal{K}_{j,1}, \dots, \mathcal{K}_{j,t}$ of $H_j - \bigcup_{z \in \mathcal{Z}_{r,i}} T_z^j$ which are pairwise $(q-i)$-disjoint **then**

pick $s \in [t]$ uniformly at random and let $\mathcal{K}_j := \mathcal{K}_{j,s}$

else

return 'unsuccessful'

end if

end for

Claim 1: If Algorithm 10.8 outputs $\mathcal{K}_1, \dots, \mathcal{K}_p$, then $\mathcal{K} := \bigcup_{j \in [p]} \tilde{\mathcal{K}}_j$ is a packing as desired, where $\tilde{\mathcal{K}}_j := (S_j \uplus \mathcal{K}_j^{(q-i)})^{\le}$.

Proof of claim: Since $z_1 + z_2 > r - i$, we have $H_j^{(r-i)} = (H_j - \bigcup_{z \in \mathcal{Z}_{r,i}} T_z^j)^{(r-i)}$. Hence, $\mathcal{K}_j$ is indeed a $K_{q-i}^{(r-i)}$-decomposition of H_j. Thus, by Fact 10.1, $\tilde{\mathcal{K}}_j$ is a $K_q^{(r)}$-packing in G covering all r-edges containing S_j. Therefore, $\mathcal{K}$ covers all $\mathcal{S}$-important r-edges of G. Now, let $j' < j$. We have to show that $\tilde{\mathcal{K}}_{j'}$ and $\tilde{\mathcal{K}}_j$ are r-disjoint. Suppose, for a contradiction, that there exist $K \in \mathcal{K}_j^{(q-i)}$ and $K' \in \mathcal{K}_{j'}^{(q-i)}$ such that $|(S_j \cup K) \cap (S_{j'} \cup K')| \ge r$. Let $z_0 := |S_j \cap S_{j'}|$ and $z_3 := |S_j \cap K'|$. Hence, we have $|K \cap (S_{j'} \cup K')| \ge r - z_0 - z_3$. Choose $X \subseteq K$ such that $|X \cap (S_{j'} \cup K')| = r - z_0 - z_3$ and let $Z_1 := X \cap S_{j'}$ and $Z_2 := X \cap K'$. We claim that $z := (z_0, |Z_1|, |Z_2|, z_3) \in \mathcal{Z}_{r,i}$. Clearly, we have $z_0 + |Z_1| + |Z_2| + z_3 = r$. Furthermore, note that $z_0 + z_3 < i$. Indeed, we clearly have $z_0 + z_3 = |S_j \cap (S_{j'} \cup K')| \le |S_j| = i$, and equality can only hold if $S_j \subseteq S_{j'} \cup K'$, which is impossible since G is r-exclusive. Similarly, we have $z_0 + |Z_1| < i$. Thus, $z \in \mathcal{Z}_{r,i}$. But this implies that $Z_1 \cup Z_2 \in T_z^j$, in contradiction to $Z_1 \cup Z_2 \subseteq K$. –

In order to prove the lemma, it is thus sufficient to prove that with positive probability, $\Delta(T_z^j) \le 2^{2r} q \rho^{1/2} |U_j|$ for all $j \in [p]$ and $z \in \mathcal{Z}_{r,i}$. Indeed, this would imply that $\Delta(\bigcup_{z \in \mathcal{Z}_{r,i}} T_z^j) \le (r+1)^3 2^{2r} q \rho^{1/2} |U_j|$, and by Proposition 5.6(v), $H_j - \bigcup_{z \in \mathcal{Z}_{r,i}} T_z^j$ would be a $(\rho^{1/6}, \xi/2, q-i, r-i)$-supercomplex. By Corollary 3.13 and since $|U_j| \ge 0.5\rho\rho_{size}n$, there are then t $(q-i)$-disjoint $K_{q-i}^{(r-i)}$-decompositions in $H_j - \bigcup_{z \in \mathcal{Z}_{r,i}} T_z^j$, so the algorithm would succeed.

In order to analyse $\Delta(T^j_z)$, we define the following variables. Suppose that $1 \le j' < j \le p$, that $z = (z_0, z_1, z_2, z_3) \in \mathcal{Z}_{r,i}$ and $f \subseteq U_j$ is a $(z_1 + z_2 - 1)$-set. Let $Y^{f,j'}_{j,z}$ denote the random indicator variable of the event that each of the following holds:

(a) there exists some $K' \in \mathcal{K}^{(q-i)}_{j'}$ with $|K' \cap S_j| = z_3$;
(b) there exist $Z_1 \subseteq S_{j'}$, $Z_2 \subseteq K'$ with $|Z_1| = z_1$, $|Z_2| = z_2$ such that $f \subseteq Z_1 \cup Z_2 \subseteq U_j$;
(c) $|S_j \cap S_{j'}| = z_0$.

We say that $v \in \binom{U_j \setminus f}{1}$ is a *witness for* j' if (a)–(c) hold with $Z_1 \uplus Z_2 = f \uplus v$. For all $j \in [p]$, $z = (z_0, z_1, z_2, z_3) \in \mathcal{Z}_{r,i}$ and $(z_1 + z_2 - 1)$-sets $f \subseteq U_j$, let $X^f_{j,z} := \sum_{j'=1}^{j-1} Y^{f,j'}_{j,z}$.

Claim 2: For all $j \in [p]$, $z = (z_0, z_1, z_2, z_3) \in \mathcal{Z}_{r,i}$ and $(z_1 + z_2 - 1)$-sets $f \subseteq U_j$, we have $|T^j_z(f)| \le 2^{2r} q X^f_{j,z}$.

Proof of claim: Let j, z and f be fixed. Clearly, if $v \in T^j_z(f)$, then by Algorithm 10.8, v is a witness for some $j' \in [j-1]$. Conversely, we claim that for each $j' \in [j-1]$, there are at most $2^{2r}q$ witnesses for j'. Clearly, this would imply that $|T^j_z(f)| \le 2^{2r} q |\{j' \in [j-1] : Y^{f,j'}_{j,z} = 1\}| = 2^{2r} q X^f_{j,z}$.

Fix $j' \in [j-1]$. If v is a witness for j', then there exists $K_v \in \mathcal{K}^{(q-i)}_{j'}$ such that (a)–(c) hold with $Z_1 \uplus Z_2 = f \uplus v$ and K_v playing the role of K'. By (b) we must have $v \subseteq Z_1 \cup Z_2 \subseteq S_{j'} \cup K_v$. Since $|S_{j'} \cup K_v| = q$, there are at most q witnesses v' for j' such that K_v can play the role of $K_{v'}$. It is thus sufficient to show that there are at most 2^{2r} $K' \in \mathcal{K}^{(q-i)}_{j'}$ such that (a)–(c) hold.

Note that for any possible choice of Z_1, Z_2, K', we must have $|f \cap Z_2| \in \{z_2, z_2 - 1\}$ and $f \cap Z_2 \subseteq Z_2 \subseteq K'$ by (b). For any $Z_2' \subseteq f$ with $|Z_2'| \in \{z_2, z_2 - 1\}$ and $Z_3 \in \binom{S_j}{z_3}$, there can be at most one $K' \in \mathcal{K}^{(q-i)}_{j'}$ with $Z_2' \subseteq K'$ and $K' \cap S_j = Z_3$. This is because $\mathcal{K}_{j'}$ is a $K^{(r-i)}_{q-i}$-decomposition and $|Z_2' \cup Z_3| \ge z_2 - 1 + z_3 \ge r - i$. Hence, there can be at most $2^{|f|}\binom{i}{z_3} \le 2^{2r}$ possible choices for K'. –

The following claim thus implies the lemma.

Claim 3: With positive probability, we have $X^f_{j,z} \le \rho^{1/2}|U_j|$ for all $j \in [p]$, $z = (z_0, z_1, z_2, z_3) \in \mathcal{Z}_{r,i}$ and $(z_1 + z_2 - 1)$-sets $f \subseteq U_j$.

Proof of claim: Fix j, z, f as above. We split $X^f_{j,z}$ into two sums. For this, let

$$\begin{aligned}
\mathcal{J}^f_{j,z} &:= \{j' \in [j-1] : |S_j \cap S_{j'}| = z_0, f \setminus S_{j'} \subseteq U_{j'}, |U_{j'} \cap S_j| \ge z_3\},\\
\mathcal{J}^f_{j,z,1} &:= \{j' \in \mathcal{J}^f_{j,z} : |f \cap S_{j'}| = z_1\},\\
\mathcal{J}^f_{j,z,2} &:= \{j' \in \mathcal{J}^f_{j,z} : |f \cap S_{j'}| = z_1 - 1, |U_j \cap (S_{j'} \setminus f)| \ge 1\}.
\end{aligned}$$

Since $\mathcal{U}$ is a (ρ_{size}, ρ, r)-focus for $\mathcal{S}$, (F3) implies that

$$|\mathcal{J}^f_{j,z,1}| \le 2^{6r} \rho^{z_2 + z_3 - 1} n^{i - z_0 - z_1}, \tag{10.3}$$

$$|\mathcal{J}^f_{j,z,2}| \le 2^{9r} \rho^{z_2 + z_3 + 1} n^{i - z_0 - z_1 + 1}. \tag{10.4}$$

Note that if $Y^{f,j'}_{j,z} = 1$, then $j' \in \mathcal{J}^f_{j,z,1} \cup \mathcal{J}^f_{j,z,2}$. Hence, we have $X^f_{j,z} = X^f_{j,z,1} + X^f_{j,z,2}$, where $X^f_{j,z,1} := \sum_{j' \in \mathcal{J}^f_{j,z,1}} Y^{f,j'}_{j,z}$ and $X^f_{j,z,2} := \sum_{j' \in \mathcal{J}^f_{j,z,2}} Y^{f,j'}_{j,z}$. We bound $X^f_{j,z,1}$ and $X^f_{j,z,2}$ separately.

Step 1: Estimating $X^f_{j,z,1}$

Consider $j' \in \mathcal{J}^f_{j,z,1}$. Let

$$\mathcal{K}^{f,j'}_{j,z} := \{K' \in \binom{U_{j'}}{q-i} : f \subseteq S_{j'} \cup K', |K' \cap U_j| \ge z_2, |K' \cap S_j| = z_3\}. \tag{10.5}$$

Note that if $Y^{f,j'}_{j,z} = 1$, then there exists $K' \in \mathcal{K}^{(q-i)}_{j'}$ with $K' \in \mathcal{K}^{f,j'}_{j,z}$. We now bound $|\mathcal{K}^{f,j'}_{j,z}|$. For all $K' \in \mathcal{K}^{f,j'}_{j,z}$, we have $f \setminus S_{j'} \subseteq K'$ and $|f \cap K'| = |f| - |f \cap S_{j'}| = z_2 - 1$, and the sets $f \cap K'$, $K' \cap S_j$, $(K' \setminus f) \cap (U_j \cap U_{j'})$ are disjoint. Moreover, we have $|(K' \setminus f) \cap (U_j \cap U_{j'})| = |(K' \setminus f) \cap U_j| \geq |K' \cap U_j| - |f \cap K'| \geq 1$. We can thus count

$$|\mathcal{K}^{f,j'}_{j,z}| \leq \binom{|S_j|}{z_3} \cdot |U_j \cap U_{j'}| \cdot |U_{j'}|^{q-i-z_2-z_3} \leq 2^i \cdot 2\rho^2 n \cdot (2\rho\rho_{size} n)^{q-i-z_2-z_3}.$$

Recall that the candidates $\mathcal{K}_{j',1}, \dots, \mathcal{K}_{j',t}$ in Algorithm 10.8 from which $\mathcal{K}_{j'}$ was chosen at random are $(q-i)$-disjoint. Let $\tilde{\rho}_1 := \rho^{z_0+z_1-i+5/3}\rho_{size} n^{1+z_0+z_1-i} \in [0,1]$. In order to apply Proposition 5.8, let $j_1, \dots, j_b$ be an enumeration of $\mathcal{J}^f_{j,z,1}$. We then have for all $k \in [b]$ and all $y_1, \dots, y_{k-1} \in \{0,1\}$ that

$$\begin{aligned}\mathbb{P}(Y^{f,j_k}_{j,z} = 1 \mid Y^{f,j_1}_{j,z} = y_1, \dots, Y^{f,j_{k-1}}_{j,z} = y_{k-1}) \leq \frac{|\mathcal{K}^{f,j_k}_{j,z}|}{t} &\leq \frac{2^i \cdot 2\rho^2 n \cdot (2\rho\rho_{size} n)^{q-i-z_2-z_3}}{\rho^{1/6}(0.5\rho\rho_{size} n)^{q-r}} \\ &= 2^{2q-r+1-z_2-z_3}\rho^{11/6}(\rho\rho_{size})^{z_0+z_1-i} n^{1+z_0+z_1-i} \\ &\leq \tilde{\rho}_1.\end{aligned}$$

Let $B_1 \sim Bin(|\mathcal{J}^f_{j,z,1}|, \tilde{\rho}_1)$ and observe that

$$\begin{aligned}7\mathbb{E}B_1 = 7|\mathcal{J}^f_{j,z,1}|\tilde{\rho}_1 &\overset{(10.3)}{\leq} 7 \cdot 2^{6r}\rho^{z_2+z_3-1} n^{i-z_0-z_1} \cdot \rho^{z_0+z_1-i+5/3}\rho_{size} n^{1+z_0+z_1-i} \\ &= 7 \cdot 2^{6r}\rho^{r-i+2/3}\rho_{size} n \leq 0.5\rho^{1/2}|U_j|.\end{aligned}$$

Thus,

$$\mathbb{P}(X^f_{j,z,1} \geq 0.5\rho^{1/2}|U_j|) \overset{\text{Proposition 5.8}}{\leq} \mathbb{P}(B_1 \geq 0.5\rho^{1/2}|U_j|) \overset{\text{Lemma 5.7(iii)}}{\leq} e^{-0.5\rho^{1/2}|U_j|}.$$

Step 2: Estimating $X^f_{j,z,2}$

Consider $j' \in \mathcal{J}^f_{j,z,2}$. Define $\mathcal{K}^{f,j'}_{j,z}$ as in (10.5). This time, since $|f \cap S_{j'}| = z_1 - 1$, we have $|K' \cap f| = |f \setminus S_{j'}| = z_2$ for all $K' \in \mathcal{K}^{f,j'}_{j,z}$. Thus, we count

$$|\mathcal{K}^{f,j'}_{j,z}| \leq \binom{|S_j|}{z_3} \cdot |U_{j'}|^{q-i-z_2-z_3} \leq 2^i \cdot (2\rho\rho_{size} n)^{q-i-z_2-z_3}.$$

Let $\tilde{\rho}_2 := \rho^{z_0+z_1-i-1/5}\rho_{size} n^{z_0+z_1-i} \in [0,1]$. In order to apply Proposition 5.8, let $j_1, \dots, j_b$ be an enumeration of $\mathcal{J}^f_{j,z,2}$. We then have for all $k \in [b]$ and all $y_1, \dots, y_{k-1} \in \{0,1\}$ that

$$\begin{aligned}\mathbb{P}(Y^{f,j_k}_{j,z} = 1 \mid Y^{f,j_1}_{j,z} = y_1, \dots, Y^{f,j_{k-1}}_{j,z} = y_{k-1}) \leq \frac{|\mathcal{K}^{f,j_k}_{j,z}|}{t} &\leq \frac{2^i \cdot (2\rho\rho_{size} n)^{q-i-z_2-z_3}}{\rho^{1/6}(0.5\rho\rho_{size} n)^{q-r}} \\ &= 2^{2q-r-z_2-z_3}\rho^{-1/6}(\rho\rho_{size} n)^{z_0+z_1-i} \\ &\leq \tilde{\rho}_2.\end{aligned}$$

Let $B_2 \sim Bin(|\mathcal{J}^f_{j,z,2}|, \tilde{\rho}_2)$ and observe that

$$\begin{aligned}7\mathbb{E}B_2 = 7|\mathcal{J}^f_{j,z,2}|\tilde{\rho}_2 &\overset{(10.4)}{\leq} 7 \cdot 2^{9r}\rho^{z_2+z_3+1} n^{i-z_0-z_1+1} \cdot \rho^{z_0+z_1-i-1/5}\rho_{size} n^{z_0+z_1-i} \\ &= 7 \cdot 2^{9r}\rho^{r-i+4/5}\rho_{size} n \leq 0.5\rho^{1/2}|U_j|.\end{aligned}$$

Thus,

$$\mathbb{P}(X^f_{j,z,2} \geq 0.5\rho^{1/2}|U_j|) \overset{\text{Proposition 5.8}}{\leq} \mathbb{P}(B_2 \geq 0.5\rho^{1/2}|U_j|) \overset{\text{Lemma 5.7(iii)}}{\leq} e^{-0.5\rho^{1/2}|U_j|}.$$

Hence,

$$\mathbb{P}(X^f_{j,z} \geq \rho^{1/2}|U_j|) \leq \mathbb{P}(X^f_{j,z,1} \geq 0.5\rho^{1/2}|U_j|) + \mathbb{P}(X^f_{j,z,2} \geq 0.5\rho^{1/2}|U_j|) \leq 2e^{-0.5\rho^{1/2}|U_j|}.$$

Since $|\mathcal{S}| \leq n^i$, a union bound easily implies Claim 3. $-$

This completes the proof of Lemma 10.7. □

10.2. **Partition pairs.** We now develop the appropriate framework to be able to state the Cover down lemma for setups (Lemma 10.22). Recall that we will consider (and cover) r-sets separately according to their type. The type of an r-set e naturally imposes constraints on the type of a q-set which covers e. We will need to track and adjust the densities of r-sets with respect to q-sets for each pair of types separately. This gives rise to the following concepts of partition pairs and partition regularity (see Section 10.3).

