HAMILTON CYCLES IN SPARSE ROBUSTLY EXPANDING DIGRAPHS

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Abstract. The notion of robust expansion has played a central role in the solution of several conjectures involving the packing of Hamilton cycles in graphs and directed graphs. These and other results usually rely on the fact that every robustly expanding (di)graph with suitably large minimum degree contains a Hamilton cycle. Previous proofs of this require Szemerédi's Regularity Lemma and so this fact can only be applied to dense, sufficiently large robust expanders. We give a proof that does not use the Regularity Lemma and, indeed, we can apply our result to suitable sparse robustly expanding digraphs.

1. Introduction

Throughout we work with simple directed graphs (also called digraphs), i.e. directed graphs with no loops and with at most two edges between each pair of vertices (one in each direction). A Hamilton cycle in a (directed) graph is a (directed) cycle that passes through every vertex. Over the last several decades, there has been intense study in finding sufficient conditions for the existence of Hamilton cycles in graphs and digraphs. The seminal result in the case of graphs is Dirac's Theorem [9] and in the case of digraphs is Ghouila–Houri's Theorem [12], each giving tight minimum degree conditions for the existence of Hamilton cycles.

In this paper we consider how Hamiltonicity is related to expansion properties of digraphs. In recent years several researchers have investigated the connection between expansion and Hamiltonicity, particularly for graphs. There are several different notions of expansion one can consider, and while we shall discuss a few of these for comparison, our main focus will be so-called *robust expansion* first introduced in [29], although implicitly present in [16, 18].

Definition 1.1. For an n-vertex digraph $D = (V, E)$, $\nu \in (0, 1)$, and $S \subseteq V$, the *robust ν-outneighbourhood* of S, denoted $\mathrm{RN}^+_\nu(S)$, is the set of vertices that have at least νn inneighbours in S. Given $0 < \nu \le \tau < 1$, we say D is a *robust (ν, τ)-outexpander* if

$$|\mathrm{RN}^+_\nu(S)| \ge |S| + \nu n$$

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for every $S \subseteq V$ satisfying $\tau n \leq |S| \leq (1-\tau)n$.

In this paper, we think of the parameters ν and τ as functions of n. Note that robust expansion is a resilience property, i.e. if D is a robust outexpander, then D remains a robust outexpander (with slightly worse parameters) even after removing a sparse subgraph.

As we shall discuss shortly, robust expansion has played a central role in the proofs of several conjectures about Hamilton cycles. The starting point of many of these proofs is the following result which says that a robust expander with linear minimum semi-degree contains a Hamilton cycle. The semi-degree $\delta^0(D)$ of a digraph D is given by $\delta^0(D) = \min(\delta^+(D), \delta^-(D))$ where $\delta^+(D)$ and $\delta^-(D)$ are respectively the minimum outdegree and minimum indegree of D.

Theorem 1.2 ([29]). *Let n_0 be a positive integer and γ, ν, τ be positive constants such that $1/n_0 \ll \nu \leq \tau \ll \gamma < 1$. Let D be a digraph on $n \geq n_0$ vertices with $\delta^0(D) \geq \gamma n$ which is a robust (ν,τ)-outexpander. Then D contains a Hamilton cycle.*

This result was first proved in [29] by Kühn, Osthus and Treglown. A simpler proof is given in [26] and an algorithmic version is given in [7]. The proofs of Theorem 1.2 presented in [29, 26, 7] all rely on the Regularity Lemma and so in particular one can only work with sufficiently large and dense digraphs.

Our contribution in this paper is to strengthen Theorem 1.2 by showing that in a robust expander with linear minimum semi-degree, every vertex is contained in a cycle of length ℓ for all $\nu n/2 \leq \ell \leq n$. Further, our proof is algorithmic and does not use the Regularity Lemma, but uses instead the recent absorption technique developed by Rödl, Ruciński and Szemerédi [35] (with special forms appearing in earlier work e.g. [20]). In fact we prove a version of Theorem 1.2 that also applies to some sparse digraphs and where we can explicitly give a reasonable value for n_0.

Theorem 1.3. *There exists an integer C such that the following holds. Suppose $n \in \mathbb{N}$ and $C\sqrt[12]{\log n/n} < \nu \leq \tau \leq \gamma/16 < 1/16$. Let D be an n-vertex digraph with $\delta^0(G) \geq \gamma n$ which is a robust (ν,τ)-outexpander. Then, for any $\nu n/2 \leq \ell \leq n$ and any vertices v of D, D contains a directed cycle of length ℓ through v.*

Theorem 1.2 has been used directly as a tool in several papers including [30, 25, 19, 27, 33, 10]. Below we discuss results that require the Regularity Lemma only because they rely (directly or indirectly) on Theorem 1.2. For some such results, we can now replace Theorem 1.2 with Theorem 1.3 to give proofs that do not require the Regularity Lemma and consequently hold for much smaller values of n.

Robust expansion was first used to prove an approximate analogue of Dirac's Theorem for oriented graphs (an oriented graph is a directed graph in which there is at most one edge between each pair of vertices).

Theorem 1.4 ([18]). *For every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that if D is an oriented graph with $n > n_0$ vertices and $\delta^0(D) > \frac{3}{8}n + \varepsilon n$ then D contains a Hamilton cycle.*

Note that the constant 3/8 cannot be improved due to examples given in [18]. The result above was proved using the Regularity Lemma and an exact version was proved later in [16] also using the Regularity Lemma. A consequence of Theorem 1.3 is that one can adapt the proof of Theorem 1.4 to avoid the use of the Regularity Lemma.

Corollary 1.5. *There exists a constant $C > 0$ such that the following holds. Suppose $\varepsilon > 0$ and $n \in \mathbb{N}$ with $n > C\varepsilon^{-25}$. If D is an n-vertex oriented graph with $\delta^0(D) > \frac{3}{8}n + \varepsilon n$ then D contains a Hamilton cycle.*

In fact, one can use Theorem 1.3 to adapt the proof of the exact version in [16] avoiding the use of the Regularity Lemma.

In [29], Kühn, Osthus and Treglown give an approximate solution to a conjecture of Nash-Williams [32] about sufficient conditions on the degree sequence of a digraph to guarantee the existence of a Hamilton cycle. Their result uses the Regularity Lemma, but Theorem 1.3 can be used to adapt their proof to avoid using the Regularity Lemma and thus give a better approximation.

For a digraph D, consider its outdegree sequence $d_1^+ \leq \cdots \leq d_n^+$ and indegree sequence $d_1^- \leq \cdots \leq d_n^-$. Note that d_i^+ and d_i^- do not necessarily correspond to the degree of the same vertex of D.

Theorem 1.6. *There exists an integer C such that the following holds. Suppose $n \in \mathbb{N}$ and $C\sqrt[24]{\log n/n} < \gamma/16 < 1/16$. Let D be an n-vertex digraph such that for all $i < n/2$,*

- $d_i^+ \geq i + \gamma n$ *or* $d_{n-i-\gamma n}^- \geq n - i$,
- $d_i^- \geq i + \gamma n$ *or* $d_{n-i-\gamma n}^+ \geq n - i$.