Let X be a set. We say that $\mathcal{P} = (X_1, \dots, X_a)$ is an ordered partition of X if the X_i are disjoint subsets of X whose union is X. We let $\mathcal{P}(i) := X_i$ and $\mathcal{P}([i]) := (X_1, \dots, X_i)$. If $\mathcal{P} = (X_1, \dots, X_a)$ is an ordered partition of X and $X' \subseteq X$, we let $\mathcal{P}[X']$ denote the ordered partition $(X_1 \cap X', \dots, X_a \cap X')$ of X'. If $\{X', X''\}$ is a partition of X, $\mathcal{P}' = (X'_1, \dots, X'_a)$ is an ordered partition of X' and $\mathcal{P}'' = (X''_1, \dots, X''_b)$ is an ordered partition of X'', we let

$$\mathcal{P}' \sqcup \mathcal{P}'' := (X'_1, \dots, X'_a, X''_1, \dots, X''_b).$$

Definition 10.9. Let G be a complex and let $q > r \geq 1$. An *(r, q)-partition pair of G* is a pair $(\mathcal{P}_{edge}, \mathcal{P}_{clique})$, where $\mathcal{P}_{edge}$ is an ordered partition of $G^{(r)}$ and $\mathcal{P}_{clique}$ is an ordered partition of $G^{(q)}$, such that for all $\mathcal{E} \in \mathcal{P}_{edge}$ and $\mathcal{Q} \in \mathcal{P}_{clique}$, every $Q \in \mathcal{Q}$ contains the same number $B(\mathcal{E}, \mathcal{Q})$ of elements from $\mathcal{E}$. We call $B \colon \mathcal{P}_{edge} \times \mathcal{P}_{clique} \to [\binom{q}{r}]_0$ the *containment function* of the partition pair. We say that $(\mathcal{P}_{edge}, \mathcal{P}_{clique})$ is *upper-triangular* if $B(\mathcal{P}_{edge}(\ell), \mathcal{P}_{clique}(k)) = 0$ whenever $\ell > k$.

Clearly, for every $\mathcal{Q} \in \mathcal{P}_{clique}$, $\sum_{\mathcal{E} \in \mathcal{P}_{edge}} B(\mathcal{E}, \mathcal{Q}) = \binom{q}{r}$. If $(\mathcal{P}_{edge}, \mathcal{P}_{clique})$ is an (r, q)-partition pair of G and $H \subseteq G^{(r')}$, we define

$$(\mathcal{P}_{edge}, \mathcal{P}_{clique})[H] := (\mathcal{P}_{edge}[G[H]^{(r)}], \mathcal{P}_{clique}[G[H]^{(q)}]).$$

Clearly, $(\mathcal{P}_{edge}, \mathcal{P}_{clique})[H]$ is an (r, q)-partition pair of $G[H]$.

Example 10.10. Suppose that G is a complex and $U \subseteq V(G)$. For $\ell \in [r]_0$, define $\mathcal{E}_\ell := \{e \in G^{(r)} : |e \cap U| = \ell\}$. For $k \in [q]_0$, define $\mathcal{Q}_k := \{Q \in G^{(q)} : |Q \cap U| = k\}$. Let $\mathcal{P}_{edge} := (\mathcal{E}_0, \dots, \mathcal{E}_r)$ and $\mathcal{P}_{clique} := (\mathcal{Q}_0, \dots, \mathcal{Q}_q)$. Then clearly $(\mathcal{P}_{edge}, \mathcal{P}_{clique})$ is an (r, q)-partition pair of G, where the containment function is given by $B(\mathcal{E}_\ell, \mathcal{Q}_k) = \binom{k}{\ell}\binom{q-k}{r-\ell}$. In particular, $B(\mathcal{E}_\ell, \mathcal{Q}_k) = 0$ whenever $\ell > k$ or $k > q - r + \ell$. We say that $(\mathcal{P}_{edge}, \mathcal{P}_{clique})$ is the *(r, q)-partition pair of G, U*.

The partition pairs we use are generalisations of the above example. More precisely, suppose that G is a complex, $\mathcal{S}$ is an i-system in $V(G)$ and $\mathcal{U}$ is a focus for $\mathcal{S}$. Moreover, assume that G is r-exclusive with respect to $\mathcal{S}$. For $r' \geq r$, let $\tau_{r'}$ denote the type function of $G^{(r')}$, $\mathcal{S}$, $\mathcal{U}$. As in the above example, if $\mathcal{E}_\ell := \tau_r^{-1}(\ell)$ for all $\ell \in [r-i]_0$ and $\mathcal{Q}_k := \tau_q^{-1}(k)$ for all $k \in [q-i]_0$, then every $Q \in \mathcal{Q}_k$ contains exactly $\binom{k}{\ell}\binom{q-i-k}{r-i-\ell}$ elements from $\mathcal{E}_\ell$. However, we also have to consider $\mathcal{S}$-unimportant edges and cliques. It turns out that it is useful to assume that the unimportant edges and cliques are partitioned into i parts each, in an upper-triangular fashion.

More formally, for $r' \geq r$, let $\mathcal{D}_{r'}$ denote the set of $\mathcal{S}$-unimportant r'-sets of G and assume that $\mathcal{P}^*_{edge}$ is an ordered partition of $\mathcal{D}_r$ and $\mathcal{P}^*_{clique}$ is an ordered partition of $\mathcal{D}_q$. We say that $(\mathcal{P}^*_{edge}, \mathcal{P}^*_{clique})$ is *admissible with respect to G, $\mathcal{S}$, $\mathcal{U}$* if the following hold:

(P1) $|\mathcal{P}^*_{edge}| = |\mathcal{P}^*_{clique}| = i$;

(P2) for all $S \in \mathcal{S}$, $h \in [r-i]_0$ and $F \subseteq G(S)^{(h)}$ with $1 \leq |F| \leq 2^h$ and all $\ell \in [i]$, there exists $D(S, F, \ell) \in \mathbb{N}_0$ such that for all $Q \in \bigcap_{f \in F} G(S \cup f)[U_S]^{(q-i-h)}$, we have that

$$|\{e \in \mathcal{P}^*_{edge}(\ell) : \exists f \in F \colon e \subseteq S \cup f \cup Q\}| = D(S, F, \ell);$$

(P3) $(\mathcal{P}^*_{edge} \sqcup \{G^{(r)} \setminus \mathcal{D}_r\}, \mathcal{P}^*_{clique} \sqcup \{G^{(q)} \setminus \mathcal{D}_q\})$ is an upper-triangular (r, q)-partition pair of G.

	$\mathcal{P}^*_{clique}(1)$	$\dots$	$\mathcal{P}^*_{clique}(i)$	$\tau_q^{-1}(0)$	$\tau_q^{-1}(1)$	$\dots$	$\dots$	$\tau_q^{-1}(q-r)$	$\dots$	$\dots$	$\tau_q^{-1}(q-i)$
$\mathcal{P}^*_{edge}(1)$	$*$										
$\dots$	0	$*$									
$\mathcal{P}^*_{edge}(i)$	0	0	$*$								
$\tau_r^{-1}(0)$	0	0	0	$*$				$*$	0	0	0
$\dots$	0	0	0	0	$*$				$*$	0	0
$\dots$	0	0	0	0	0	$*$				$*$	0
$\tau_r^{-1}(r-i)$	0	0	0	0	0	0	$*$				$*$

FIGURE 1. The above table sketches the containment function of an (r,q)-partition pair induced by $(\mathcal{P}^*_{edge}, \mathcal{P}^*_{clique})$ and $\mathcal{U}$. The fields marked with $*$ and the shaded subtable will play an important role later on.

Note that for $i = 0$, $(\emptyset, \emptyset)$ trivially satisfies these conditions. Also note that (P2) can be viewed as an analogue of the containment function (from Definition 10.9) which is suitable for dealing with supercomplexes.

Assume that $(\mathcal{P}^*_{edge}, \mathcal{P}^*_{clique})$ is admissible with respect to G, $\mathcal{S}$, $\mathcal{U}$. Define

$$\mathcal{P}_{edge} := \mathcal{P}^*_{edge} \sqcup (\tau_r^{-1}(0), \dots, \tau_r^{-1}(r-i)),$$
$$\mathcal{P}_{clique} := \mathcal{P}^*_{clique} \sqcup (\tau_q^{-1}(0), \dots, \tau_q^{-1}(q-i)).$$

It is not too hard to see that $(\mathcal{P}_{edge}, \mathcal{P}_{clique})$ is an (r,q)-partition pair of G. Indeed, $\mathcal{P}_{edge}$ clearly is a partition of $G^{(r)}$ and $\mathcal{P}_{clique}$ is a partition of $G^{(q)}$. Suppose that B is the containment function of $(\mathcal{P}^*_{edge} \sqcup \{G^{(r)} \setminus \mathcal{D}_r\}, \mathcal{P}^*_{clique} \sqcup \{G^{(q)} \setminus \mathcal{D}_q\})$. Then B' as defined below is the containment function of $(\mathcal{P}_{edge}, \mathcal{P}_{clique})$:

- For all $\mathcal{E} \in \mathcal{P}^*_{edge}$ and $\mathcal{Q} \in \mathcal{P}^*_{clique}$, let $B'(\mathcal{E}, \mathcal{Q}) := B(\mathcal{E}, \mathcal{Q})$.
- For all $\ell \in [r-i]_0$ and $\mathcal{Q} \in \mathcal{P}^*_{clique}$, let $B'(\tau_r^{-1}(\ell), \mathcal{Q}) := 0$.
- For all $\mathcal{E} \in \mathcal{P}^*_{edge}$ and $k \in [q-i]_0$, define $B'(\mathcal{E}, \tau_q^{-1}(k)) := B(\mathcal{E}, \{G^{(q)} \setminus \mathcal{D}_q\})$.
- For all $\ell \in [r-i]_0$, $k \in [q-i]_0$, let

$$B'(\tau_r^{-1}(\ell), \tau_q^{-1}(k)) := \binom{k}{\ell}\binom{q-i-k}{r-i-\ell}. \tag{10.6}$$

We say that $(\mathcal{P}_{edge}, \mathcal{P}_{clique})$ as defined above is *induced by* $(\mathcal{P}^*_{edge}, \mathcal{P}^*_{clique})$ *and* $\mathcal{U}$.

Finally, we say that $(\mathcal{P}_{edge}, \mathcal{P}_{clique})$ is an *(r,q)-partition pair of G, $\mathcal{S}$, $\mathcal{U}$*, if

- $(\mathcal{P}_{edge}([i]), \mathcal{P}_{clique}([i]))$ is admissible with respect to G, $\mathcal{S}$, $\mathcal{U}$;
- $(\mathcal{P}_{edge}, \mathcal{P}_{clique})$ is induced by $(\mathcal{P}_{edge}([i]), \mathcal{P}_{clique}([i]))$ and $\mathcal{U}$.

Proposition 10.11. *Let $0 \le i < r < q$ and suppose that G is a complex, $\mathcal{S}$ is an i-system in $V(G)$ and $\mathcal{U}$ is a focus for $\mathcal{S}$. Moreover, assume that G is r-exclusive with respect to $\mathcal{S}$. Let $(\mathcal{P}_{edge}, \mathcal{P}_{clique})$ be an (r,q)-partition pair of G, $\mathcal{S}$, $\mathcal{U}$ with containment function B. Then the following hold:*

(P1′) $|\mathcal{P}_{edge}| = r+1$ *and* $|\mathcal{P}_{clique}| = q+1$;

(P2′) *for* $i < \ell \le r+1$, $\mathcal{P}_{edge}(\ell) = \tau_r^{-1}(\ell - i - 1)$, *and for* $i < k \le q+1$, $\mathcal{P}_{clique}(k) = \tau_q^{-1}(k-i-1)$;

(P3′) $(\mathcal{P}_{edge}, \mathcal{P}_{clique})$ *is upper-triangular;*

(P4′) $B(\mathcal{P}_{edge}(\ell), \mathcal{P}_{clique}(k)) = 0$ *whenever both* $\ell > i$ *and* $k > q - r + \ell$;

(P5′) (P2) *holds for all* $\ell \in [r+1]$, *with* $\mathcal{P}_{edge}$ *playing the role of* $\mathcal{P}^*_{edge}$.

(P6′) *if* $i = 0$, $\mathcal{S} = \{\emptyset\}$ *and* $\mathcal{U} = \{U\}$ *for some* $U \subseteq V(G)$, *then the (unique) (r,q)-partition pair of G, $\mathcal{S}$, $\mathcal{U}$ is the (r,q)-partition pair of G, U (cf. Example 10.10);*

(P7′) *for all* $H \subseteq G^{(r)}$, $(\mathcal{P}_{edge}, \mathcal{P}_{clique})[H]$ *is an (r,q)-partition pair of* $G[H]$, $\mathcal{S}$, $\mathcal{U}$.

Proof. Clearly, (P1′), (P2′) and (P6′) hold, and it is also straightforward to check (P7′). Moreover, (P3′) holds because of (P3) and (10.6). The latter also implies (P4′).

Finally, consider (P5′). For $\ell \in [i]$, this holds since $(\mathcal{P}_{edge}([i]), \mathcal{P}_{clique}([i]))$ is admissible, so assume that $\ell > i$. We have $\mathcal{P}_{edge}(\ell) = \tau_r^{-1}(\ell - i - 1)$. Let $S \in \mathcal{S}$, $h \in [r-i]_0$ and $F \subseteq G(S)^{(h)}$ with $1 \leq |F| \leq 2^h$.

For $Q \in \bigcap_{f \in F} G(S \cup f)[U_S]^{(q-i-h)}$, let

$$\mathcal{D}_Q := \{e \in G^{(r)} \,:\, S \subseteq e, |e \cap U_S| = \ell - i - 1, \exists f \in F \colon e \setminus S \subseteq f \cup Q\},$$

and for $e \in \mathcal{D}_Q$, let $\sigma_{Q,e}$ be some $f \in F$ such that $e \setminus S \subseteq f \cup Q$.

It is easy to see that

$$\{e \in \mathcal{P}_{edge}(\ell) \,:\, \exists f \in F \colon e \subseteq S \cup f \cup Q\} = \mathcal{D}_Q.$$

Note that for every $e \in \mathcal{D}_Q$, we have $e = S \cup (\bigcup F \cap e) \cup (Q \cap e)$.

It remains to show that for all $Q, Q' \in \bigcap_{f \in F} G(S \cup f)[U_S]^{(q-i-h)}$, we have $|\mathcal{D}_Q| = |\mathcal{D}_{Q'}|$. Let $\pi \colon Q \to Q'$ be any bijection. For each $e \in \mathcal{D}_Q$, define $\pi'(e) := S \cup (\bigcup F \cap e) \cup \pi(Q \cap e)$. It is straightforward to check that $\pi' \colon \mathcal{D}_Q \to \mathcal{D}_{Q'}$ is a bijection. □

10.3. **Partition regularity.**

Definition 10.12. Let G be a complex on n vertices and $(\mathcal{P}_{edge}, \mathcal{P}_{clique})$ an (r,q)-partition pair of G with $a := |\mathcal{P}_{edge}|$ and $b := |\mathcal{P}_{clique}|$. Let $A = (a_{\ell,k}) \in [0,1]^{a \times b}$. We say that G is *(ε, A, q, r)-regular with respect to $(\mathcal{P}_{edge}, \mathcal{P}_{clique})$* if for all $\ell \in [a]$, $k \in [b]$ and $e \in \mathcal{P}_{edge}(\ell)$, we have

$$|(\mathcal{P}_{clique}(k))(e)| = (a_{\ell,k} \pm \varepsilon) n^{q-r}, \tag{10.7}$$

where we view $\mathcal{P}_{clique}(k)$ as a subgraph of $G^{(q)}$. If $\mathcal{E} = \mathcal{P}_{edge}(\ell)$ and $\mathcal{Q} = \mathcal{P}_{clique}(k)$, we may often write $A(\mathcal{E}, \mathcal{Q})$ instead of $a_{\ell,k}$.

For $A \in [0,1]^{a \times b}$ with $1 \leq t \leq a \leq b$, we define

- $\min^{\setminus}(A) := \min\{a_{j,j} \,:\, j \in [a]\}$ as the minimum value on the diagonal,
- $\min^{\setminus t}(A) := \min\{a_{j,j+b-a} \,:\, j \in \{a-t+1, \dots, a\}\}$ and
- $\min^{\setminus\setminus t}(A) := \min\{\min^{\setminus}(A), \min^{\setminus t}(A)\}$.

Note that $\min^{\setminus\setminus r-i+1}(A)$ is the minimum value of the entries in A that correspond to the entries marked with $*$ in Figure 1.

Example 10.13. Suppose that G is a complex and that $U \subseteq V(G)$ is $(\varepsilon, \mu, \xi, q, r)$-random in G (see Definition 7.1). Let $(\mathcal{P}_{edge}, \mathcal{P}_{clique})$ be the (r,q)-partition pair of G, U (cf. Example 10.10). Let $Y \subseteq G^{(q)}$ and $d \geq \xi$ be such that (R2) holds. Define the matrix $A \in [0,1]^{(r+1) \times (q+1)}$ as follows: for all $\ell \in [r+1]$ and $k \in [q+1]$, let

$$a_{\ell,k} := bin(q-r, \mu, k-\ell)d.$$

For all $\ell \in [r+1]$, $k \in [q+1]$ and $e \in \mathcal{P}_{edge}(\ell) = \{e \in G^{(r)} \,:\, |e \cap U| = \ell - 1\}$, we have that

$$\begin{aligned} |(\mathcal{P}_{clique}[Y](k))(e)| &= |\{Q \in G[Y]^{(q)}(e) \,:\, |(e \cup Q) \cap U| = k-1\}| \\ &= |\{Q \in G[Y]^{(q)}(e) \,:\, |Q \cap U| = k-\ell\}| \\ &\overset{(R2)}{=} (1 \pm \varepsilon) bin(q-r, \mu, k-\ell) d n^{q-r} = (a_{\ell,k} \pm \varepsilon) n^{q-r}. \end{aligned}$$

In other words, $G[Y]$ is (ε, A, q, r)-regular with respect to $(\mathcal{P}_{edge}, \mathcal{P}_{clique}[Y])$. Note also that $\min^{\setminus\setminus r+1}(A) = \min\{bin(q-r, \mu, 0), bin(q-r, \mu, q-r)\}d \geq (\min\{\mu, 1-\mu\})^{q-r}\xi$.

In the proof of the Cover down lemma for setups, we face (amongst others) the following two challenges: (i) given an (ε, A, q, r)-regular complex G for some suitable A, we need to find an efficient $K_q^{(r)}$-packing in G; (ii) if A is not suitable for (i), we need to find a 'representative' subcomplex G' of G which is (ε, A', q, r)-regular for some A' that is suitable for (i). The strategy to implement (i) is similar to that of the Boost lemma (Lemma 6.3): We randomly sparsify $G^{(q)}$ according to a suitably chosen (non-uniform) probability distribution in order to find $Y^* \subseteq G^{(q)}$ such that $G[Y^*]$ is (ε, d, q, r)-regular. We can then apply the Boosted nibble lemma (Lemma 6.4). The desired probability distribution arises from a non-negative solution to the equation $Ax = \mathbb{1}$. The following condition on A allows us to find such a solution (cf. Proposition 10.15).

Definition 10.14. We say that $A \in [0,1]^{a\times b}$ is *diagonal-dominant* if $a_{\ell,k} \leq a_{k,k}/2(a-\ell)$ for all $1 \leq \ell < k \leq \min\{a,b\}$.

Definition 10.14 also allows us to achieve (ii). Given some A, we can find a 'representative' subcomplex G' of G which is (ε, A', q, r)-regular for some A' that is diagonal-dominant (cf. Lemma 10.19).

Proposition 10.15. *Let $A \in [0,1]^{a\times b}$ be upper-triangular and diagonal-dominant with $a \leq b$. Then there exists $x \in [0,1]^b$ such that $x \geq \min^\backslash(A)/4b$ and $Ax = \min^\backslash(A)\mathbb{1}$.*

Proof. If $\min^\backslash(A) = 0$, we can take $x = 0$, so assume that $\min^\backslash(A) > 0$. For $k > a$, let $y_k := 1/4b$. For k from a down to 1, let $y_k := a_{k,k}^{-1}(1 - \sum_{j=k+1}^{b} a_{k,j}y_j)$. Since A is upper-triangular, we have $Ay = \mathbb{1}$. We claim that $1/4b \leq y_k \leq a_{k,k}^{-1}$ for all $k \in [b]$. This clearly holds for all $k > a$. Suppose that for some $k \in [a]$, we have already checked that $1/4b \leq y_j \leq a_{j,j}^{-1}$ for all $j > k$. We now check that

$$1 \geq 1 - \sum_{j=k+1}^{b} a_{k,j}y_j \geq 1 - \sum_{j=k+1}^{a} \frac{a_{j,j}}{2(a-k)}y_j - \frac{b-a}{4b} \geq \frac{3}{4} - \frac{a-k}{2(a-k)} = \frac{1}{4}$$

and so $1/4b \leq y_k \leq a_{k,k}^{-1}$. Thus we can take $x := \min^\backslash(A)y$. □

Corollary 10.16. *Let $1/n \ll \varepsilon \ll \xi, 1/q$ and $r \in [q-1]$. Suppose that G is a complex on n vertices and $(\mathcal{P}_{edge}, \mathcal{P}_{clique})$ is an upper-triangular (r,q)-partition pair of G with $|\mathcal{P}_{edge}| \leq |\mathcal{P}_{clique}| \leq q+1$. Let $A \in [0,1]^{|\mathcal{P}_{edge}|\times|\mathcal{P}_{clique}|}$ be diagonal-dominant with $d := \min^\backslash(A) \geq \xi$. Suppose that G is (ε, A, q, r)-regular with respect to $(\mathcal{P}_{edge}, \mathcal{P}_{clique})$ and $(\xi, q+r, r)$-dense. Then there exists $Y^* \subseteq G^{(q)}$ such that $G[Y^*]$ is $(2q\varepsilon, d, q, r)$-regular and $(0.9\xi(\xi/4(q+1))^{\binom{q+r}{q}}, q+r, r)$-dense.*

Proof. Since $(\mathcal{P}_{edge}, \mathcal{P}_{clique})$ is upper-triangular, we may assume that A is upper-triangular too. By Proposition 10.15, there exists a vector $x \in [0,1]^{|\mathcal{P}_{clique}|}$ with $x \geq \min^\backslash(A)/4(q+1) \geq \xi/4(q+1)$ and $Ax = d\mathbb{1}$.

Obtain $Y^* \subseteq G^{(q)}$ randomly by including every $Q \in G^{(q)}$ that belongs to $\mathcal{P}_{clique}(k)$ with probability x_k, all independently. Let $e \in \mathcal{P}_{edge}(\ell)$ for any $\ell \in [|\mathcal{P}_{edge}|]$. We have

$$\mathbb{E}|G[Y^*]^{(q)}(e)| = \sum_{k=1}^{|\mathcal{P}_{clique}|} x_k(a_{\ell,k} \pm \varepsilon)n^{q-r} = (d \pm (q+1)\varepsilon)n^{q-r}.$$

Then, combining Lemma 5.7(ii) with a union bound, we conclude that whp $G[Y^*]$ is $(2q\varepsilon, d, q, r)$-regular.

Let $e \in G^{(r)}$. Since $|G^{(q+r)}(e)| \geq \xi n^q$ and every $Q \in G^{(q+r)}(e)$ belongs to $G[Y^*]^{(q+r)}(e)$ with probability at least $(\xi/4(q+1))^{\binom{q+r}{q}}$, we conclude with Corollary 5.11 that with probability at least $1 - \mathrm{e}^{-n^{1/6}}$, we have

$$|G[Y^*]^{(q+r)}(e)| \geq 0.9(\xi/4(q+1))^{\binom{q+r}{q}}|G^{(q+r)}(e)| \geq 0.9\xi(\xi/4(q+1))^{\binom{q+r}{q}}n^q.$$

Applying a union bound shows that whp $G[Y^*]$ is $(0.9\xi(\xi/4(q+1))^{\binom{q+r}{q}}, q+r, r)$-dense. □

The following concept of a setup turns out to be the appropriate generalisation of Definition 7.1 to i-systems and partition pairs.

Definition 10.17 (Setup)**.** Let G be a complex on n vertices and $0 \le i < r < q$. We say that $\mathcal{S},\mathcal{U},(\mathcal{P}_{edge},\mathcal{P}_{clique})$ form an $(\varepsilon,\mu,\xi,q,r,i)$-*setup* for G if there exists a q-graph Y on $V(G)$ such that the following hold:

- (S1) $\mathcal{S}$ is an i-system in $V(G)$ such that G is r-exclusive with respect to $\mathcal{S}$; $\mathcal{U}$ is a μ-focus for $\mathcal{S}$ and $(\mathcal{P}_{edge},\mathcal{P}_{clique})$ is an (r,q)-partition pair of G, $\mathcal{S}$, $\mathcal{U}$;
- (S2) there exists a matrix $A \in [0,1]^{(r+1)\times(q+1)}$ with $\min^{\backslash\backslash r-i+1}(A) \ge \xi$ such that $G[Y]$ is (ε,A,q,r)-regular with respect to $(\mathcal{P}_{edge},\mathcal{P}_{clique})[Y] = (\mathcal{P}_{edge},\mathcal{P}_{clique}[Y])$;
- (S3) every $\mathcal{S}$-unimportant $e \in G^{(r)}$ is contained in at least $\xi(\mu n)^q$ $\mathcal{S}$-unimportant $Q \in G[Y]^{(q+r)}$, and for every $\mathcal{S}$-important $e \in G^{(r)}$ with $e \supseteq S \in \mathcal{S}$, we have $|G[Y]^{(q+r)}(e)[U_S]| \ge \xi(\mu n)^q$;
- (S4) for all $S \in \mathcal{S}$, $h \in [r-i]_0$ and all $F \subseteq G(S)^{(h)}$ with $1 \le |F| \le 2^h$ we have that $\bigcap_{f\in F} G(S\cup f)[U_S]$ is an $(\varepsilon,\xi,q-i-h,r-i-h)$-complex.

Moreover, if (S1)–(S4) are true and A is diagonal-dominant, then we say that $\mathcal{S},\mathcal{U},(\mathcal{P}_{edge},\mathcal{P}_{clique})$ form a *diagonal-dominant* $(\varepsilon,\mu,\xi,q,r,i)$-*setup* for G.

Note that (S4) implies that $G(S)[U_S]$ is an $(\varepsilon,\xi,q-i,r-i)$-supercomplex for every $S \in \mathcal{S}$, but is stronger in the sense that F is not restricted to U_S. We will now see that Definition 10.17 does indeed generalise Definition 7.1.

Proposition 10.18. *Let G be a complex on n vertices and suppose that $U \subseteq V(G)$ is $(\varepsilon,\mu,\xi,q,r)$-random in G. Let $\mathcal{S} := \{\emptyset\}$, $\mathcal{U} := \{U\}$ and let $(\mathcal{P}_{edge},\mathcal{P}_{clique})$ be the (r,q)-partition pair of G,U. Then $\mathcal{S},\mathcal{U},(\mathcal{P}_{edge},\mathcal{P}_{clique})$ form an $(\varepsilon,\mu,\tilde{\mu}\xi,q,r,0)$-setup for G, where $\tilde{\mu} := (\min\{\mu,1-\mu\})^{q-r}$.*

Proof. We first check (S1). Clearly, $\mathcal{S}$ is a 0-system in $V(G)$. Moreover, G is trivially r-exclusive with respect to $\mathcal{S}$ since $|\mathcal{S}| < 2$. Moreover, by (R1), $\mathcal{U}$ is a μ-focus for $\mathcal{S}$, and $(\mathcal{P}_{edge},\mathcal{P}_{clique})$ is an (r,q)-partition pair of $G,\mathcal{S},\mathcal{U}$ by (P6′) in Proposition 10.11. Note that (S4) follows immediately from (R4). In order to check (S2) and (S3), assume that $Y \subseteq G^{(q)}$ and $d \ge \xi$ are such that (R2) and (R3) hold. Clearly, all $e \in G^{(r)}$ are $\mathcal{S}$-important, and by (R3), we have for all $e \in G^{(r)}$ that $|G[Y]^{(q+r)}(e)[U]| \ge \xi(\mu n)^q$, so (S3) holds. Finally, we have seen in Example 10.13 that there exists a matrix $A \in [0,1]^{(r+1)\times(q+1)}$ with $\min^{\backslash\backslash r-i+1}(A) \ge \tilde{\mu}\xi$ such that $G[Y]$ is (ε,A,q,r)-regular with respect to $(\mathcal{P}_{edge},\mathcal{P}_{clique}[Y])$. □

The following lemma shows that we can (probabilistically) sparsify a given setup so that the resulting setup is diagonal-dominant.

Lemma 10.19. *Let $1/n \ll \varepsilon \ll \nu \ll \mu,\xi,1/q$ and $0 \le i < r < q$. Let $\xi' := \nu^{8^q\cdot q+1}$. Let G be a complex on n vertices and suppose that*

$$\mathcal{S},\mathcal{U},(\mathcal{P}_{edge},\mathcal{P}_{clique}) \textit{ form an } (\varepsilon,\mu,\xi,q,r,i)\textit{-setup for } G.$$

Then there exists a subgraph $H \subseteq G^{(r)}$ with $\Delta(H) \le 1.1\nu n$ such that for all $L \subseteq G^{(r)}$ with $\Delta(L) \le \varepsilon n$,

$$\mathcal{S},\mathcal{U},(\mathcal{P}_{edge},\mathcal{P}_{clique})[H \triangle L] \textit{ form a diagonal-dominant } (\sqrt{\varepsilon},\mu,\xi',q,r,i)\textit{-setup for } G[H \triangle L].$$

Proof. Let $Y \subseteq G^{(q)}$ and $A \in [0,1]^{(r+1)\times(q+1)}$ be such that (S1)–(S4) hold for G. Let $B\colon \mathcal{P}_{edge} \times \mathcal{P}_{clique} \to [\binom{q}{r}]_0$ be the containment function of $(\mathcal{P}_{edge},\mathcal{P}_{clique})$. We will write $b_{\ell,k} := B(\mathcal{P}_{edge}(\ell),\mathcal{P}_{clique}(k))$ for all $\ell \in [r+1]$ and $k \in [q+1]$. We may assume that $a_{\ell,k} = 0$ whenever $b_{\ell,k} = 0$ (and $\min^{\backslash\backslash r-i+1}(A) \ge \xi$ still holds).