Then, for any $\nu n/2 \leq \ell \leq n$ and any vertex v of D, D contains a directed cycle of length ℓ through v.

In [23, 24], Kühn, Osthus, Staden and the first author prove the one remaining case of a conjecture of Bollobás and Haggvist, making (indirect) use of the Regularity Lemma: they prove that there exits n_0 such that every 3-connected D-regular graph on $n \geq n_0$ vertices with $D \geq n/4$ is Hamiltonian. Replacing the use of Theorem 1.2 by Theorem 1.3 in [23, 24] gives a proof of the result avoiding the Regularity Lemma.

Robust expansion has also been central in the recent solutions of several long-standing conjectures about the decomposition of dense graphs and digraphs into Hamilton cycles. Beginning with the proof of Kelly's Conjecture (for sufficiently large tournaments) by Kühn and Osthus [27], the robust expanders technique was applied by Csaba et al. in [8] to solve three more long-standing conjectures including the 1-factorization conjecture and Hamilton decomposition conjecture. Further discussion of robust expansion and its applications can be found in [28] and Section 6 of [26]. We remark that, although these results use Theorem 1.2

as a tool, they also make use of the Regularity Lemma at other points in their proof and so we cannot immediately use Theorem 1.3 to avoid the Regularity Lemma in these results.

1.1. **Related Notions of Expansion.** In this section we discuss different notions of expansion, how they are related to Hamiltonicity, and how they compare with robust expansion. Most results in this direction are for graphs rather than digraphs, so we compare with robust expansion of undirected graphs, which is defined in the obvious way (see Section 3 for the formal definition).

The strongest expansion properties are enjoyed by random and pseudorandom graphs and digraphs. For a d-regular graph G, let $\lambda_1, \dots, \lambda_n$ denote the eigenvalues of the adjacency matrix of G ordered such that $|\lambda_1| \geq \cdots \geq |\lambda_n|$. It is well known that $|\lambda_1| = d$, and that $\lambda(G) := |\lambda_2|$ is a measure of the expansion of G: indeed the expander mixing lemma (see [2]) implies that for any $A, B \subseteq V(G)$

$$\left| e_G(A,B) - \frac{d}{n}|A||B| \right| \leq \lambda(G)\sqrt{|A||B|},$$

where $e_G(A,B)$ denotes the number of edges that have one endpoint in A and the other in B (and edges in $A \cap B$ are counted twice). Krivelevich and Sudakov [21] showed that every d-regular n-vertex graph satisfying $\lambda(G) \leq d(\log\log n)^2/1000 \log n \log\log\log n$ has a Hamilton cycle. This shows Hamiltonicity for quite sparse regular graphs satisfying a relatively strong expansion property. Butler and Chung [6] generalised this giving a sufficient condition for Hamiltonicity in terms of the spectral gap of the graph Laplacian.

A related notion of expansion (which in particular includes graphs that are not regular) is that of jumbled graphs introduced by Thomason [37, 38]. An n-vertex graph is (p,α)-jumbled if, for all disjoint $A, B \subseteq V(G)$,

$$\left| e_G(A,B) - p|A||B| \right| \leq \alpha\sqrt{|A||B|}. \tag{1}$$

Thomason [37] showed that every $(p, \beta p^2 n)$-jumbled graph with minimum degree βpn has a Hamilton cycle. Robust expansion is in general a weaker form of expansion than the pseudorandom forms described above. One can easily show (see Proposition 3.1) that a (p,α)-jumbled graph with $\alpha < \varepsilon pn$ for some $\varepsilon > 0$ is a robust (ν,τ)-expander if $\nu < p(\tau - \varepsilon)$ (note that (1) becomes almost vacuous if e.g. $\alpha > pn/2$ since one gets no lower bound on $e_G(A,B)$ in this case). On the other hand, it is easy to construct strong robust expanders in which two disjoint linear-sized sets of vertices have no edges passing between them, e.g. a blow-up of C_5, and such graphs can only be weakly jumbled by (1).

Different notions of expansion, not directly comparable to robust expansion, were considered in [5] and [13]. It was shown in the latter that any graph in which (i) there is at least one edge between any two large subsets of vertices and (ii) every small set of vertices expands is Hamiltonian. This is a fairly weak notion of expansion, but again, a robust expander may have linear-sized sets of vertices between which there is no edge. Recently, a quasirandom notation for finding Hamilton cycles in hypergraphs was considered in [31].

There are far fewer results on Hamilton cycles in directed expanders. We have already mentioned [30, 33] where Theorem 1.2 is proved in the dense setting using the Regularity Lemma. The only other results in this direction that we are aware of concern Hamiltonian resilience. Improving on results in [14], Ferber et al. [11, Theorem 3.2] show that quite sparse directed graphs satisfying a certain pseudorandom condition contain a Hamilton cycle (in fact they show that such graphs remain Hamiltonian even after deleting about half the edges at each vertex). However, their pseudorandom condition again requires one to control the number of edges between every pair of vertex subsets of size greater than approximately $\log n$, which robust expansion does not require.

The study of Hamilton cycles in expanding and pseudorandom graphs has been applied to packing problems (as discussed earlier) but also to positional games [13], Caley graphs [21] and resilience [11]. Finally we list (by no means complete) a few papers which show how expansion leads to other spanning structures, e.g. triangle factors [22], powers of a Hamilton cycle [1], long cycles in directed graphs [3], all bounded-degree spanning trees [15], and arbitrary orientations of spanning cycles (in directed graphs) [36, 17].

1.2. **Outline.** In the next section we collect some notation and in Section 3, we prove some simple facts about robustly expanding digraphs. Section 4 is devoted to describing and constructing an 'absorbing structure' H in a robustly expanding digraph D. Informally, one can think of H as a set of edges of D which have the property that (almost) any small collection of vertex-disjoint cycles of D can be connected together into a long cycle using the edges of H. In Section 5 we show that the vertices of any robustly expanding digraph can be covered by a small number of cycles. In Section 6 we combine these results to prove Theorem 1.3, and we give some concluding remarks in Section 7.

We mention here that during the course of various proofs, several straightforward calculations, which we feel detract from the main argument, are suppressed and can be found at the end of the paper.

2. Notation

The digraphs considered in this paper do not have loops and we allow up to two edges between any pair x, y of distinct vertices, at most one in each direction. Given a digraph $D = (V, E)$, we sometimes write $V(D) := V$ for its vertex set and $E(D) := E$ for its edge set and $|D|$ for the number of its vertices. We write xy for an edge directed from x to y.

We write $H \subseteq D$ to mean that H is a subdigraph of D, i.e. $V(H) \subseteq V(D)$ and $E(H) \subseteq E(D)$. Given $X \subseteq V(D)$, we write $D - X$ for the digraph obtained from D by deleting all vertices in X, and $D[X]$ for the subdigraph of D induced by X. Given $F \subseteq E(D)$, we write $D - F$ for the digraph obtained from D by deleting all edges in F. If H is a subdigraph of D, we write $D - H$ for $D - E(H)$. For two subdigraphs H_1 and H_2 of D, we write $H_1 \cup H_2$ for the subdigraph with vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2)$. For a set U, U^2 means the set of all ordered pairs of U, and $U^{[2]}$ means the set of all ordered pairs of U except pairs of the form (x,x).

If x is a vertex of a digraph D, then $N^+_D(x)$ denotes the *outneighbourhood* of x, i.e. the set of all those vertices y for which $xy \in E(D)$. Similarly, $N^-_D(x)$ denotes the *inneighbourhood* of x, i.e. the set of all those vertices y for which $yx \in E(D)$. We write $d^+_D(x) := |N^+_D(x)|$ for the *outdegree* of x and $d^-_D(x) := |N^-_D(x)|$ for its *indegree*. We denote the *minimum outdegree* of D by $\delta^+(D) := \min\{d^+_D(x) : x \in V(D)\}$ and the *minimum indegree* $\delta^-(D) := \min\{d^-_D(x) : x \in V(D)\}$. The *minimum semi-degree* of D is $\delta^0(D) := \min\{\delta^+(D), \delta^-(D)\}$.

Unless stated otherwise, when we refer to paths and cycles in digraphs, we mean directed paths and cycles, i.e. the edges on these paths and cycles are oriented consistently. We write $P = x_1x_2\cdots x_t$ to indicate that P is a path with edges $x_1x_2, x_2x_3, \dots, x_{t-1}x_t$, where $x_1, \dots, x_t$ are distinct vertices. We occasionally denote such a path P by x_1Px_t to indicate that it starts at x_1 and ends at x_t. We write $\mathring{P}$ for the interior of P, i.e. $\mathring{P} = x_2\cdots x_{t-1}$. For two paths $P = a\cdots b$ and $Q = b\cdots c$, we write $aPbQc$ for the concatenation of the paths P and Q and this notation generalises to cycles in the obvious ways.

Throughout, logarithms are taken base e.

3. Preliminaries

In this section, we prove some basic properties of robust expanders. We begin by defining robust expansion for graphs and showing how it is related to jumbled graphs (Proposition 3.1) in order to complete the discussion in the introduction; however these will not be needed in the rest of the paper.

The notion of robust expansion extends to graphs in the obvious way. More precisely, an undirected graph is a (ν,τ)-expander if for each $S \subseteq V$ satisfying $\tau n \le |S| \le (1-\tau)n$, there is a set of at least $|S| + \nu n$ vertices of G each of which has at least νn neighbours in S.

Proposition 3.1. *Let $G = (V,E)$ be a (p,α)-jumbled n-vertex graph for some $p \in (0,1)$ and some $\alpha > 0$ and assume $\alpha = \varepsilon pn$ for some $\varepsilon > 0$. Then G is a robust (ν,τ)-expander if $0 < \nu < p(\frac{1}{4}\tau - \varepsilon)$.*

Proof. Assume $\tau \le 1$; otherwise the statement above is vacuous. Assume also that τn is an integer for convenience. Let $S \subseteq V$ be any set with $\tau n \le |S| \le (1-\tau)n$. Arbitrarily choose some $S' \subseteq S$ with $|S'| = \frac{1}{4}\tau n$. Set T to be the set of all vertices in $V \setminus S'$ that have fewer than νn neighbours in S'. If $|T| > \frac{1}{4}\tau n$ then arbitrarily choose $T' \subseteq T$ with $|T'| = \frac{1}{4}\tau n$. Then

$$\frac{1}{16}p\tau^2n^2 - \varepsilon pn \cdot \frac{1}{4}\tau n = p|S'||T'| - \varepsilon pn\sqrt{|S'||T'|} \le e_G(S',T') \le \frac{1}{4}\tau n \cdot \nu n,$$

which, after cancellation, implies $p(\frac{1}{4}\tau - \varepsilon) \le \nu$, a contradiction. Hence $|T| \le \frac{1}{4}\tau n$, so there are at least $n - |S'| - |T| \ge (1-\frac{1}{2}\tau)n \ge |S| + \nu n$ vertices with more than νn neighbours in S' and hence in S. Thus the set of vertices of G each of which has at least νn neighbours in S is at least $|S| + \nu n$. $\square$

It will be convenient to work with the following slightly more general definition of robust expansion.

Definition 3.2. Let $0 < \mu, \nu \leq \tau < 1$ and let $D = (V, E)$ be a digraph with $|V| = n$. For $S \subseteq V$, the *robust ν-outneighbourhood* (resp. *robust ν-inneighbourhood*) denoted $\mathrm{RN}^+_\nu(S)$ (resp. $\mathrm{RN}^-_\nu(S)$) is given by

$$\mathrm{RN}^+_\nu(S) = \{v \in V : |N^-(v) \cap S| \geq \nu n\}$$
$$\mathrm{RN}^-_\nu(S) = \{v \in V : |N^+(v) \cap S| \geq \nu n\}.$$

We say that D is a *robust (μ, ν, τ)-outexpander (resp. -inexpander)* if for every $S \subseteq V$ satisfying $\tau n \leq |S| \leq (1-\tau)n$, we have $\mathrm{RN}^-_\nu(S) \geq |S| + \mu n$ (resp. $\mathrm{RN}^+_\nu(S) \geq |S| + \mu n$).

Finally we say D is a *robust (μ, ν, τ)-expander* if it is both a robust (μ, ν, τ)-inexpander and a robust (μ, ν, τ)-outexpander.

Note that if D is a robust (μ, ν, τ)-expander and $S \subseteq V$ satisfies $|S| \geq (1-\tau)n$, then $|\mathrm{RN}^+_\nu(S)|, |\mathrm{RN}^-_\nu(S)| \geq (1-\tau)n$ (simply consider a subset of S of size $\lfloor (1-\tau)n \rfloor$). Throughout we shall usually be concerned with digraphs D that are robust (μ, ν, τ)-expanders with $\delta^0(D) > \gamma n$, where the parameters μ, ν, τ, and γ should be thought of as functions of n.

The following proposition follows immediately from the definition of a robust expander.

Proposition 3.3. *Suppose $D = (V, E)$ is a robust (μ, ν, τ)-expander and $S \subseteq V$ with $|S| \leq \varepsilon n$. Then $D - S$ is a $(\mu - \varepsilon, \nu - \varepsilon, \tau/(1-\varepsilon))$-expander.*

The following observation of DeBiasio, which can be found in [36], says that robust inexpansion is essentially equivalent to robust outexpansion; thus we can and will restrict ourselves to digraphs that are robust (μ, ν, τ)-expanders. We reproduce the proof explicitly quantifying the relationships between the various parameters.