Define the matrix $A' \in [0,1]^{(r+1)\times(q+1)}$ by letting $a'_{\ell,k} := a_{\ell,k}\nu^{-\ell}\prod_{\ell'\in[r+1]}\nu^{\ell' b_{\ell',k}}$. Note that we always have $a'_{\ell,k} \le a_{\ell,k}$.

Claim 1: A' is diagonal-dominant and $\min^{\backslash\backslash r-i+1}(A') \ge \xi'$.

Proof of claim: For $1 \leq \ell < k \leq r+1$,

$$\frac{a'_{\ell,k}}{a'_{k,k}} = \frac{a_{\ell,k}\nu^{-\ell}}{a_{k,k}\nu^{-k}} \leq \frac{\nu^{k-\ell}}{\xi} \leq \frac{1}{2(r+1-\ell)}.$$

Moreover, we have $\min^{\backslash\backslash r-i+1}(A') \geq \xi\nu^{(r+1)\binom{q}{r}-1} \geq \xi'$. –

We choose H randomly by including independently each $e \in \mathcal{P}_{edge}(\ell)$ with probability ν^{ℓ}, for all $\ell \in [r+1]$. Clearly, whp $\Delta(H) \leq 1.1\nu n$. Moreover, for any $L \subseteq G^{(r)}$, $G[H \triangle L]$ is r-exclusive with respect to $\mathcal{S}$ and $(\mathcal{P}_{edge}, \mathcal{P}_{clique})[H \triangle L]$ is an (r,q)-partition pair of $G[H \triangle L]$, $\mathcal{S}, \mathcal{U}$ by (P7′) in Proposition 10.11. Thus, (S1) holds.

We now consider (S2). Let $\ell \in [r+1]$, $k \in [q+1]$ and $e \in \mathcal{P}_{edge}(\ell)$. Define

$$\mathcal{Q}_{e,k} := (\mathcal{P}_{clique}[Y](k))(e).$$

By (10.7) and (S2) for $\mathcal{S}, \mathcal{U}, (\mathcal{P}_{edge}, \mathcal{P}_{clique})$, we have that $|\mathcal{Q}_{e,k}| = (a_{\ell,k} \pm \varepsilon)n^{q-r}$. We view $\mathcal{Q}_{e,k}$ as a $(q-r)$-graph and consider the random subgraph $\mathcal{Q}'_{e,k}$ that contains all $Q \in \mathcal{Q}_{e,k}$ with $\binom{Q\cup e}{r} \setminus \{e\} \subseteq H$. For all $Q \in \mathcal{Q}_{e,k}$, we have

$$\mathbb{P}(Q \in \mathcal{Q}'_{e,k}) = \nu^{-\ell} \prod_{\ell' \in [r+1]} \nu^{\ell' b_{\ell',k}} = \frac{a'_{\ell,k}}{a_{\ell,k}}.$$

Thus, $\mathbb{E}|\mathcal{Q}'_{e,k}| = (a'_{\ell,k} \pm \varepsilon)n^{q-r}$. Using Corollary 5.11 and a union bound, we thus conclude that with probability at least $1 - \mathrm{e}^{-n^{1/7}}$, we have $|\mathcal{Q}'_{e,k}| = (a'_{\ell,k} \pm \varepsilon^{2/3})n^{q-r}$ for all $\ell \in [r+1]$, $k \in [q+1]$ and $e \in \mathcal{P}_{edge}(\ell)$. (Technically, we can only apply Corollary 5.11 if $|\mathcal{Q}_{e,k}| \geq 2\varepsilon n^{q-r}$, say. Note that the result holds trivially if $|\mathcal{Q}_{e,k}| \leq 2\varepsilon n^{q-r}$.) Assuming that this holds for H, Proposition 5.4 implies that any $L \subseteq G^{(r)}$ with $\Delta(L) \leq \varepsilon n$ results in $G[H \triangle L][Y]$ being $(\sqrt{\varepsilon}, A', q, r)$-regular with respect to $(\mathcal{P}_{edge}, \mathcal{P}_{clique})[H \triangle L][Y]$.

We now check (S3). Let $e \in G^{(r)}$. If e is $\mathcal{S}$-unimportant then let $\mathcal{Q}_e$ be the set of all $Q \in G[Y]^{(q+r)}(e)$ such that $Q \cup e$ is $\mathcal{S}$-unimportant, otherwise let $\mathcal{Q}_e := G[Y]^{(q+r)}(e)[U_S]$. By (S3) for $\mathcal{S}, \mathcal{U}, (\mathcal{P}_{edge}, \mathcal{P}_{clique})$, we have that $|\mathcal{Q}_e| \geq \xi(\mu n)^q$. We view $\mathcal{Q}_e$ as a q-graph and consider the random subgraph $\mathcal{Q}'_e$ containing all $Q \in \mathcal{Q}_e$ such that $\binom{Q\cup e}{r} \setminus \{e\} \subseteq H$. For each $Q \in \mathcal{Q}_e$, we have

$$\mathbb{P}(Q \in \mathcal{Q}'_e) \geq \nu^{(r+1)\binom{q+r}{r}-1} \geq \nu^{q(4^q)},$$

thus $\mathbb{E}|\mathcal{Q}'_e| \geq \nu^{q(4^q)}\xi(\mu n)^q$. Using Corollary 5.11 and a union bound, we conclude that whp $|\mathcal{Q}'_e| \geq 2\xi'(\mu n)^q$ for all $e \in G^{(r)}$. Assuming that this holds for H, Proposition 5.4 implies that for any admissible choice of L, (S3) still holds.

Finally, we check (S4). Let $S \in \mathcal{S}$, $h \in [r-i]_0$ and $F \subseteq G(S)^{(h)}$ with $1 \leq |F| \leq 2^h$. By assumption, $G_{S,F} := \bigcap_{f \in F} G(S \cup f)[U_S]$ is an $(\varepsilon, \xi, q-i-h, r-i-h)$-complex. We intend to apply Proposition 5.15 with $i+h$, $G[U_S \cup S \cup \bigcup F]$, $\mathcal{P}_{edge}[G^{(r)}[U_S \cup S \cup \bigcup F]]$, $\{f \cup S : f \in F\}$, ν^{r+1}, $2\varepsilon/\mu$ playing the roles of $i, G, \mathcal{P}, F, p, \gamma$. Note that for every $f \in F$ and all $e \in G^{(r-i-h)}_{S,F}$, $S \cup f \cup e$ is $\mathcal{S}$-important and $\tau_r(S \cup f \cup e) = |(S \cup f \cup e) \cap U_S| = |f \cap U_S| + r - i - h$. Hence, $S \cup f \cup e \in \mathcal{P}_{edge}(|f \cap U_S| + r - h + 1)$. Thus, condition (I) in Proposition 5.15 is satisfied. Moreover, (II) is also satisfied because of (P5′) in Proposition 10.11. Therefore, by Proposition 5.15, with probability at least $1 - \mathrm{e}^{-|U_S|^{1/8}}$, for any $L \subseteq G^{(r)}$ with $\Delta(L) \leq \varepsilon n \leq 2\varepsilon|U_S|/\mu$, $\bigcap_{f \in F} G[H \triangle L](S \cup f)[U_S]$ is an $(\sqrt{\varepsilon}, \xi', q-i-h, r-i-h)$-complex. A union bound now shows that with probability at least $1 - \mathrm{e}^{-n^{1/10}}$, (S4) holds.

Thus, there exists H with the desired properties. □

We also need a similar (but simpler) result which 'sparsifies' the neighbourhood complexes of an i-system.

Lemma 10.20. *Let $1/n \ll \varepsilon \ll \mu, \beta, \xi, 1/q$ and $1 \le i < r < q$. Let $\xi' := 0.9\xi\beta^{(8^q)}$. Let G be a complex on n vertices and let $\mathcal{S}$ be an i-system in G such that G is r-exclusive with respect to $\mathcal{S}$. Let $\mathcal{U}$ be a μ-focus for $\mathcal{S}$. Suppose that*

$$G(S)[U_S] \text{ is an } (\varepsilon, \xi, q-i, r-i)\text{-supercomplex for every } S \in \mathcal{S}.$$

Then there exists a subgraph $H \subseteq G^{(r)}$ with $\Delta(H) \le 1.1\beta n$ such that for all $L \subseteq G^{(r)}$ with $\Delta(L) \le \varepsilon n$, the following holds for $G' := G[H \triangle L]$:

$$G'(S)[U_S] \text{ is a } (\sqrt{\varepsilon}, \xi', q-i, r-i)\text{-supercomplex for every } S \in \mathcal{S}.$$

Proof. Choose H randomly by including each $e \in G^{(r)}$ independently with probability β. Clearly, whp $\Delta(H) \le 1.1\beta n$. Now, consider $S \in \mathcal{S}$. Let $h \in [r-i]_0$ and $F \subseteq G(S)[U_S]^{(h)}$ with $1 \le |F| \le 2^h$. By assumption, $G_{S,F} := \bigcap_{f \in F} G(S)[U_S](f) = \bigcap_{f \in F} G(S \cup f)[U_S]$ is an $(\varepsilon, \xi, q-i-h, r-i-h)$-complex. Proposition 5.15 (applied with $G[U_S \cup S \cup \bigcup F] =: G_1, \{f \cup S : f \in F\}, i+h, \{G_1^{(r)}\}, \beta, 2\varepsilon/\mu$ playing the roles of $G, F, i, \mathcal{P}, p, \gamma$) implies that with probability at least $1 - \mathrm{e}^{-|U_S|^{1/8}}$, H has the property that for all $L \subseteq G^{(r)}$ with $\Delta(L) \le \varepsilon n \le 2\varepsilon |U_S|/\mu$, $\bigcap_{f \in F} G[H \triangle L](S \cup f)[U_S] = \bigcap_{f \in F} G'(S)[U_S](f)$ is a $(\sqrt{\varepsilon}, \xi', q-i-h, r-i-h)$-complex.

Therefore, applying a union bound to all $S \in \mathcal{S}$, $h \in [r-i]_0$ and $F \subseteq G(S)[U_S]^{(h)}$ with $1 \le |F| \le 2^h$, we conclude that whp H has the property that for all $L \subseteq G^{(r)}$ with $\Delta(L) \le \varepsilon n$, $G'(S)[U_S]$ is a $(\sqrt{\varepsilon}, \xi', q-i, r-i)$-supercomplex for every $S \in \mathcal{S}$. Thus, there exists an H with the desired properties. □

10.4. **Proof of the Cover down lemma.** In this subsection, we state and prove the Cover down lemma for setups and deduce the Cover down lemma.

Definition 10.21. Let G be an r-graph, let $\mathcal{S}$ be an i-system in $V(G)$, and let $\mathcal{U}$ be a focus for $\mathcal{S}$. We say that G is (q,r)-*divisible with respect to* $\mathcal{S}, \mathcal{U}$, if for all $S \in \mathcal{S}$ and all $f \subseteq V(G) \setminus S$ with $|f| \le r-i-1$ and $|f \setminus U_S| \ge 1$, we have $\binom{q-i-|f|}{r-i-|f|} \mid |G(S \cup f)|$.

Recall that a setup for G was defined in Definition 10.17 and density with respect to H in Definition 7.3. We will prove the Cover down lemma for setups by induction on $r-i$. We will deduce the Cover down lemma by applying this lemma with $i = 0$.

Lemma 10.22 (Cover down lemma for setups)**.** *Let $1/n \ll \gamma \ll \varepsilon \ll \nu \ll \mu, \xi, 1/q$ and $0 \le i < r < q$. Assume that $(*)_\ell$ is true for all $\ell \in [r-i-1]$. Let G be a complex on n vertices and suppose that $\mathcal{S}, \mathcal{U}, (\mathcal{P}_{edge}, \mathcal{P}_{clique})$ form an $(\varepsilon, \mu, \xi, q, r, i)$-setup for G. For $r' \ge r$, let $\tau_{r'}$ denote the type function of $G^{(r')}$, $\mathcal{S}$, $\mathcal{U}$. Then the following hold.*

- (i) *Let $\tilde{G}$ be a complex on $V(G)$ with $G \subseteq \tilde{G}$ such that $\tilde{G}$ is (ε, q, r)-dense with respect to $G^{(r)} - \tau_r^{-1}(0)$. Then there exists a subgraph $H^* \subseteq G^{(r)} - \tau_r^{-1}(0)$ with $\Delta(H^*) \le \nu n$ such that for any $L^* \subseteq \tilde{G}^{(r)}$ with $\Delta(L^*) \le \gamma n$ and $H^* \cup L^*$ being (q,r)-divisible with respect to $\mathcal{S}, \mathcal{U}$, there exists a $K_q^{(r)}$-packing in $\tilde{G}[H^* \cup L^*]$ which covers all edges of L^*, and all $\mathcal{S}$-important edges of H^* except possibly some from $\tau_r^{-1}(r-i)$.*
- (ii) *If $G^{(r)}$ is (q,r)-divisible with respect to $\mathcal{S}, \mathcal{U}$ and the setup is diagonal-dominant, then there exists a $K_q^{(r)}$-packing in G which covers all $\mathcal{S}$-important r-edges except possibly some from $\tau_r^{-1}(r-i)$.*

Before proving Lemma 10.22, we show how it implies the Cover down lemma (Lemma 7.4). Note that we only need part (i) of Lemma 10.22 to prove Lemma 7.4. (ii) is used in the inductive proof of Lemma 10.22 itself.

Proof of Lemma 7.4. Let $\mathcal{S} := \{\emptyset\}, \mathcal{U} := \{U\}$ and let $(\mathcal{P}_{edge}, \mathcal{P}_{clique})$ be the (r,q)-partition pair of G, U. By Proposition 10.18, $\mathcal{S}, \mathcal{U}, (\mathcal{P}_{edge}, \mathcal{P}_{clique})$ form a $(\varepsilon, \mu, \mu^{q-r}\xi, q, r, 0)$-setup for G. We can thus apply Lemma 10.22(i) with $\mu^{q-r}\xi$ playing the role of ξ. Recall that all r-edges of G are $\mathcal{S}$-important. Moreover, let τ_r denote the type function of $G^{(r)}, \mathcal{S}, \mathcal{U}$. We then have $\tau_r^{-1}(0) = G^{(r)}[\bar{U}]$ and $\tau_r^{-1}(r) = G^{(r)}[U]$, where $\bar{U} := V(G) \setminus U$. □

Proof of Lemma 10.22. The proof is by induction on $r-i$. For $i=r-1$, we will prove the statement directly (and Steps 1 and 2 below will be vacuous in this case). For $i<r-1$, we assume that the statement is true for $i' \in \{i+1,\dots,r-1\}$.

If $i<r-1$, choose new constants $\nu_1,\rho_1,\beta_1,\dots,\nu_{r-i-1},\rho_{r-i-1},\beta_{r-i-1}$ such that

$$1/n \ll \gamma \ll \varepsilon \ll \nu_1 \ll \rho_1 \ll \beta_1 \ll \dots \ll \nu_{r-i-1} \ll \rho_{r-i-1} \ll \beta_{r-i-1} \ll \nu \ll \mu,\xi,1/q.$$

Let $Y \subseteq G^{(q)}$ and $A \in [0,1]^{(r+1)\times(q+1)}$ be such that (S1)–(S4) hold. Moreover, write $\mathcal{S}=\{S_1,\dots,S_p\}$ and $U_j := U_{S_j}$ for all $j \in [p]$.