Proposition 3.4 (DeBiasio)**.** *Suppose $D = (V, E)$ is an n-vertex robust (ν, τ)-outexpander with $\delta^0(D) \geq \gamma n$, where $\gamma > 2\tau$, $\tau\gamma > \nu^2/2$ and $\nu < 1/2$. Then D is a robust $(\nu^2/2, 2\tau)$-inexpander.*

Proof. Suppose that D is not a robust $(\nu^2/2, 2\tau)$-inexpander. Then there is a set $S \subseteq V$ with $2\tau n \leq |S| \leq (1-2\tau)n$ such that $|\mathrm{RN}^-_{\nu^2/2}(S)| < |S| + \nu^2/2$. Let $T = V \setminus RN^-_{\nu^2/2}(S)$. Observe that

$$|S|\gamma n \leq e(V, S) \leq |RN^-_{\nu^2/2}(S)||S| + \nu^2 n^2/2,$$

so $|RN^-_{\nu^2/2}(S)| \geq \gamma n/2$, where we used that $|S|\gamma n/2 \geq \tau\gamma n^2 \geq \nu^2 n^2/2$. Therefore

$$\tau n < n - (1 - 2\tau + \nu^2/2)n < |T| < (1-\gamma/2)n < (1-\tau)n,$$

where the first and last inequalities follow from our choice of parameters. By the definition of T, we have that $e(T, S) < |T|\nu^2 n/2$ and so $|\mathrm{RN}^+_\nu(T) \cap S| <$

$|T|\nu/2 < \nu n/2$. Hence

$$\begin{aligned}|\mathrm{RN}^+_\nu(T)| &= |\mathrm{RN}^+_\nu(T) \setminus S| + |\mathrm{RN}^+_\nu(T) \cap S| \\ &< (n - |S|) + \nu n/2 \leq n - (|S| + \nu^2 n/2) + \nu n/4 \\ &< n - |\mathrm{RN}^-_{\nu^2/2}(S)| + \nu n/4 = |T| + \nu n,\end{aligned}$$

where we used that $\nu < \frac{1}{2}$ on the second line. Thus D is not a robust (ν, τ)-outexpander, a contradiction. $\square$

The next two lemmas show that robust expansion allows us to construct short paths between prescribed pairs of vertices.

Lemma 3.5. *Let $0 < \mu, \nu \leq \tau \leq \gamma/2 < 1/2$ and $n \in \mathbb{N}$ satisfying $n \geq 4\mu^{-1}\nu^{-1}$. Suppose that D is an n-vertex digraph which is a robust (μ, ν, τ)-expander and $\delta^0(D) \geq \gamma n$. Given distinct vertices $u, v \in V(D)$, there exists a path $P = x_0 \cdots x_{t+1}$ in D where $x_0 = u$, $x_{t+1} = v$ and $t \leq \mu^{-1} + 1$. (Note that P consists of at most $\mu^{-1} + 3$ vertices.)*

Proof. Let $N_1 := N^+(u)$ and inductively define $N_{i+1} := RN^+_\nu(N_i)$ Note that $|N_1| \geq \gamma n > \tau n$, so for all $i \geq 1$ if $|N_i| < (1 - \tau)n$ then $|N_{i+1}| \geq |N_i| + \mu n$. Observe that, for some $t \leq \mu^{-1} + 1$, we have $|N_t| \geq (1 - \tau)n$.

Now we obtain the sequence of vertices $x_1, \ldots, x_t$ by backwards induction as follows. Since $|N^-(v)| > \gamma n \geq 2\tau n$ and $|N_t| \geq (1 - \tau)n$, then $|N_t \cap N^-(v)| \geq \tau n \geq \mu\nu n \geq 4$, so there exists a vertex $x_t \in N_t \setminus \{u, v\}$ such that $x_t x_{t+1} \in E(D)$. By induction we assume we have found distinct vertices $x_{i+1}, \ldots, x_t$ (which are also distinct from u, v) with $x_j \in N_j$ and $x_j x_{j+1} \in E(D)$ for each $j = i+1, \ldots, t$. Since $x_{i+1} \in N_{i+1} = \mathrm{RN}^+_\nu(N_i)$, then x_{i+1} has at least νn inneighbours in N_i and since $\nu n > \mu^{-1} + 3 \geq t + 2$, we can choose $x_i \in N_i$ to be an inneighbour of x_{i+1} distinct from $x_{i+1}, \ldots, x_t, u, v$.

By induction, we obtain the vertices $x_1, \ldots, x_t$, and $P = x_0 \cdots x_{t+1}$, where $x_0 = u$ and $x_{t+1} = v$ is a directed path in D. $\square$

Lemma 3.6. *Let $0 < \mu, \nu \leq \tau \leq \gamma/4 < 1/4$ and $n, r \in \mathbb{N}$ satisfying $n \geq (6r + 11)\mu^{-1} \max(\mu^{-1}, \nu^{-1})$. Suppose that D is an n-vertex digraph which is a robust (μ, ν, τ)-expander and $\delta^0(D) \geq \gamma n$. Given distinct vertices $u_1 \ldots, u_r, v_1, \ldots, v_r \in V(D)$, there exists vertex disjoint paths $P_1, \ldots, P_r$ in D where P_i is from u_i to v_i and $|P_i| \leq 2\mu^{-1} + 3$.*

Proof. By induction assume that we have constructed vertex-disjoint paths $P_1, \ldots, P_{k-1}$ in D for some $k < r$, where, for each $i = 1, \ldots, k - 1$, P_i is from u_i to v_i and $|P_i| \leq 2\mu^{-1} + 3 \leq 3\mu^{-1}$ and $V(P_i) \cap \{u_{i+1}, \ldots, u_r, v_{i+1}, \ldots, v_r\} = \emptyset$. Let D_{k-1} be the digraph obtained from D by deleting all vertices in $P_1, \ldots, P_{k-1}$ and all vertices $u_{k+1}, \ldots, u_r, v_{k+1}, \ldots, v_r$. Note that D_{k-1} is obtained from D by deleting at most $3r\mu^{-1} \leq \frac{1}{2}\min(\mu, \nu)n$ vertices, so by Proposition 3.3, D_{k-1} is a robust $(\frac{1}{2}\mu, \frac{1}{2}\nu, \frac{8}{7}\tau)$-expander with $\delta^0(D_{k-1}) \geq \frac{7}{8}\gamma n \geq \frac{7}{8}\gamma|D_{k-1}|$. Note that $|D_{k-1}| \geq n - 3r\mu^{-1} \geq n - r\mu^{-1}\nu^{-1} \geq 16\mu^{-1}\nu^{-1}$. Apply Lemma 3.5 to D_{k-1} giving a path P_k in D_{k-1} of length at most $2\mu^{-1}+3$ from u_k to v_k. Thus P_k (as a path in D) is vertex-disjoint from $P_1, \dots, P_{k-1}$ and $\{u_{k+1}, \dots, u_r, v_{k+1}, \dots, v_r\}$ as required. Thus by induction we can find the paths $P_1, \dots, P_r$. □

We give a simple inequality that we shall use several times.

Proposition 3.7. *Fix $k, a > 0$. Then $e^x > ax^k$ for all $x \geq \max(3k(\log k + 1) + 3\log a, 0)$. Similarly for $c, d > 0$ we have $x > c\log x + d$ if $x > 3c(\log c + 1) + 3d$.*

Proof. We start by showing that for all $a > 0$ and $x \geq \max(3\log a + 3, 0)$, we have $e^x \geq ax$. This is clearly true if $0 \leq a \leq 1$. If $a > 1$, set $f(x) = e^x - ax$ and set $x_0 := 3\log a + 3 > 0$. We have $f(x_0) = e^3a^3 - 3a\log a - 3a > 0$ and $f'(x) = e^x - a > 0$ for all $x \geq x_0$. Hence $f(x) > 0$ for all $x \geq x_0$ and so $e^x > ax$ for all $x \geq \max(3\log a + 3, 0)$.