Convention: Throughout the proof, we will use the variables $\ell \in [r-i-1]$ and $i' \in \{i+1,\dots,r-1\}$ simultaneously, but always assume that $i'=r-\ell$.

Step 1: Defining i'-systems and partition pairs

As part of our inductive argument, instead of considering $\mathcal{S}$ directly, we will work with a suitable collection of i'-systems $\mathcal{T}^{(i')}$ for $i' \in \{i+1,\dots,r-1\}$. To this end, fix $i' \in \{i+1,\dots,r-1\}$. For every $j \in [p]$, let $\mathcal{S}_j^{i'}$ be the set of all $(i'-i)$-subsets S of $V(G)\setminus(U_j \cup S_j)$ with the property that $S_j \cup S \subseteq e$ for some $e \in G^{(r)}$. Define the collection

$$\mathcal{T}^{(i')} := \{S_j \cup S \,:\, j \in [p], S \in \mathcal{S}_j^{i'}\}.$$

Clearly, all elements of $\mathcal{T}^{(i')}$ have size i'. Moreover, note that since G is r-exclusive, all elements are distinct, that is, for every $T \in \mathcal{T}^{(i')}$ there is a unique $j \in [p]$ and a unique $S \in \mathcal{S}_j^{i'}$ such that $T = S_j \cup S$. Thus, $\mathcal{T}^{(i')}$ is an i'-system in G.

Let $\ell \in [r-i-1]$ and define

$$G_\ell := G - \{e \in G^{(r)} \,:\, e \text{ is } \mathcal{S}\text{-important and } \tau_r(e) < \ell\}.$$

So if $e \in G^{(r)}$ is $\mathcal{S}$-important and $\tau_r(e)=\ell$, then $e \in G_\ell^{(r)}$ and e is $\mathcal{T}^{(i')}$-important. Roughly speaking, H^* will consist of suitable subgraphs of $G_1,\dots,G_{r-i-1}$ and the edges of G_ℓ are relevant when covering the edges of $H^* \cup L^*$ of type ℓ.

Claim 1: G_ℓ is r-exclusive with respect to $\mathcal{T}^{(i')}$.

Proof of claim: Suppose, for a contradiction, that there is some $f \in G_\ell$ with $|f| \geq r$ and distinct $T,T' \in \mathcal{T}^{(i')}$ such that f contains both T and T'. By definition of $\mathcal{T}^{(i')}$, we have unique j,S,j',S' such that $T=S_j\cup S$ and $T'=S_{j'}\cup S'$. But since G is r-exclusive with respect to $\mathcal{S}$, we must have $j=j'$ and hence $S\cup S' \subseteq V(G)\setminus(U_j\cup S_j)$. Let e be a set obtained by including all vertices from S_j, choosing $i'-i+1$ vertices from $S\cup S'$ and choosing $r-i'-1$ other vertices from f. Hence, $e \in G_\ell^{(r)}$. But since $S_j \subseteq e$, e is $\mathcal{S}$-important. However, $\tau_r(e) = |e \cap U_j| \leq r-(i'+1) < \ell$, contradicting the definition of G_ℓ. –

Claim 2: Let $f \in G$ with $|f| \geq r$. Then we have

$$f \notin G_\ell \quad \Leftrightarrow \quad f \text{ is } \mathcal{S}\text{-important and } \tau_{|f|}(f) < |f| - i'.$$

Proof of claim: Indeed, let $\mathcal{E}_f$ be the set of $\mathcal{S}$-important r-sets in f. By definition of G_ℓ, we have $f \notin G_\ell$ if and only if f is $\mathcal{S}$-important, $\mathcal{E}_f \neq \emptyset$ and $\min_{e\in\mathcal{E}_f}\tau_r(e) < \ell$. Then Fact 10.4(ii) implies the claim. –

As a consequence, we have for each $r' \geq r$

$$G_\ell^{(r')} = G^{(r')} \setminus \bigcup_{k=0}^{r'-r+\ell-1} \tau_{r'}^{-1}(k). \tag{10.8}$$

Claim 3: For $r' \geq r$, the $\mathcal{T}^{(i')}$-important elements of $G_\ell^{(r')}$ are precisely the elements of $\tau_{r'}^{-1}(r'-r+\ell)$.

Proof of claim: Suppose first that $f \in G_\ell^{(r')}$ is $\mathcal{T}^{(i')}$-important. Clearly, we have $\tau_{r'}(f) \leq r'-i' = r'-r+\ell$. Also, since f must also be $\mathcal{S}$-important, but $f \in G_\ell$, Claim 2 implies that $\tau_{r'}(f) \geq r'-i'$. Hence, $f \in \tau_{r'}^{-1}(r'-r+\ell)$. Now, suppose that $f \in \tau_{r'}^{-1}(r'-r+\ell)$.

	$\mathcal{P}_{clique}([i])$			$\tau_q^{-1}(q-i)$	$\dots$	$\tau_q^{-1}(q-r+\ell+1)$	$\tau_q^{-1}(q-r+\ell)$
$\mathcal{P}_{edge}([i])$	*						
	0	*					
	0	0	*				
$\tau_r^{-1}(r-i)$		0		*			
$\dots$		0		0	*		
$\tau_r^{-1}(\ell+1)$		0		0	0	*	
$\tau_r^{-1}(\ell)$		0		0	0	0	*

FIGURE 2. The above table sketches the containment function of $(\mathcal{P}^{\ell*}_{edge} \sqcup \{\tau_r^{-1}(\ell)\}, \mathcal{P}^{\ell*}_{clique} \sqcup \{\tau_q^{-1}(q-r+\ell)\})$. Note that the shaded subtable corresponds to the shaded subtable in Figure 1, but has been flipped to make it upper-triangular instead of lower-triangular.

Since $\tau_{r'}(f) = r' - i'$, Claim 2 implies that $f \in G_\ell$. Let $j \in [p]$ be such that $S_j \subseteq f$. Define $S := f \setminus (U_j \cup S_j)$. We have $|S| = r' - \tau_{r'}(f) - i = i' - i$ and hence $S \in \mathcal{S}_j^{i'}$. Thus, f contains $S_j \cup S \in \mathcal{T}^{(i')}$. –

In what follows, we aim to obtain an (r,q)-partition pair for G_ℓ. Recall that every element of a class from $\mathcal{P}_{edge}([i])$ and $\mathcal{P}_{clique}([i])$ is $\mathcal{S}$-unimportant, and thus $\mathcal{T}^{(i')}$-unimportant as well. By (10.8) and Claim 3, the $\mathcal{T}^{(i')}$-unimportant r-sets of G_ℓ that are $\mathcal{S}$-important are precisely the elements of $\tau_r^{-1}(\ell+1), \dots, \tau_r^{-1}(r-i)$, and the $\mathcal{T}^{(i')}$-unimportant q-sets of G_ℓ that are $\mathcal{S}$-important are precisely the elements of $\tau_q^{-1}(q-r+\ell+1), \dots, \tau_q^{-1}(q-i)$. Thus, we aim to attach these classes to $\mathcal{P}_{edge}([i])$ and $\mathcal{P}_{clique}([i])$, respectively, in order to obtain partitions of the $\mathcal{T}^{(i')}$-unimportant r-sets and q-sets of G_ℓ. When doing so, we reverse their order. This will ensure that the new partition pair is again upper-triangular (cf. Figure 2).

Note that we can view $\mathcal{U}$ as a μ-focus for $\mathcal{T}^{(i')}$, by associating $T \in \mathcal{T}^{(i')}$ with U_j, where j is the unique $j \in [p]$ with $S_j \subseteq T$. Define

$$\mathcal{P}^{\ell*}_{edge} := \mathcal{P}_{edge}([i]) \sqcup (\tau_r^{-1}(r-i), \dots, \tau_r^{-1}(\ell+1)), \tag{10.9}$$

$$\mathcal{P}^{\ell*}_{clique} := \mathcal{P}_{clique}([i]) \sqcup (\tau_q^{-1}(q-i), \dots, \tau_q^{-1}(q-r+\ell+1)). \tag{10.10}$$

Claim 4: $(\mathcal{P}^{\ell*}_{edge}, \mathcal{P}^{\ell*}_{clique})$ is admissible with respect to G_ℓ, $\mathcal{T}^{(i')}$, $\mathcal{U}$.

Proof of claim: By (10.8) and Claim 3, we have that $\mathcal{P}^{\ell*}_{edge}$ is a partition of the $\mathcal{T}^{(i')}$-unimportant elements of $G_\ell^{(r)}$ and $\mathcal{P}^{\ell*}_{clique}$ is a partition of the $\mathcal{T}^{(i')}$-unimportant elements of $G_\ell^{(q)}$. Moreover, note that $|\mathcal{P}^{\ell*}_{edge}| = i + (r-i-\ell) = i'$ and $|\mathcal{P}^{\ell*}_{clique}| = i + (q-i) - (q-r+\ell) = i'$, so (P1) holds.

We proceed with checking (P3). By Claim 3, $\tau_r^{-1}(\ell)$ consists of all $\mathcal{T}^{(i')}$-important edges of $G_\ell^{(r)}$, and $\tau_q^{-1}(q-r+\ell)$ consists of all $\mathcal{T}^{(i')}$-important q-sets of $G_\ell^{(q)}$. Thus, $(\mathcal{P}^{\ell*}_{edge} \sqcup \{\tau_r^{-1}(\ell)\}, \mathcal{P}^{\ell*}_{clique} \sqcup \{\tau_q^{-1}(q-r+\ell)\})$ clearly is an (r,q)-partition pair of G_ℓ. If $0 \le k' < \ell' \le i'-i$, then no $Q \in \tau_q^{-1}(q-i-k')$ contains any element from $\tau_r^{-1}(r-i-\ell')$ by Fact 10.4(ii), so $(\mathcal{P}^{\ell*}_{edge} \sqcup \{\tau_r^{-1}(\ell)\}, \mathcal{P}^{\ell*}_{clique} \sqcup \{\tau_q^{-1}(q-r+\ell)\})$ is upper-triangular (cf. Figure 2).

It remains to check (P2). Let $T \in \mathcal{T}^{(i')}$, $h' \in [r-i']_0$ and $F' \subseteq G_\ell(T)^{(h')}$ with $1 \le |F'| \le 2^{h'}$. Thus $T = S_j \cup S$ for some unique $j \in [p]$. Let $h := h'+i'-i \in [r-i]_0$ and $F := \{S \cup f' \,:\, f' \in F'\}$. Clearly, $F \subseteq G(S_j)^{(h)}$ with $1 \le |F| \le 2^h$. Thus, by (P5′) in Proposition 10.11, we have for all $\mathcal{E} \in \mathcal{P}_{edge}$ that there exists $D(S_j, F, \mathcal{E}) \in \mathbb{N}_0$ such that for all $Q \in \bigcap_{f \in F} G(S_j \cup f)[U_j]^{(q-i-h)}$, we have that

$$|\{e \in \mathcal{E} \,:\, \exists f \in F \colon e \subseteq S_j \cup f \cup Q\}| = D(S_j, F, \mathcal{E}).$$

For each $\mathcal{E} \in \mathcal{P}^{\ell*}_{edge}$, define $D'(T, F', \mathcal{E}) := D(S_j, F, \mathcal{E})$. Thus, we have for all $Q \in \bigcap_{f' \in F'} G_\ell(T \cup f')[U_j]^{(q-i'-h')}$ that

$$|\{e \in \mathcal{E} \,:\, \exists f' \in F' \colon e \subseteq T \cup f' \cup Q\}| = D'(T, F', \mathcal{E}).$$

–

Step 2: Defining focuses

We will now define a focus $\mathcal{U}_{i'}$ for $\mathcal{T}^{(i')}$ for every $i' \in \{i+1,\dots,r-1\}$ (as noted above, $\mathcal{U}$ is also a focus for $\mathcal{T}^{(i')}$, but it is not suitable for our induction argument as the intersections $U_j \cap U_{j'}$ may be too large). Fix $i' \in \{i+1,\dots,r-1\}$. For each $T \in \mathcal{T}^{(i')}$, we will have $U_T \subseteq U_j$, where j is the unique $j \in [p]$ with $S_j \subseteq T$. In what follows, we already refer to the resulting type functions of $\mathcal{U}_{i'}$. The relevant claims do not depend on the specific choice of U_T, we only need to know that $U_T \subseteq U_j$.

First, note that this means that $(\mathcal{P}^{\ell*}_{edge}, \mathcal{P}^{\ell*}_{clique})$ will be admissible with respect to G_ℓ, $\mathcal{T}^{(i')}$, $\mathcal{U}_{i'}$. Recall that by Claim 1, G_ℓ is r-exclusive with respect to $\mathcal{T}^{(i')}$. For $r' \geq r$, let $\tau_{\ell,r'}$ denote the type function of $G_\ell^{(r')}$, $\mathcal{T}^{(i')}$, $\mathcal{U}_{i'}$. Let $(\mathcal{P}^{\ell}_{edge}, \mathcal{P}^{\ell}_{clique})$ be the (r,q)-partition pair of G_ℓ induced by $(\mathcal{P}^{\ell*}_{edge}, \mathcal{P}^{\ell*}_{clique})$ and $\mathcal{U}_{i'}$.

Define the matrix $A_\ell \in [0,1]^{(r+1)\times(q+1)}$ such that the following hold:

- For all $\mathcal{E} \in \mathcal{P}^{\ell*}_{edge}$ and $\mathcal{Q} \in \mathcal{P}^{\ell*}_{clique}$, let $A_\ell(\mathcal{E},\mathcal{Q}) := A(\mathcal{E},\mathcal{Q})$.
- For all $\ell' \in [r-i']_0$ and $\mathcal{Q} \in \mathcal{P}^{\ell*}_{clique}$, let $A_\ell(\tau^{-1}_{\ell,r}(\ell'),\mathcal{Q}) := 0$.
- For all $\mathcal{E} \in \mathcal{P}^{\ell*}_{edge}$ and $k' \in [q-i']_0$, define
$$A_\ell(\mathcal{E}, \tau^{-1}_{\ell,q}(k')) := bin(q-i', \rho_\ell, k') A(\mathcal{E}, \tau_q^{-1}(q-r+\ell)).$$
- For all $\ell' \in [r-i']_0$, $k' \in [q-i']_0$, let
$$A_\ell(\tau^{-1}_{\ell,r}(\ell'), \tau^{-1}_{\ell,q}(k')) := bin(q-r, \rho_\ell, k'-\ell') A(\tau_r^{-1}(\ell), \tau_q^{-1}(q-r+\ell)).$$

Note that $\min^{\backslash\backslash r-i'+1}(A_\ell) \geq \rho_\ell^{q-r}\xi$.

We will show that $\mathcal{U}_{i'}$ can be chosen as a (μ,ρ_ℓ,r)-focus for $\mathcal{T}^{(i')}$ such that

$$\mathcal{T}^{(i')}, \mathcal{U}_{i'}, (\mathcal{P}^{\ell}_{edge}, \mathcal{P}^{\ell}_{clique}) \text{ form a } (1.1\varepsilon, \rho_\ell\mu, \rho_\ell^{q-r}\xi, q, r, i')\text{-setup for } G_\ell, \tag{10.11}$$

$$G_\ell(T)[U_T] \text{ is a } (1.1\varepsilon, 0.9\xi, q-i', r-i')\text{-supercomplex for every } T \in \mathcal{T}^{(i')}. \tag{10.12}$$

For every $T \in \mathcal{T}^{(i')}$, choose a random subset U_T of U_j by including every vertex from U_j independently with probability ρ_ℓ, where j is the unique $j \in [p]$ with $S_j \subseteq T$. We claim that with positive probability, $\mathcal{U}_{i'} := \{U_T : T \in \mathcal{T}^{(i')}\}$ is the desired focus.