Finally, making the transformation $X = kx$ and $A = a^k/k^k$, and assuming $A, k > 0$, the inequality above becomes $e^X \geq AX^k$ for all $X \geq \max(3k\log k + 3\log A + 3k, 0)$.

For the other inequality, note that $x > c\log x + d$ if and only if $e^x > e^dx^c$, which holds if $x > \max(3c\log c + 3d + 3c, 0)$. □

4. The absorbing structure

In this section, we describe what we mean by an *absorbing structure* and show how to find one in a robustly expanding digraph with large minimum in- and outdegree. We begin by informally describing the properties we desire our absorbing structure to have. Given a digraph D we shall seek a subdigraph $S \subseteq D$ with the properties that

- $|V(S)|$ is small;
- S contains a Hamilton cycle (on $V(S)$);
- In D, given a small number of any vertex-disjoint paths $P_1, \dots, P_d$ that are also vertex-disjoint from S, we can use S to absorb $P_1, \dots, P_d$ into C i.e. we can find a Hamilton cycle C' on $V(S) \cup (\bigcup_{i=1}^d V(P_i))$.

The sequence of definitions that follow will lead to a precise description of our absorbing structure. We start by defining an alternating path.

Definition 4.1. Let D be a digraph, and let $x_1, \dots, x_t$ be distinct vertices of D with t even. An *alternating path* $P = [x_1x_2 \cdots x_t]$ is a subgraph of D with vertex set $\{x_1, \dots, x_t\}$ (where $x_1, \dots, x_t$ are distinct vertices) and edge set

$$\{x_ix_{i+1} \mid i = 1, 3, 5, \dots, t-1\} \cup \{x_{j+1}x_j \mid j = 2, 4, 6, \dots, t-2\}.$$

We say P is an alternating path from x_1 to x_t.

An alternating path is thus a path where the directions of the edges alternate. It will be important for us that the number of vertices in an alternating path is even so that the first vertex has outdegree 1 and the last vertex has indegree 1.

As with paths, robust expansion allows us to construct alternating paths between prescribed vertices.

Lemma 4.2. *Let $0 < \mu, \nu \le \tau \le \gamma/2 < 1/2$ and $n \in \mathbb{N}$ satisfying $n \ge 4\mu^{-1}\nu^{-1}$. Suppose that D is an n-vertex digraph which is a robust (μ, ν, τ)-expander and $\delta^0(D) \ge \gamma n$. Given distinct vertices $u, v \in V(D)$, there exists an alternating path $P = [x_0 \cdots x_t x_t^* \cdots x_0^*]$ in D where $x_0 = u$, $x_0^* = v$ and $t \le (\mu^{-1} + 4)/2$ is even. (Thus P consists of at most $\mu^{-1} + 6$ vertices.)*

Proof. Let $N_1 := N^+(u)$ and inductively define

$$N_{i+1} := \begin{cases} \mathrm{RN}^+_\nu(N_i) & \text{if } i \text{ even;} \\ \mathrm{RN}^-_\nu(N_i) & \text{if } i \text{ odd.} \end{cases}$$

Note that $|N_1| \ge \gamma n > \tau n$, so for all $i \ge 1$ if $|N_i| < (1-\tau)n$ then $|N_{i+1}| \ge |N_i| + \mu n$. Set $r := \lceil \mu^{-1} \rceil$ and observe that $|N_r| \ge (1-\tau)n$. Choose r' to be the smallest integer that is at least r and divisible by 4; thus $r' \le \mu^{-1} + 4$ and $|N_{r'}| \ge (1-\tau)n$.

Now we obtain a sequence of vertices $y_1, \ldots, y_{r'}$ by backwards induction as follows. Since $|N^-(v)| > \gamma n \ge 2\tau n$ and $|N_{r'}| \ge (1-\tau)n$, then $|N_{r'} \cap N^-(v)| \ge \tau n \ge 3$, so there exists a vertex $y_{r'} \in N_{r'} \setminus \{u, v\}$ such that $y_{r'} y_{r'+1} \in E(D)$ (where $y_{r'+1} := v$). By induction we assume we have found distinct vertices $y_{i+1}, \ldots, y_{r'}$ (which are also distinct from u, v) such that for each $j = i+1, \ldots, r'$, we have $y_j \in N_j$ with $y_j y_{j+1} \in E(D)$ if j is even and $y_{j+1} y_j \in E(D)$ if j is odd. Since

$$y_{i+1} \in \begin{cases} \mathrm{RN}^+_\nu(N_i) & \text{if } i \text{ even;} \\ \mathrm{RN}^-_\nu(N_i) & \text{if } i \text{ odd} \end{cases}$$

then y_{i+1} has at least νn in- (respectively out-) neighbours in N_i if i is even (respectively odd), and since $\nu n \ge 4\mu^{-1} \ge \mu^{-1}+6 \ge r'+2$, we can choose $y_i \in N_i$ to be an in- (respectively out-) neighbour of y_{i+1} distinct from $y_{i+1}, \ldots, y_{r'}, u, v$ for i odd (respectively even).

Thus we obtain distinct vertices $y_1, \ldots, y_{r'}$ such that $y_{i+1} y_i, y_j y_{j+1} \in E(D)$ for $i = 1, 3, 5 \ldots, r'-1$ and $j = 2, 4, 6, \ldots, r'-2$. Then relabelling $y_1, \ldots, y_{r'}$ to $x_1, \ldots, x_t x_t^*, \ldots, x_1^*$ respectively and $x_0 := u, x_0^* := v$ gives the desired alternating path. Since r' is divisible by 4, we have that t is even as required. □

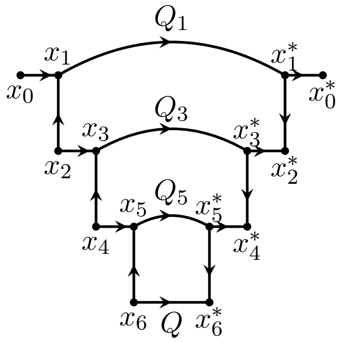
Next we define ladders, which will be the key structures that allow us to absorb paths.

Definition 4.3. Let D be a digraph and let $u, v \in V(D)$ be distinct vertices. A *ladder* L from u to v is a subdigraph of D given by

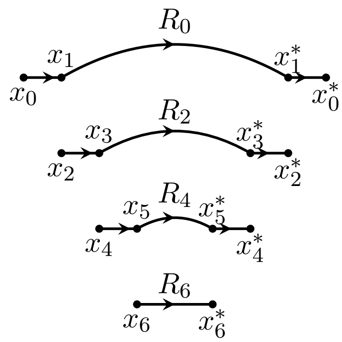
$$L = Q \cup Q_1 \cup Q_3 \cup Q_5 \cup \cdots \cup Q_{t-1},$$

where

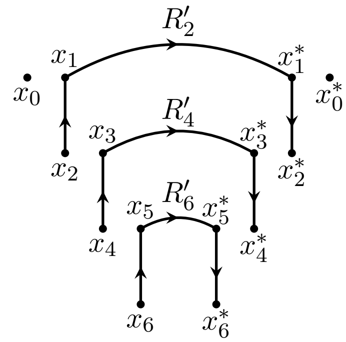
(i) $Q = [x_0 x_1 \cdots x_t x_t^* \cdots x_1^* x_0^*]$ is an alternating path (with t even) and $x_0 = u$ and $x_0^* = v$;
(ii) Q_i is a directed path from x_i to x_i^* for each $i = 1, 3, \ldots, t-1$; and
(iii) $Q_1, Q_3, \ldots, Q_{t-1}$ are vertex-disjoint paths and are each internally vertex disjoint from Q.



(A) L.



(B) Rung paths of L.



(C) Alternative rung paths of L.

FIGURE 1. A Ladder $L = Q \cup Q_1 \cup Q_3 \cup Q_5$

We call Q the alternating path of L.

- For $i = 0, 2, 4, \ldots, t-2$, we define $R_i \subseteq L$ to be the path $R_i := x_i x_{i+1} Q_{i+1} x_{i+1}^* x_i^*$ and $R_t := x_t x_t^*$. We call these the *rung paths* of L.
- For $i = 2, 4, \ldots, t$, define $R_i' \subseteq L$ to be the path $R_i' := x_i x_{i-1} Q_{i-1} x_{i-1}^* x_i^*$. We call these the *alternative rung paths* of L.

We say the ladder L is *embedded* in the cycle C if $R_i \subseteq C$ for all $i = 0, 2, 4, \ldots, t$.

It is relatively easy to construct ladders in robust expanders. First we show how a ladder embedded in a cycle can be used to absorb a path into the cycle.

Lemma 4.4. *Let D be a digraph and let $u, v \in V(D)$ be distinct vertices. Let $L \subseteq D$ be a ladder from u to v embedded in a cycle $C \subseteq D$. For any path $P \subseteq D$ from u to v that is internally vertex-disjoint from C there exists a cycle $C' \subseteq D$ such that*

(i) $P \subseteq C'$,
(ii) $V(L) \subseteq V(C')$,
(iii) *for any path $P' \subseteq C$ with $V(P') \cap V(L) = \emptyset$, we have $P' \subseteq C'$, and*
(iv) *if $x \in V(D)$ satisfies $x \notin V(C) \cup V(P)$, then $x \notin V(C')$.*

In particular, (i), (ii), (iii), and (iv) immediately imply

(v) $V(C) \cup V(P) = V(C')$.

Proof. Let $Q = [x_0 \cdots x_t x_t^* \cdots x_0^*]$ be the alternating path of L, and let Q_i be the corresponding paths of L from x_i to x_i^* for $i = 1, 3, \ldots, t-1$. Let $R_0, R_2, \ldots, R_t$ be the rung paths of L and $R_2', R_4', \ldots, R_t'$ the alternative rung paths. Set $R_0' := P$. We simply replace R_i with R_i' in C one at a time to obtain the desired cycle C'. We spell out the details of the induction below.

We define cycles $C_0, C_2, C_4, \ldots, C_t$ as follows. Set $C_0 := C$. By induction, we assume that C_{i-2} is a cycle with $R_0', \ldots, R_{i-2}', R_i, \ldots, R_t \subseteq C_{i-2}$ (implicitly noting these paths are vertex-disjoint) and that $\mathring{R}'_i = Q_{i-1}$ is vertex-disjoint from C_{i-2}. We obtain C_i by deleting R_i from C_{i-2} and replacing it with R'_i. Since R_i and R'_i are internally vertex-disjoint and both are paths from x_i to x_i^*, then C_i is a cycle. Clearly we have $R'_0, \dots, R'_i, R_{i+2}, \dots, R_t \subseteq C_i$. Since $\mathring{R}_i = \mathring{R}'_{i+2} = Q_{i+1}$ is vertex-disjoint from C_i (since we deleted $\mathring{R}_i$), then $\mathring{R}'_{i+2}$ is vertex-disjoint from C_i.

Thus by induction, we have that $C' := C_t$ is a cycle with $R'_0, \dots, R'_t \subseteq C'$. Therefore $P = R'_0 \subseteq C'$ proving (i). Furthermore, since

$$V(\bigcup_{\substack{i=0\\ i \text{ even}}}^{t} R_i) = V(\bigcup_{\substack{i=2\\ i \text{ even}}}^{t} R'_i) = V(L)$$

then $V(L) \subseteq V(C')$ proving (ii). In the above induction, we note that if $P' \subseteq C_{i-2}$ is a path vertex-disjoint from L, then $P' \subseteq C_i$, so by induction if $P' \subseteq C = C_0$ is a path vertex-disjoint from L, then $P' \subseteq C_t = C'$ proving (iii). Finally, we note that, in the above induction, for any vertex $x \in V(D) \setminus (V(P) \cup V(L))$, if $x \notin C_{i-2}$ then $x \notin C_i$, proving (iv) and completing the proof. □

From the previous lemma, we now see that embedding several carefully chosen ladders into a cycle can give us the absorbing structure we desire. The next definition makes precise what we mean by 'carefully' in the previous sentence.

Definition 4.5. Given a digraph D and distinct vertices $x, y, u, v \in V(D)$, we say that the ordered pair $(u, v) \in V(D)^{[2]}$ *covers* $(x, y) \in V(D)^2$ if $ux, yv \in E(D)$. Given $K \subseteq V(D)^{[2]}$ and $U \subseteq V(D)$, we say that K *d-covers* U if for every $(x, y) \in U^2$ there exist d distinct elements of K each of which covers (x, y). We say K is vertex-disjoint if no two elements of K share a vertex.

Our motivation for this definition is the following. Suppose L is a ladder from u to v embedded in a cycle C and P is a path from x to y that is vertex-disjoint from C, and suppose further that (u, v) covers (x, y). Then we can extend P to the path $uxPyv$ and use the previous lemma to absorb P into C. For a digraph D, if we can find a small set $K \subseteq V(D)^{[2]}$ which d-covers $V(D)$, then we might hope to construct vertex-disjoint ladders from u to v for each $(u, v) \in K$ and embed all those ladders into a cycle C. This structure would then have the property that any d vertex-disjoint paths of D (that are also vertex-disjoint from C) could be absorbed into C. This will be our absorbing structure.

Definition 4.6. Given a digraph D and $d \in \mathbb{N}$, a *d-absorber* S of D is a triple $S = (K, \mathcal{L}, C)$, where

- $K \subseteq V(D)^{[2]}$ is a set of vertex-disjoint pairs which d-covers $V(D)$,
- $\mathcal{L}$ is a set of vertex-disjoint ladders such that for each $(u, v) \in K$, we have a ladder $L \in \mathcal{L}$ from u to v,
- $C \subseteq D$ is a cycle such that each $L \in \mathcal{L}$ is embedded in C.

We sometimes abuse notation by also writing S for the subgraph $(\cup_{L \in \mathcal{L}} L) \cup C$ of D. Note that $V(C) = V(S)$.

It follows from Lemma 4.4 that a d-absorber can absorb d vertex-disjoint paths into its cycle.