By Lemma 10.6 whp $\mathcal{U}_{i'}$ is a (μ,ρ_ℓ,r)-focus for $\mathcal{T}^{(i')}$. In particular, whp $\mathcal{U}_{i'}$ is a $\rho_\ell\mu$-focus for $\mathcal{T}^{(i')}$, implying that (S1) holds for G_ℓ with $\mathcal{T}^{(i')}$, $\mathcal{U}_{i'}$ and $(\mathcal{P}^{\ell}_{edge}, \mathcal{P}^{\ell}_{clique})$. We now check (S2)–(S4).

Claim 5: Whp $G_\ell[Y]$ is $(1.1\varepsilon, A_\ell, q, r)$-regular with respect to $(\mathcal{P}^{\ell}_{edge}, \mathcal{P}^{\ell}_{clique})[Y]$.

Proof of claim: For $\mathcal{E} \in \mathcal{P}_{edge}$ and $\mathcal{Q} \in \mathcal{P}_{clique}$, we write $A(\mathcal{E}, \mathcal{Q}\cap Y) := A(\mathcal{E},\mathcal{Q})$, and similarly for A_ℓ. By definition of $(\mathcal{P}^{\ell*}_{edge}, \mathcal{P}^{\ell*}_{clique})$, we have for all $\mathcal{E} \in \mathcal{P}^{\ell*}_{edge} \sqcup \{\tau_r^{-1}(\ell)\}$ and $\mathcal{Q} \in (\mathcal{P}^{\ell*}_{clique} \sqcup \{\tau_q^{-1}(q-r+\ell)\})[Y]$ that $\mathcal{E} \in \mathcal{P}_{edge}$ and $\mathcal{Q} \in \mathcal{P}_{clique}[Y]$. Since $G[Y]$ is (ε, A, q, r)-regular with respect to $(\mathcal{P}_{edge}, \mathcal{P}_{clique})[Y]$, we have thus for all $e \in \mathcal{E}$ that

$$|\mathcal{Q}(e)| = (A(\mathcal{E},\mathcal{Q}) \pm \varepsilon) n^{q-r}. \tag{10.13}$$

We have to show that for all $\mathcal{E} \in \mathcal{P}^{\ell}_{edge}$, $\mathcal{Q} \in \mathcal{P}^{\ell}_{clique}[Y]$ and $e \in \mathcal{E}$, we have $|\mathcal{Q}(e)| = (A_\ell(\mathcal{E},\mathcal{Q}) \pm 1.1\varepsilon) n^{q-r}$. We distinguish four cases as in the definition of A_ℓ.

Firstly, for all $\mathcal{E} \in \mathcal{P}^{\ell*}_{edge}$, $\mathcal{Q} \in \mathcal{P}^{\ell*}_{clique}[Y]$ and $e \in \mathcal{E}$, we have by (10.13) that $|\mathcal{Q}(e)| = (A(\mathcal{E},\mathcal{Q}) \pm \varepsilon) n^{q-r} = (A_\ell(\mathcal{E},\mathcal{Q}) \pm \varepsilon) n^{q-r}$ with probability 1.

Also, for all $\ell' \in [r-i']_0$, $\mathcal{Q} \in \mathcal{P}^{\ell*}_{clique}[Y]$ and $e \in \tau^{-1}_{\ell,r}(\ell')$, we have $|\mathcal{Q}(e)| = 0 = A_\ell(\tau^{-1}_{\ell,r}(\ell'),\mathcal{Q}) n^{q-r}$ with probability 1.

Let $\mathcal{E} \in \mathcal{P}^{\ell*}_{edge} \sqcup \{\tau_r^{-1}(\ell)\}$ and consider $e \in \mathcal{E}$. Let $\mathcal{Q}_e := (Y \cap \tau_q^{-1}(q-r+\ell))(e)$. By (10.13), we have that $|\mathcal{Q}_e| = (A(\mathcal{E}, \tau_q^{-1}(q-r+\ell)) \pm \varepsilon) n^{q-r}$.

First, assume that $e \in \mathcal{E} \in \mathcal{P}^{\ell*}_{edge}$. For each $k' \in [q-i']_0$, we consider the random subgraph $\mathcal{Q}_e^{k'}$ of $\mathcal{Q}_e$ that contains all $Q \in \mathcal{Q}_e$ with $Q \cup e \in \tau^{-1}_{\ell,q}(k')$. Hence, $\mathcal{Q}_e^{k'} = (Y \cap \tau^{-1}_{\ell,q}(k'))(e)$.

For each $Q \in \mathcal{Q}_e$, there are unique $T_Q \in \mathcal{T}^{(i')}$ and $j_Q \in [p]$ with $S_{j_Q} \subseteq T_Q \subseteq Q \cup e$ and $(Q \cup e) \setminus T_Q \subseteq U_{j_Q}$.

For each $Q \in \mathcal{Q}_e$, we then have

$$\mathbb{P}(Q \in \mathcal{Q}_e^{k'}) = \mathbb{P}(\tau_{\ell,q}(Q \cup e) = k') = \mathbb{P}(|(Q \cup e) \cap U_{T_Q}| = k') = bin(q - i', \rho_\ell, k').$$

Thus, $\mathbb{E}|\mathcal{Q}_e^{k'}| = bin(q - i', \rho_\ell, k')|\mathcal{Q}_e|$. For each $T \in \mathcal{T}^{(i')}$, let $\mathcal{Q}_T$ be the set of all those $Q \in \mathcal{Q}_e$ for which $T_Q = T$. Since e is $\mathcal{T}^{(i')}$-unimportant, we have $|T \setminus e| > 0$ and thus $|\mathcal{Q}_T| \le n^{q-r-1}$ for all $T \in \mathcal{T}^{(i')}$. Thus we can partition $\mathcal{Q}_e$ into n^{q-r-1} subgraphs such that each of them intersects each $\mathcal{Q}_T$ in at most one element. For all Q lying in the same subgraph, the events $Q \in \mathcal{Q}_e^{k'}$ are now independent. Hence, by Lemma 5.9, we conclude that with probability at least $1 - \mathrm{e}^{-n^{1/6}}$ we have that

$$\begin{aligned} |\mathcal{Q}_e^{k'}| &= (1 \pm \varepsilon^2)\mathbb{E}|\mathcal{Q}_e^{k'}| = (1 \pm \varepsilon^2) bin(q - i', \rho_\ell, k')|\mathcal{Q}_e| \\ &= (1 \pm \varepsilon^2) bin(q - i', \rho_\ell, k')(A(\mathcal{E}, \tau_q^{-1}(q - r + \ell)) \pm \varepsilon)n^{q-r} \\ &= (A_\ell(\mathcal{E}, \tau_{\ell,q}^{-1}(k')) \pm 1.1\varepsilon)n^{q-r}. \end{aligned} \tag{10.14}$$

(Technically, we can only apply Lemma 5.9 if $|\mathcal{Q}_e| \ge 0.1\varepsilon n^{q-r}$, say. Note that (10.14) holds trivially if $|\mathcal{Q}_e| \le 0.1\varepsilon n^{q-r}$.)

Finally, consider the case $e \in \mathcal{E} = \tau_r^{-1}(\ell)$. By Claim 3, e is $\mathcal{T}^{(i')}$-important, so let $T \in \mathcal{T}^{(i')}$ be such that $T \subseteq e$. Note that for every $Q \in \mathcal{Q}_e$, we have $Q \subseteq U_j$, where j is the unique $j \in [p]$ with $S_j \subseteq T$. For every $x \in [q - r]_0$, let $\mathcal{Q}_e^x$ be the random subgraph of $\mathcal{Q}_e$ that contains all $Q \in \mathcal{Q}_e$ with $|Q \cap U_T| = x$. By the random choice of U_T, for each $Q \in \mathcal{Q}$ and $x \in [q - r]_0$, we have

$$\mathbb{P}(Q \in \mathcal{Q}_e^x) = bin(q - r, \rho_\ell, x).$$

Using Corollary 5.11 we conclude that for $x \in [q - r]_0$, with probability at least $1 - \mathrm{e}^{-n^{1/6}}$ we have that

$$\begin{aligned} |\mathcal{Q}_e^x| &= (1 \pm \varepsilon^2)\mathbb{E}|\mathcal{Q}_e^x| = (1 \pm \varepsilon^2) bin(q - r, \rho_\ell, x)|\mathcal{Q}_e| \\ &= (1 \pm \varepsilon^2) bin(q - r, \rho_\ell, x)(A(\tau_r^{-1}(\ell), \tau_q^{-1}(q - r + \ell)) \pm \varepsilon)n^{q-r} \\ &= (bin(q - r, \rho_\ell, x)A(\tau_r^{-1}(\ell), \tau_q^{-1}(q - r + \ell)) \pm 1.1\varepsilon)n^{q-r}. \end{aligned}$$

Thus for all $\ell' \in [r - i']_0$, $k' \in [q - i']_0$ and $e \in \tau_{\ell,r}^{-1}(\ell')$ with $k' \ge \ell'$, with probability at least $1 - \mathrm{e}^{-n^{1/6}}$ we have

$$|(Y \cap \tau_{\ell,q}^{-1}(k'))(e)| = |\mathcal{Q}_e^{k'-\ell'}| = (A_\ell(\tau_{\ell,r}^{-1}(\ell'), \tau_{\ell,q}^{-1}(k')) \pm 1.1\varepsilon)n^{q-r},$$

and if $\ell' > k'$ then trivially $|(Y \cap \tau_{\ell,q}^{-1}(k'))(e)| = 0 = A_\ell(\tau_{\ell,r}^{-1}(\ell'), \tau_{\ell,q}^{-1}(k'))n^{q-r}$. Thus, a union bound implies the claim. —

Claim 6: Whp every $\mathcal{T}^{(i')}$-unimportant $e \in G_\ell^{(r)}$ is contained in at least $0.9\xi(\rho_\ell \mu n)^q$ $\mathcal{T}^{(i')}$-unimportant $Q \in G_\ell[Y]^{(q+r)}$, and for every $\mathcal{T}^{(i')}$-important $e \in G_\ell^{(r)}$ with $e \supseteq T \in \mathcal{T}^{(i')}$, we have $|G_\ell[Y]^{(q+r)}(e)[U_T]| \ge 0.9\xi(\rho_\ell \mu n)^q$.

Proof of claim: Let $e \in G_\ell^{(r)}$ be $\mathcal{T}^{(i')}$-unimportant. By Claim 3, we thus have that e is $\mathcal{S}$-unimportant or $\tau_r(e) > \ell$. In the first case, we have that e is contained in at least $\xi(\mu n)^q$ $\mathcal{S}$-unimportant $Q \in G[Y]^{(q+r)}$ by (S3) for $\mathcal{U}, G, \mathcal{S}$. But each such Q is clearly $\mathcal{T}^{(i')}$-unimportant as well and contained in $G_\ell[Y]$. If the second case applies, assume that e contains $S_j \in \mathcal{S}$. By (S3) for $\mathcal{U}, G, \mathcal{S}$, we have that $|G[Y]^{(q+r)}(e)[U_j]| \ge \xi(\mu n)^q$. For every $Q \in G[Y]^{(q+r)}(e)[U_j]$, we have that $\tau_{q+r}(Q \cup e) = |(Q \cup e) \cap U_j| = q + \tau_r(e) > q + \ell$. Thus, Claim 2 implies that $Q \cup e \in G_\ell[Y]$, and by Claim 3 we have that $Q \cup e$ is $\mathcal{T}^{(i')}$-unimportant. Altogether, every $\mathcal{T}^{(i')}$-unimportant edge $e \in G_\ell^{(r)}$ is contained in at least $\xi(\mu n)^q \ge 0.9\xi(\rho_\ell \mu n)^q$ $\mathcal{T}^{(i')}$-unimportant $Q \in G_\ell[Y]^{(q+r)}$.

Let $e \in G_\ell^{(r)}$ be $\mathcal{T}^{(i')}$-important. Assume that e contains $T \in \mathcal{T}^{(i')}$ and let j be the unique $j \in [p]$ with $S_j \subseteq T$. By (S3) for $\mathcal{U}, G, \mathcal{S}$, we have that $|G[Y]^{(q+r)}(e)[U_j]| \ge \xi(\mu n)^q$. As before, for every $Q \in G[Y]^{(q+r)}(e)[U_j]$, we have $Q \cup e \in G_\ell[Y]$. Moreover, $\mathbb{P}(Q \subseteq U_T) = \rho_\ell^q$. Thus, by Corollary 5.11, with probability at least $1-\mathrm{e}^{-n^{1/6}}$ we have that $|G_\ell[Y]^{(q+r)}(e)[U_T]| \geq 0.9\xi(\rho_\ell\mu n)^q$. A union bound hence implies the claim. –

Claim 7: Whp for all $T \in \mathcal{T}^{(i')}$, $h' \in [r-i']_0$ and $F' \subseteq G_\ell(T)^{(h')}$ with $1 \leq |F'| \leq 2^{h'}$ we have that $\bigcap_{f' \in F'} G_\ell(T \cup f')[U_T]$ is an $(1.1\varepsilon, 0.9\xi, q-i'-h', r-i'-h')$-complex.

Proof of claim: Let $T \in \mathcal{T}^{(i')}$, $h' \in [r-i']_0$ and $F' \subseteq G_\ell(T)^{(h')}$ with $1 \leq |F'| \leq 2^{h'}$. Let j be the unique $j \in [p]$ with $T = S_j \cup S$. We claim that

$$\bigcap_{f' \in F'} G_\ell(T \cup f')[U_j] \text{ is an } (\varepsilon, \xi, q-i'-h', r-i'-h')\text{-complex.} \tag{10.15}$$

If $\bigcap_{f' \in F'} G_\ell(T \cup f')[U_j]^{(r-i'-h')}$ is empty, then there is nothing to prove, thus assume the contrary. We claim that we must have $f' \subseteq U_j$ for all $f' \in F'$. Indeed, let $f' \in F'$ and $g_0 \in G_\ell(T \cup f')[U_j]^{(r-i'-h')}$. Hence, $g_0 \cup T \cup f' \in G_\ell^{(r)}$. By Claim 2, we must have $|(g_0 \cup T \cup f') \cap U_j| \geq |g_0 \cup T \cup f'| - i'$. But since $T \cap U_j = \emptyset$, we must have $f' \subseteq U_j$.

Let $h := h' + i' - i \in [r-i]_0$ and $F := \{S \cup f' : f' \in F'\} \subseteq G(S_j)^{(h)}$. (S4) for $\mathcal{U}, G, \mathcal{S}$ implies that $\bigcap_{f \in F} G(S_j \cup f)[U_j]$ is an $(\varepsilon, \xi, q-i-h, r-i-h)$-complex. It thus suffices to show that $G(S_j \cup S \cup f')[U_j]^{(r')} = G_\ell(T \cup f')[U_j]^{(r')}$ for all $r' \geq r-i-h$ and $f' \in F'$. To this end, let $f' \in F'$, $r' \geq r-i-h$ and suppose that $g \in G(S_j \cup S \cup f')[U_j]^{(r')}$. Observe that $|(g \cup T \cup f') \cap U_j| = |g \cup T \cup f'| - i'$, so Claim 2 implies that $g \cup T \cup f' \in G_\ell$ and thus $g \in G_\ell(T \cup f')[U_j]^{(r')}$. This proves (10.15).

By Proposition 5.13, with probability at least $1 - \mathrm{e}^{-|U_j|/8}$, $\bigcap_{f' \in F'} G_\ell(T \cup f')[U_T]$ is an $(1.1\varepsilon, 0.9\xi, q-i'-h', r-i'-h')$-complex.

Applying a union bound to all $T \in \mathcal{T}^{(i')}$, $h' \in [r-i']_0$ and $F' \subseteq G_\ell(T)^{(h')}$ with $1 \leq |F'| \leq 2^{h'}$ then establishes the claim. –

By the above claims, $\mathcal{U}_{i'}$ satisfies (10.11) whp. Moreover, Claim 7 implies that whp (10.12) holds. Thus, there exists a (μ, ρ_ℓ, r)-focus $\mathcal{U}_{i'}$ for $\mathcal{T}^{(i')}$ such that (10.11) and (10.12) hold.