Corollary 4.7. *Let D be a digraph and let $S \subseteq D$ be a d-absorber. Suppose $P_1, \ldots, P_r$ are vertex-disjoint paths in D that are also vertex-disjoint from $V(S)$ and $r \le d$. Then there exists a cycle C^* in D such that $V(C^*) = V(S) \cup V(P_1) \cup \cdots \cup V(P_r)$.*

Proof. Let x_i and y_i be such that P_i is a path from x_i to y_i for $i = 1, \ldots, r$ and let $S = (K, \mathcal{L}, C)$. Since S is a d-absorber, for each $i = 1, \ldots, r$, there exists $(u_i, v_i) \in K$ and $L_i \in \mathcal{L}$ such that (u_i, v_i) covers (x_i, y_i) and L_i is a ladder from u_i to v_i, and where $u_1, \ldots, u_r, v_1, \ldots, v_r$ are distinct vertices. For each i, observe that $Q_i := u_i x_i P_i y_i v_i$ is a path in D and that $Q_1, \ldots, Q_r$ are vertex disjoint.

Set $C_0 := C$ and assume by induction that there is a cycle $C_{i-1} \subseteq D$ with the property that $V(C_{i-1}) = V(C_0) \cup V(Q_1) \cup \cdots \cup V(Q_{i-1})$ and where $L_i, \ldots, L_r$ are embedded in C_{i-1}. Since L_i is a ladder from u_i to v_i embedded in C_{i-1} and Q_i is a path from u_i to v_i internally vertex-disjoint from C_{i-1}, Lemma 4.4 implies that there exists a cycle C_i such that $V(C_i) = V(C_{i-1}) \cup V(Q_i) = V(C) \cup V(Q_1) \cup \cdots \cup V(Q_i)$. Furthermore, by Lemma 4.4, since $L_{i+1}, \ldots, L_r$ are vertex-disjoint from L_i, and are embedded in C_{i-1}, so they are embedded in C_i.

This completes the induction step and so we obtain a cycle $C^* := C_r$ of D where $V(C_r) = V(C) \cup V(Q_1) \cup \cdots \cup V(Q_r) = V(S) \cup V(P_1) \cup \cdots \cup V(P_r)$. $\square$

The sequence of lemmas that follow show how to build a d-absorber in a robust expander. The first lemma shows how to find a d-cover in a digraph.

Lemma 4.8. *Let $\gamma \in (0,1)$ and $n, d \in \mathbb{N}$ with $d \ge 3$ and*

$$n > 10^4 d^2 \gamma^{-5} \log^2(100 d \gamma^{-2}).$$

If D is an n-vertex digraph with $\delta^0(D) \ge \gamma n$ and $U \subseteq V(D)$, then there exists a vertex-disjoint $K \subseteq V(D)^{[2]}$ with $|K| = \lceil 24 d \gamma^{-2} \log(24 d \gamma^{-2}) + 48 \gamma^{-2} \log n \rceil$ which d-covers U.

Proof. Set $m := \lceil 24 d \gamma^{-2} \log(24 d \gamma^{-2}) + 48 \gamma^{-2} \log n \rceil$ and construct K^* randomly by taking a set of m elements, each picked independently and uniformly at random, from $V(D)^{[2]}$; thus K^* may not be vertex-disjoint. We have that

$$\begin{aligned}
\mathbb{P}(K^* \text{ is vertex-disjoint}) &= \prod_{i=0}^{m-1} \left(\binom{n-2i}{2} \Big/ \left(\binom{n}{2} - i \right) \right) \\
&\ge \prod_{i=0}^{m-1} \left(\binom{n-2i}{2} \Big/ \left(n^2/2 \right) \right) = \prod_{i=0}^{2m-1} \left(1 - \frac{i}{n} \right) \\
&\ge 1 - \sum_{i=1}^{2m-1} \frac{i}{n} \ge 1 - \frac{2m^2}{n} > \frac{1}{2},
\end{aligned}$$

our choice of m and n and applying Proposition 3.7[1].

For $(x,y) \in U^2$, let $\mathrm{cov}(x,y)$ be the set of elements in $V(D)^{[2]}$ that cover (x,y). For a uniformly random element (u,v) of $V(D)^{[2]}$, set

$$p := \mathbb{P}((u,v) \in \mathrm{cov}(x,y)) \geq \frac{\gamma n(\gamma n-1)}{n(n-1)} \geq \frac{\gamma^2}{2},$$

where the last inequality follows by our choice of n. Let $E_{x,y}$ be the number of distinct elements of K^* that cover (x,y) so that $E_{x,y} \sim \mathrm{bin}(m,p)$. In particular,

$$\begin{aligned}
\mathbb{P}(E_{x,y} < d) &= \sum_{i=0}^{d-1} \binom{m}{i} p^i (1-p)^{m-i} \leq (1-p)^{m-d} \sum_{i=0}^{d-1} \frac{m^i}{i!} \\
&\leq em^d (1-p)^{m-d} \leq em^d (1-\gamma^2/2)^{m-d} \\
&\leq m^d \exp(-\gamma^2(m-d)/2) \leq m^d \exp(-\gamma^2 m/8) \leq n^{-2}/2.
\end{aligned}$$

by our choice of m and applying Proposition 3.7[2]. Let X be the number of elements of U^2 not d-covered by K^*. Then

$$\mathbb{P}(X \geq 0) \leq \mathbb{E}(X) = \sum_{(x,y)\in U^2} \mathbb{P}(E_{x,y} < d) \leq 1/2.$$

Therefore $\mathbb{P}(X = 0$ and K^* is vertex-disjoint$) > 0$. Hence there exists some vertex-disjoint $K \subseteq V(D)^{[2]}$ of size m that d-covers U. □

Next we show how to build a ladder in a robust expander.

Lemma 4.9. *Let $0 < \mu, \nu \leq \tau \leq \gamma/8 < 1/8$ and $n \in \mathbb{N}$ satisfying $\gamma > 8\tau$ and*

$$n \geq 57\mu^{-2} \max(\mu^{-1}, \nu^{-1}).$$

Let D be a robust (μ,ν,τ)-expander on n vertices with $\delta^0(D) \geq \gamma n$ and let u, v be distinct vertices of D. Then there exists a ladder L from u to v with $|L| \leq 3\mu^{-2}$ and where the alternating path of L has at most $2\mu^{-1}$ vertices.

Proof. By Lemma 4.2, we can find an alternating path $Q = [x_0 \cdots x_t x_t^* \cdots x_0^*]$, where $x_0 = u$, $x_0^* = v$, and $t \leq (\mu^{-1}+4)/2$ is even (so this alternating path has at most $\mu^{-1}+6 \leq 2\mu^{-1}$ vertices). Next, as in the definition of ladders, we construct vertex-disjoint paths $Q_1, Q_3, \ldots, Q_{t-1}$, where Q_i is from x_i to x_i^* and is vertex-disjoint from P (except at its end points). We do this using Lemma 3.6.

Let D' be the digraph obtained from D by deleting x_i and x_i^* for each even value $i = 0, \ldots, t$; thus we delete $t+2 \leq (\mu^{-1}+8)/2 \leq \mu^{-1}$ vertices and by our choice of large n, Proposition 3.3 implies[3] that D' is a robust $(\frac{1}{2}\mu, \frac{1}{2}\nu, \frac{16}{15}\tau)$-expander with $\delta^0(D') \geq \frac{15}{16}\gamma n$. By our choice of parameters and sufficiently large n, we can apply Lemma 3.6[4] with $r = t/2$ to obtain vertex disjoint paths $Q_1, Q_3, \ldots, Q_{t-1}$ in D' with each Q_i from x_i to x_i^* and of length at most $4\mu^{-1}+3$. As paths in D, these paths are also vertex-disjoint from Q except at their end-points.