Step 3: Reserving subgraphs

In this step, we will find a number of subgraphs of $G^{(r)} - \tau_r^{-1}(0)$ whose union will be the r-graph H^* we seek in (i). Let $\tilde{G}$ be a complex as specified in (i). Let $\beta_0 := \varepsilon$. Let H_0 be a subgraph of $G^{(r)} - \tau_r^{-1}(0)$ with $\Delta(H_0) \leq 1.1\beta_0 n$ such that for all $e \in \tilde{G}^{(r)}$, we have

$$|\tilde{G}[H_0 \cup \{e\}]^{(q)}(e)| \geq 0.9\beta_0^{\binom{q}{r}} n^{q-r}. \tag{10.16}$$

(H_0 will be used to greedily cover L^*.) That such a subgraph exists can be seen by a probabilistic argument: let H_0 be obtained by including every edge of $G^{(r)} - \tau_r^{-1}(0)$ with probability β_0. Clearly, whp $\Delta(H_0) \leq 1.1\beta_0 n$. Also, since $\tilde{G}$ is (ε, q, r)-dense with respect to $G^{(r)} - \tau_r^{-1}(0)$ by assumption, we have for all $e \in \tilde{G}^{(r)}$ that

$$\mathbb{E}|\tilde{G}[H_0 \cup \{e\}]^{(q)}(e)| = \beta_0^{\binom{q}{r}-1} |\tilde{G}[(G^{(r)} - \tau_r^{-1}(0)) \cup \{e\}]^{(q)}(e)| \geq \beta_0^{\binom{q}{r}-1} \varepsilon n^{q-r}.$$

Using Corollary 5.11 and a union bound, it is then easy to see that whp H_0 satisfies (10.16) for all $e \in \tilde{G}^{(r)}$.

Step 3.1: Defining 'sparse' induction graphs H_ℓ.

Consider $\ell \in [r-i-1]$. Let $\xi_\ell := \nu_\ell^{8^q \cdot q+1}$. By (10.11) and Lemma 10.19 (with $3\beta_{\ell-1}, \nu_\ell, \rho_\ell\mu, \rho_\ell^{q-r}\xi, i'$ playing the roles of $\varepsilon, \nu, \mu, \xi, i$), there exists a subgraph $H_\ell \subseteq G_\ell^{(r)}$ with $\Delta(H_\ell) \leq 1.1\nu_\ell n$ such that for all $L \subseteq G_\ell^{(r)}$ with $\Delta(L) \leq 3\beta_{\ell-1} n$, we have that

$$\begin{aligned} &\mathcal{T}^{(i')}, \mathcal{U}_{i'}, (\mathcal{P}^\ell_{edge}, \mathcal{P}^\ell_{clique})[H_\ell \triangle L] \text{ form a diagonal-dominant} \\ &(\sqrt{3\beta_{\ell-1}}, \rho_\ell\mu, \xi_\ell, q, r, i')\text{-setup for } G_\ell[H_\ell \triangle L]. \end{aligned} \tag{10.17}$$

Step 3.2: Defining 'localised' cleaning graphs J_ℓ.

Let

$$G^*_\ell := G_\ell - \bigcup_{\ell'=0}^{\ell-1} \tau_{\ell,r}^{-1}(\ell'). \tag{10.18}$$

By (10.12), for every $T \in \mathcal{T}^{(i')}$, $G_\ell(T)[U_T]$ is a $(1.1\varepsilon, 0.9\xi, q-i', r-i')$-supercomplex. Note that $G_\ell(T)[U_T] = G^*_\ell(T)[U_T]$. Thus, by Lemma 10.20 (with $G^*_\ell, 3\nu_\ell, \rho_\ell\mu, \beta_\ell, 0.9\xi$ playing the roles of $G, \varepsilon, \mu, \beta, \xi$), there exists a subgraph $J_\ell \subseteq G^{*(r)}_\ell$ with $\Delta(J_\ell) \le 1.1\beta_\ell n$ such that for all $L \subseteq G^{*(r)}_\ell$ with $\Delta(L) \le 3\nu_\ell n$, the following holds for $G^* := G^*_\ell[J_\ell \triangle L]$:

(10.19) $G^*(T)[U_T]$ is a $(\sqrt{3\nu_\ell}, 0.81\xi\beta_\ell^{(8^q)}, q-i', r-i')$-supercomplex for every $T \in \mathcal{T}^{(i')}$.

We have defined subgraphs $H_0, H_1, \dots, H_{r-i-1}, J_1, \dots, J_{r-i-1}$ of $G^{(r)} - \tau_r^{-1}(0)$. Note that they are not necessarily edge-disjoint. Let $H^*_0 := H_0$ and for $\ell \in [r-i-1]$ define inductively

$$\begin{aligned} H'_\ell &:= H^*_{\ell-1} \cup H_\ell, \\ H^*_\ell &:= H^*_{\ell-1} \cup H_\ell \cup J_\ell = H'_\ell \cup J_\ell, \\ H^* &:= H^*_{r-i-1}. \end{aligned}$$

Clearly, $\Delta(H^*_\ell) \le 2\beta_\ell n$ for all $\ell \in [r-i-1]_0$ and $\Delta(H'_\ell) \le 2\nu_\ell n$ for all $\ell \in [r-i-1]$. In particular, $\Delta(H^*) \le 2\beta_{r-i-1} n \le \nu n$, as desired.

Step 4: Covering down — Proof of (i)

Let L^* be any subgraph of $\tilde{G}^{(r)}$ with $\Delta(L^*) \le \gamma n$ such that $H^* \cup L^*$ is (q,r)-divisible with respect to $\mathcal{S}, \mathcal{U}$. We need to find a $K_q^{(r)}$-packing $\mathcal{K}$ in $\tilde{G}[H^* \cup L^*]$ which covers all edges of L^*, and covers all $\mathcal{S}$-important edges of H^* except possibly some from $\tau_r^{-1}(r-i)$.

Let $H'_0 := H_0 \cup L^*$. By (10.16), for all $e \in L^*$ we have that

$$|\tilde{G}[H'_0]^{(q)}(e)| \ge |\tilde{G}[H_0 \cup e]^{(q)}(e)| \ge 0.9\beta_0^{\binom{q}{r}} n^{q-r}.$$

By Corollary 6.7, there is a $K_q^{(r)}$-packing $\mathcal{K}^*_0$ in $\tilde{G}[H'_0]$ covering all edges of L^*. If $i = r-1$, we can take $\mathcal{K}^*_0$ and complete the proof of (i). So assume that $i < r-1$ and that Lemma 10.22 holds for larger values of i.

We will now inductively show that the following holds for all $\ell \in [r-i]$.

$(\#)_\ell$ There exists a $K_q^{(r)}$-packing $\mathcal{K}^*_{\ell-1}$ in $\tilde{G}[H^*_{\ell-1} \cup L^*]$ covering all edges of L^*, and all $\mathcal{S}$-important $e \in H^*_{\ell-1}$ with $\tau_r(e) < \ell$.

By the above, $(\#)_1$ is true. Clearly, $(\#)_{r-i}$ establishes (i). Suppose that for some $\ell \in [r-i-1]$, $(\#)_\ell$ is true. We will now find a $K_q^{(r)}$-packing $\mathcal{K}_\ell$ in $\tilde{G}[H^*_\ell \cup L^*] - \mathcal{K}^{*(r)}_{\ell-1}$ such that $\mathcal{K}^*_\ell := \mathcal{K}^*_{\ell-1} \cup \mathcal{K}_\ell$ covers all edges of L^* and all $\mathcal{S}$-important $e \in H^*_\ell$ with $\tau_r(e) \le \ell$, implying that $(\#)_{\ell+1}$ is true.

Let $H''_\ell := H'_\ell - \mathcal{K}^{*(r)}_{\ell-1}$. The packing $\mathcal{K}_\ell$ must cover all edges of $H^*_\ell - \mathcal{K}^{*(r)}_{\ell-1}$ that belong to $\tau_r^{-1}(\ell)$. By Claim 3, all those edges are $\mathcal{T}^{(i')}$-important. We will obtain $\mathcal{K}_\ell$ as the union of $\mathcal{K}^\circ_\ell$ and $\mathcal{K}^\dagger_\ell$, where

(a) $\mathcal{K}^\circ_\ell$ covers all $\mathcal{T}^{(i')}$-important edges of H''_ℓ except possibly some from $\tau_{\ell,r}^{-1}(r-i')$;

(b) $\mathcal{K}^\dagger_\ell$ covers the remaining $\mathcal{T}^{(i')}$-important edges of H^*_ℓ.

We will obtain $\mathcal{K}^\circ_\ell$ by induction and $\mathcal{K}^\dagger_\ell$ by an application of the Localised cover down lemma (Lemma 10.7). Recall that (q,r)-divisibility with respect to $\mathcal{T}^{(i')}, \mathcal{U}_{i'}$ was defined in Definition 10.21.

Claim 8: H''_ℓ is (q,r)-divisible with respect to $\mathcal{T}^{(i')}, \mathcal{U}_{i'}$.

Proof of claim: Let $T \in \mathcal{T}^{(i')}$ and $f' \subseteq V(G) \setminus T$ with $|f'| \le r - i' - 1$ and $|f' \setminus U_T| \ge 1$. We have to show that $\binom{q-i'-|f'|}{r-i'-|f'|} \mid |H''_\ell(T \cup f')|$. Let $j \in [p]$ and $S \in \mathcal{S}_j^{i'}$ with $T = S_j \cup S$. Define $f := f' \cup S$. Clearly, $f \subseteq V(G) \setminus S_j$, $|f| \le r - i - 1$ and $|f \setminus U_j| \ge |S| \ge 1$. Hence, since $H^* \cup L^*$ is (q,r)-divisible with respect to $\mathcal{S}, \mathcal{U}$, we have $\binom{q-i-|f|}{r-i-|f|} \mid |(H^* \cup L^*)(S_j \cup f)|$. Clearly, we have $H''_\ell \subseteq H^* - \mathcal{K}^{*(r)}_{\ell-1}$. Conversely, observe that every $e \in H^* \cup L^*$ that contains $T \cup f'$ and is not covered by $\mathcal{K}^*_{\ell-1}$ must belong to H''_ℓ. Indeed, since e contains T, we have that $\tau_r(e) \le r - i' = \ell$, so $e \in H^*_\ell$. Moreover, by $(\#)_\ell$ we must have $\tau_r(e) \ge \ell$. Hence, $\tau_r(e) = \ell$. But since $|f' \setminus U_T| \ge 1$, we have $\tau_{\ell,r}(e) < \ell$. By (10.18), $e \notin J_\ell$. Thus, $e \in H'_\ell - \mathcal{K}^{*(r)}_{\ell-1} = H''_\ell$. Hence, $H''_\ell(T \cup f') = ((H^* \cup L^*) - \mathcal{K}^{*(r)}_{\ell-1})(S_j \cup f)$. This implies the claim. –

Let $L'_\ell := H''_\ell \triangle H_\ell$. So $H''_\ell = H_\ell \triangle L'_\ell$.

Claim 9: $L'_\ell \subseteq G^{(r)}_\ell$ *and* $\Delta(L'_\ell) \le 3\beta_{\ell-1}n$.

Proof of claim: Suppose, for a contradiction, that there is $e \in H''_\ell \triangle H_\ell$ with $e \notin G^{(r)}_\ell$. Since $H_\ell \subseteq G^{(r)}_\ell$, we must have $e \in H''_\ell$. Thus e is not covered by $\mathcal{K}^*_{\ell-1}$, implying that $\tau(e) \ge \ell$ or e is $\mathcal{S}$-unimportant, contradicting $e \notin G^{(r)}_\ell$ (cf. Claim 2).

In order to see the second part, observe that $L'_\ell = ((H^*_{\ell-1} \cup H_\ell) - \mathcal{K}^{*(r)}_{\ell-1}) \triangle H_\ell \subseteq H^*_{\ell-1} \cup L^*$ since $\mathcal{K}^{*(r)}_{\ell-1} \subseteq L^* \cup H^*_{\ell-1}$. Thus, $\Delta(L'_\ell) \le \Delta(H^*_{\ell-1}) + \Delta(L^*) \le 3\beta_{\ell-1}n$. –

Hence, by (10.17), $\mathcal{T}^{(i')}, \mathcal{U}_{i'}, (\mathcal{P}^\ell_{edge}, \mathcal{P}^\ell_{clique})[H''_\ell]$ form a diagonal-dominant $(\sqrt{3\beta_{\ell-1}}, \rho_\ell\mu, \xi_\ell, q, r, i')$-setup for $G[H''_\ell] = G_\ell[H''_\ell]$.

We can thus apply Lemma 10.22(ii) inductively with the following objects/parameters.

object/parameter	$G[H''_\ell]$	$\sqrt{3\beta_{\ell-1}}$	$\rho_\ell\mu$	ξ_ℓ	i'	$\mathcal{T}^{(i')}$	$\mathcal{U}_{i'}$	$(\mathcal{P}^\ell_{edge}, \mathcal{P}^\ell_{clique})[H''_\ell]$
playing the role of	G	ε	μ	ξ	i	$\mathcal{S}$	$\mathcal{U}$	$(\mathcal{P}_{edge}, \mathcal{P}_{clique})$

Hence, there exists a $K^{(r)}_q$-packing $\mathcal{K}^\circ_\ell$ in $G[H''_\ell]$ covering all $\mathcal{T}^{(i')}$-important $e \in H''_\ell$ except possibly some from $\tau^{-1}_{\ell,r}(r - i') = \tau^{-1}_{\ell,r}(\ell)$. Thus $\mathcal{K}^\circ_\ell$ is as required in (a).

We will now use J_ℓ to cover these remaining edges of H''_ℓ. Let $J'_\ell := H^*_\ell - \mathcal{K}^{*(r)}_{\ell-1} - \mathcal{K}^{\circ(r)}_\ell$.

Claim 10: $J'_\ell(T)[U_T]$ *is* $K^{(r-i')}_{q-i'}$*-divisible for every* $T \in \mathcal{T}^{(i')}$.

Proof of claim: Let $T \in \mathcal{T}^{(i')}$ and $j \in [p]$ and $S \in \mathcal{S}^{i'}_j$ be such that $T = S_j \cup S$. Let $f' \subseteq U_T$ with $|f'| \le r - i' - 1$. We have to show that $\binom{q-i'-|f'|}{r-i'-|f'|} \mid |J'_\ell(T)[U_T](f')|$. Note that for every $e \in J'_\ell \subseteq G^{*(r)}_\ell$ containing T, we have $\tau_{\ell,r}(e) = r - i'$. Thus, $J'_\ell(T)[U_T]$ is identical with $J'_\ell(T)$ except for the different vertex sets. It is thus sufficient to show that $\binom{q-i'-|f'|}{r-i'-|f'|} \mid |J'_\ell(T \cup f')|$. Let $f := f' \cup S$. Since $H^* \cup L^*$ is (q,r)-divisible with respect to $\mathcal{S}, \mathcal{U}$ and $|f \setminus U_j| \ge |S| \ge 1$, we have that $\binom{q-i-|f|}{r-i-|f|} \mid |(H^* \cup L^*)(S_j \cup f)|$. It is thus sufficient to prove that $J'_\ell(T \cup f') = ((H^* \cup L^*) - \mathcal{K}^{*(r)}_{\ell-1} - \mathcal{K}^{\circ(r)}_\ell)(S_j \cup f)$. Clearly, $J'_\ell \subseteq H^* - \mathcal{K}^{*(r)}_{\ell-1} - \mathcal{K}^{\circ(r)}_\ell$ by definition. Conversely, observe that every $e \in H^* \cup L^* - \mathcal{K}^{*(r)}_{\ell-1} - \mathcal{K}^{\circ(r)}_\ell$ that contains $T \cup f'$ must belong to J'_ℓ. Indeed, since $L^* \subseteq \mathcal{K}^{*(r)}_{\ell-1}$, we have $e \in H^*$, and since e contains T, we have $\tau_r(e) \le \ell$. Hence, $e \in H^*_\ell$ and thus $e \in J'_\ell$. This implies the claim. –

Let $L''_\ell := J'_\ell \triangle J_\ell$. So $J'_\ell = L''_\ell \triangle J_\ell$.

Claim 11: $L''_\ell \subseteq G^{*(r)}_\ell$ *and* $\Delta(L''_\ell) \le 3\nu_\ell n$.

Proof of claim: Suppose, for a contradiction, that there is $e \in J'_\ell \triangle J_\ell$ with $e \notin G^{*(r)}_\ell$. By Claim 2 and (10.18), the latter implies that e is $\mathcal{S}$-important with $\tau_r(e) < \ell$ or $\mathcal{T}^{(i')}$-important with $\tau_{\ell,r}(e) < \ell$. However, since $J_\ell \subseteq G^{*(r)}_\ell$, we must have $e \in J'_\ell \setminus J_\ell$ and thus $e \in H'_\ell$ and $e \notin \mathcal{K}^{*(r)}_{\ell-1} \cup \mathcal{K}^{\circ(r)}_\ell$. In particular, $e \in H''_\ell$. Now, if e was $\mathcal{S}$-important with $\tau_r(e) < \ell$, then

$e \in H'_\ell - H_\ell \subseteq H^*_{\ell-1}$. But then e would be covered by $\mathcal{K}^*_{\ell-1}$, a contradiction. So e must be $\mathcal{T}^{(i')}$-important with $\tau_{\ell,r}(e) < \ell$. But since $e \in H''_\ell$, e would be covered by $\mathcal{K}^\circ_\ell$ unless $\tau_{\ell,r}(e) = r - i' = \ell$, a contradiction.

In order to see the second part, observe that

$$L''_\ell = ((H'_\ell \cup J_\ell) - \mathcal{K}^{*(r)}_{\ell-1} - \mathcal{K}^{\circ(r)}_\ell) \triangle J_\ell \subseteq H'_\ell \cup L^*$$

since $\mathcal{K}^{*(r)}_{\ell-1} \cup \mathcal{K}^{\circ(r)}_\ell \subseteq H'_\ell \cup L^*$. Thus, $\Delta(L''_\ell) \leq \Delta(H'_\ell) + \Delta(L^*) \leq 3\nu_\ell n$. –

Thus, by (10.19), $G[J'_\ell](T)[U_T] = G^*_\ell[J'_\ell](T)[U_T]$ is a $(\rho_\ell, \beta_\ell^{(8^q)+1}, q-i', r-i')$-supercomplex for every $T \in \mathcal{T}^{(i')}$. (Here we also use that $J'_\ell \subseteq G^{*(r)}_\ell$ by Claim 11 and the definition of J_ℓ.)

We can therefore apply the Localised cover down lemma (Lemma 10.7) with the following objects/parameters.

object/parameter	ρ_ℓ	μ	$\beta_\ell^{(8^q)+1}$	i'	$G[J'_\ell]$	$\mathcal{T}^{(i')}$	$\mathcal{U}_{i'}$
playing the role of	ρ	ρ_{size}	ξ	i	G	$\mathcal{S}$	$\mathcal{U}$

This yields a $K^{(r)}_q$-packing $\mathcal{K}^\dagger_\ell$ in $G[J'_\ell]$ covering all $\mathcal{T}^{(i')}$-important r-edges. Thus $\mathcal{K}^\dagger_\ell$ is as required in (b). As observed before, this completes the proof of $(\#)_{\ell+1}$ and thus the proof of (i).

Step 5: Covering down — Proof of (ii)

Now, suppose that $G^{(r)}$ is (q,r)-divisible with respect to $\mathcal{S},\mathcal{U}$ and A is diagonal-dominant.

Claim 12: G is $(\xi - \varepsilon, q, r)$-dense with respect to $G^{(r)} - \tau_r^{-1}(0)$.

Proof of claim: Let $e \in G^{(r)}$ and let $\ell' \in [r+1]$ be such that $e \in \mathcal{P}_{edge}(\ell')$. Suppose first that $\ell' \leq i$. Then no q-set from $\mathcal{P}_{clique}(\ell')$ contains any edge from $\tau_r^{-1}(0)$ (as such a q-set is $\mathcal{S}$-unimportant). Recall from (S2) for $\mathcal{S},\mathcal{U},(\mathcal{P}_{edge},\mathcal{P}_{clique})$ that $G[Y]$ is (ε, A, q, r)-regular with respect to $(\mathcal{P}_{edge},\mathcal{P}_{clique})[Y]$ and $\min\backslash\backslash^{r-i+1}(A) \geq \xi$. Thus,

$$|G[(G^{(r)} - \tau_r^{-1}(0)) \cup e]^{(q)}(e)| \geq |(Y \cap \mathcal{P}_{clique}(\ell'))(e)| \geq (a_{\ell',\ell'} - \varepsilon)n^{q-r} \geq (\xi - \varepsilon)n^{q-r}.$$

If $\ell' > i+1$, then by (P2′) in Proposition 10.11, no q-set from $\mathcal{P}_{clique}(q-r+\ell')$ contains any edge from $\tau_r^{-1}(0)$. Thus, we have

$$|G[(G^{(r)} - \tau_r^{-1}(0)) \cup e]^{(q)}(e)| \geq (a_{\ell',q-r+\ell'} - \varepsilon)n^{q-r} \geq (\xi - \varepsilon)n^{q-r}.$$

If $\ell' = i+1$, then $\mathcal{P}_{edge}(\ell') = \tau_r^{-1}(0)$ by (P2′). However, every q-set from $\tau_q^{-1}(q-r) = \mathcal{P}_{clique}(q-r+\ell')$ that contains e contains no other edge from $\tau_r^{-1}(0)$. Thus,

$$|G[(G^{(r)} - \tau_r^{-1}(0)) \cup e]^{(q)}(e)| \geq (a_{\ell',q-r+\ell'} - \varepsilon)n^{q-r} \geq (\xi - \varepsilon)n^{q-r}.$$

–

By Claim 12, we can choose $H^* \subseteq G^{(r)} - \tau_r^{-1}(0)$ such that (i) holds with G playing the role of $\tilde{G}$. Let

$$H_{nibble} := G^{(r)} - H^*.$$

Recall that by (S2), $G[Y]$ is (ε, A, q, r)-regular with respect to $(\mathcal{P}_{edge},\mathcal{P}_{clique})[Y]$, and (S3) implies that $G[Y]$ is $(\mu^q\xi, q+r, r)$-dense. Let

$$G_{nibble} := (G[Y])[H_{nibble}].$$

Using Proposition 5.4, it is easy to see that G_{nibble} is $(2^{r+1}\nu, A, q, r)$-regular with respect to $(\mathcal{P}_{edge},\mathcal{P}_{clique})[Y][H_{nibble}]$. Moreover, by Proposition 5.6(ii), G_{nibble} is $(\mu^q\xi/2, q+r, r)$-dense. Thus, by Corollary 10.16, there exists $Y^* \subseteq G^{(q)}_{nibble}$ such that $G_{nibble}[Y^*]$ is $(\sqrt{\nu}, d, q, r)$-regular for some $d \geq \xi$ and $(0.45\mu^q\xi(\mu^q\xi/8(q+1))^{\binom{q+r}{q}}, q+r, r)$-dense. Thus, by the Boosted nibble lemma (Lemma 6.4) there is a $K^{(r)}_q$-packing $\mathcal{K}_{nibble}$ in $G_{nibble}[Y^*]$ such that $\Delta(L_{nibble}) \leq \gamma n$, where $L_{nibble} := G_{nibble}[Y^*]^{(r)} - \mathcal{K}^{(r)}_{nibble} = H_{nibble} - \mathcal{K}^{(r)}_{nibble}$. Since $G^{(r)}$ is (q,r)-divisible with respect to $\mathcal{S},\mathcal{U}$, we clearly have that $H^* \cup L_{nibble}$ is (q,r)-divisible with respect to $\mathcal{S},\mathcal{U}$. Thus, by (i), there exists a $K_q^{(r)}$-packing $\mathcal{K}^*$ in $G[H^* \cup L_{nibble}]$ which covers all edges of L_{nibble}, and all $\mathcal{S}$-important edges of H^* except possibly some from $\tau_r^{-1}(r-i)$. But then, $\mathcal{K}_{nibble} \cup \mathcal{K}^*$ is a $K_q^{(r)}$-packing in G which covers all $\mathcal{S}$-important r-edges except possibly some from $\tau_r^{-1}(r-i)$, completing the proof. □

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References

[1] N. Alon, J.-H. Kim and J. Spencer, Nearly perfect matchings in regular simple hypergraphs, *Isr. J. Math.* **100** (1997), 171–187.
[2] N. Alon and R. Yuster, On a hypergraph matching problem, *Graphs and Combinatorics* **21** (2005), 377–384.
[3] Z. Baranyai, On the factorization of the complete uniform hypergraph, In: A. Hajnal, R. Rado, V.T. Sós, *Infinite and Finite Sets*, Proc. Coll. Keszthely, 1973, Coll. Math. J. Bolyai, 10, pages 91–108, North-Holland, Amsterdam, 1975.
[4] B. Barber, D. Kühn, A. Lo, R. Montgomery and D. Osthus, Fractional clique decompositions of dense graphs and hypergraphs, *arXiv:1507.04985*, 2015.
[5] B. Barber, D. Kühn, A. Lo and D. Osthus, Edge-decompositions of graphs with high minimum degree, *Adv. Math.* **288** (2016), 337–385.
[6] B. Barber, D. Kühn, A. Lo, D. Osthus and A. Taylor, Clique decompositions of multipartite graphs and completion of Latin squares, *arXiv:1603.01043*, 2016.
[7] F.C. Bowditch and P.J. Dukes, Fractional triangle decompositions of dense 3-partite graphs, *arXiv:1510.08998*, 2015.
[8] D. Bryant and D. Horsley, A proof of Lindner's conjecture on embeddings of partial Steiner triple systems, *J. Combinatorial Designs* **17** (2009), 63–89.
[9] C.J. Colbourn and J.H. Dinitz, *Handbook of Combinatorial Designs*, 2nd edition, CRC Press, 2006.
[10] D. Conlon and W.T. Gowers, Combinatorial theorems in sparse random sets, *Ann. Math.* **184** (2016), 367–454.
[11] F. Dross, Fractional triangle decompositions in graphs with large minimum degree, *SIAM J. Discr. Math.* **30** (2016), 36–42.
[12] P. Erdős and H. Hanani, On a limit theorem in combinatorial analysis, *Publicationes Mathematicae Debrecen* **10** (1963), 10–13.
[13] A. Ferber, R. Hod, M. Krivelevich and B. Sudakov, A construction of almost Steiner systems, *J. Combinatorial Designs* **22** (2014), 488–494.
[14] S. Glock, D. Kühn, A. Lo, R. Montgomery and D. Osthus, On the decomposition threshold of a given graph, *arXiv:1603.04724*, 2016.
[15] J.E. Graver and W.B. Jurkat, The module structure of integral designs, *J. Combinatorial Theory Ser. A* **15** (1973), 75–90.
[16] T. Gustavsson, *Decompositions of large graphs and digraphs with high minimum degree*, PhD thesis, Univ. of Stockholm, 1991.
[17] H. Hàn, Y. Person and M. Schacht, On perfect matchings in uniform hypergraphs with large minimum vertex degree, *SIAM J. Discr. Math.* **23** (2009), 732–748.
[18] S. Janson, T. Łuczak, and A. Ruciński, *Random graphs*, Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York, 2000.
[19] F. Joos, J. Kim, D. Kühn and D. Osthus, Optimal packings of bounded degree trees, *arXiv:1606.03953*, 2016.
[20] P. Keevash, The existence of designs, *arXiv:1401.3665*, 2014.
[21] P. Keevash, Counting designs, *arXiv:1504.02909*, 2015.
[22] T.P. Kirkman, On a problem in combinatorics, *Cambridge Dublin Math. J.* **2** (1847), 191–204.
[23] F. Knox, D. Kühn and D. Osthus, Edge-disjoint Hamilton cycles in random graphs, *Random Structures Algorithms* **46** (2015), 397–445.
[24] A. Kostochka, D. Mubayi and J. Verstraëte, On independent sets in hypergraphs, *Random Structures Algorithms* **44** (2014), 224–239.
[25] M. Krivelevich, Triangle factors in random graphs, *Combin. Probab. Comput.* **6** (1997), 337–347.
[26] D. Kühn and D. Osthus, Hamilton decompositions of regular expanders: A proof of Kelly's conjecture for large tournaments, *Adv. Math.* **237** (2013), 62–146.
[27] G. Kuperberg, S. Lovett and R. Peled, Probabilistic existence of regular combinatorial objects, *arXiv:1302.4295*, 2013. (Extended abstract version appeared in STOC 2012.)
[28] M. Kwan, Almost all Steiner triple systems have perfect matchings, *arXiv:1611.02246*, 2016.

[29] N. Linial and Z. Luria, Upper bounds on the number of Steiner triple systems and 1-factorizations, *Random Structures Algorithms* **43** (2013), 399–406.
[30] R. Montgomery, Fractional clique decompositions of dense partite graphs, *arXiv:1603.01039*, 2016.
[31] C.St.J.A. Nash-Williams, An unsolved problem concerning decomposition of graphs into triangles, In: *Combinatorial Theory and its Applications III*, pages 1179–1183, North Holland, 1970.
[32] N. Pippenger and J. Spencer, Asymptotic behaviour of the chromatic index for hypergraphs, *J. Combinatorial Theory Ser. A* **51** (1989), 24–42.
[33] R. Raman, The power of collision: randomized parallel algorithms for chaining and integer sorting, In: *Foundations of software technology and theoretical computer science (Bangalore, 1990)*, volume 472 of *Lecture Notes in Comput. Sci.*, pages 161–175, Springer, Berlin, 1990.
[34] V. Rödl, On a packing and covering problem, *European J. Comb.* **5** (1985), 69–78.
[35] V. Rödl and A. Ruciński, Dirac-type questions for hypergraphs – a survey (or more problems for Endre to solve), An Irregular Mind (Szemerédi is 70), *Bolyai Soc. Math. Studies* **21** (2010), 561–590.
[36] V. Rödl, A. Ruciński and E. Szemerédi, A Dirac-type theorem for 3-uniform hypergraphs, *Combin. Probab. Comput.* **15** (2006), 229–251.
[37] V. Rödl and E. Šiňajová, Note on independent sets in Steiner systems, *Random Structures Algorithms* **5** (1994), 183–190.
[38] M. Schacht, Extremal results for random discrete structures, *Ann. Math.* **184** (2016), 333–365.
[39] B. Sudakov and V. Vu, Local resilience of graphs, *Random Structures Algorithms* **33** (2008), 409–433.
[40] L. Teirlinck, Non-trivial t-designs without repeated blocks exist for all t, *Discrete Math.* **65** (1987), 301–311.
[41] V.H. Vu, New bounds on nearly perfect matchings in hypergraphs: higher codegrees do help, *Random Structures Algorithms* **17** (2000), 29–63.
[42] R.M. Wilson, An existence theory for pairwise balanced designs I. Composition theorems and morphisms, *J. Combinatorial Theory Ser. A* **13** (1972), 220–245.
[43] R.M. Wilson, An existence theory for pairwise balanced designs II. The structure of PBD-closed sets and the existence conjectures, *J. Combinatorial Theory Ser. A* **13** (1972), 246–273.
[44] R.M. Wilson, An existence theory for pairwise balanced designs III. Proof of the existence conjectures, *J. Combinatorial Theory Ser. A* **18** (1975), 71–79.
[45] R. Yuster, Combinatorial and computational aspects of graph packing and graph decomposition, *Computer Science Review* **1** (2007), 12–26.
[46] Y. Zhao, Recent advances on Dirac-type problems for hypergraphs, In: A. Beveridge, J.R. Griggs, L. Hogben, G. Musiker and P. Tetali, *Recent Trends in Combinatorics*, the IMA Volumes in Mathematics and its Applications **159**, pages 145–165, Springer, New York, 2016.

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