Thus the union of the alternating path Q with the paths $Q_1, Q_3, \ldots, Q_{t-1}$ gives a ladder L from u to v. We have $|Q| \leq \mu^{-1}+6 \leq \mu^{-2}/2$, $|Q_i| \leq 4\mu^{-1}+3 \leq 9\mu^{-1}/2$ for each odd $i < t \leq \mu^{-1}$. Thus $|L| \leq 9\mu^{-2}/4 + |Q| \leq 3\mu^{-2}$. □

Next we show that we can build several ladders (between prescribed vertices) in a robustly expanding digraph (for a suitable choice of parameters).

Lemma 4.10. *Let $0 < \mu, \nu \leq \tau \leq \gamma/16 < 1/16$ and $n, k \in \mathbb{N}$ satisfying $n \geq 460k\mu^{-2}\max(\mu^{-1}, \nu^{-1})$. Let D be a robust (μ, ν, τ)-expander on n vertices with $\delta^0(D) \geq \gamma n$ and let $u_1, \ldots, u_k, v_1, \ldots, v_k$ be distinct vertices of D. Then we can construct vertex-disjoint ladders $L_1, \ldots, L_k$ from u_i to v_i such that $|L_i| \leq 12\mu^{-2}$ and $|P_i| \leq 4\mu^{-1}$, where P_i is the alternating path of L_i.*

Proof. By induction, suppose we have constructed vertex-disjoint ladders $L_1, \ldots, L_{i-1}$ for some $i \leq k$ where L_j is from u_j to v_j and $|L_j| \leq 12\mu^{-2}$ for all $j \leq i$, where the alternating path P_j of L_j satisfies $|P_j| \leq 4\mu^{-1}$ for all $j \leq i$, and where $L_1, \ldots, L_{i-1}$ are disjoint from $S_i := \{u_i, \ldots, u_k, v_i, \ldots, v_k\}$. Let D_i be obtained from D by deleting all the vertices of $L_1, \ldots, L_{i-1}$ and S_{i+1} (so $u_i, v_i \in V(D_i)$); thus the number of vertices deleted is at most

$$12\mu^{-2}(i-1) + 2(k-i) \leq 12k\mu^{-2} \leq \min(\mu, \nu)n/2,$$

where the last inequality follows from our choice of n. The inequality above together with Proposition 3.3 implies that D_i is a robust $(\frac{1}{2}\mu, \frac{1}{2}\nu, \frac{32}{31}\tau)$-expander with $\delta(D_i) \geq \frac{31}{32}\gamma n$. By our choice of parameters and n, we can apply[5] Lemma 4.9 to obtain a ladder from u_i to v_i in D_i with $|L_i| \leq 12\mu^{-2}$. and with alternating path P_i satisfying $|P_i| \leq 4\mu^{-1}$. By our choice of D_i, we see that $L_1, \ldots, L_i$ are vertex-disjoint ladders disjoint from S_{i+1}, where L_i is from u_i to v_i, completing the induction step and the proof. □

Finally we combine our various constructions to show how to build a d-absorber in a robustly expanding digraph.

Theorem 4.11. *Let $0 < \mu, \nu \leq \tau \leq \gamma/16 < 1/16$ and $n, d \in \mathbb{N}$ and set $T := \max(\mu^{-1}, \nu^{-1})$. Suppose $d \geq 3$ and*

$$n > \max\left(10^4 d^2\gamma^{-5}\log^2(100d\gamma^{-2}), 10^5 d\gamma^{-2}\mu^{-2}T\log(1500d\gamma^{-2}T)\right).$$

If D is a robust (μ, ν, τ)-expander on n vertices with $\delta^0(D) \geq \gamma n$ then we can find a d-absorber S in D such that $|V(S)| \leq 1600\mu^{-2}\gamma^{-2}(d\log(d\gamma^{-2}) + \log n)$.

Proof. For our choice of γ, d, n, we can apply Lemma 4.8 to D to obtain a vertex-disjoint $K \subseteq V(D)^{[2]}$ which d-covers $V(D)$, and moreover $m := |K| = \lceil 24d\gamma^{-2}\log(24d\gamma^{-2}) + 48\gamma^{-2}\log n \rceil$.

Next, by our choice of n, we can apply Lemma 4.10[6] (taking $k = m$) to construct a ladder from a to b for every $(a, b) \in K$ such that each ladder has at most $12\mu^{-2}$ vertices, the alternating path of each ladder has length at most $4\mu^{-1}$, and the ladders are vertex-disjoint.

Let $\mathcal{L} = \{L_1, \ldots, L_m\}$ be the set of constructed ladders and let $R_1, \ldots, R_s$ be the collection of all rung paths of all the ladders constructed; thus $s \leq 4\mu^{-1}m$. Let x_i and y_i be the initial and final vertices of R_i and let D' be the digraph obtained from D by deleting all internal vertices of $R_1, \ldots, R_s$. So we have deleted at most $12\mu^{-2}m \leq \max(\mu, \nu)n/2$ vertices[7]. Then D' is a $(\frac{1}{2}\mu, \frac{1}{2}\nu, \frac{32}{31}\tau)$-expander by Proposition 3.3. By our choice[8] of parameters and

n, we can apply Lemma 3.6 (with $n = |D'|$, $r = s$ and μ, ν, τ, γ replaced by $\frac{1}{2}\mu$, $\frac{1}{2}\nu$, $\frac{32}{31}\tau$, $\frac{31}{32}\gamma$) to find paths U_i from y_i to x_{i+1} for each $i = 1, \ldots, s$, where indices are understood to be modulo s and each path has length at most $4\mu^{-1}+3$. Then $C = x_1R_1U_1 \cdots R_sU_sx_1$ is a cycle in which all the ladders $L_1, \ldots, L_m$ are embedded. Thus $S = (K, \mathcal{L}, C)$ is a d-absorber of D. Also $|V(S)| \leq 12\mu^{-2}m + s(4\mu^{-1}+3) \leq 32\mu^{-2}m$. Recall that $m = \lceil 24d\gamma^{-2}\log(24d\gamma^{-2}) + 48\gamma^{-2}\log n\rceil \leq 25d\gamma^{-2}\log(24d\gamma^{-2}) + 48\gamma^{-2}\log n$ as $d\gamma^{-2}\log(24d\gamma^{-2}) > 1$. Therefore $|V(S)| \leq 1600\mu^{-2}\gamma^{-2}(d\log(d\gamma^{-2}) + \log n)$ as required. □

5. Rotation-extension: 1-factors with few cycles

Let D be a digraph. Throughout this section, a *factor* U of D refers to a 1-*factor* of D, i.e. a spanning subgraph of D in which every vertex has in- and outdegree 1. Thus a factor consists of a collection of vertex-disjoint cycles. We shall think of U interchangeably as both a set of vertex-disjoint cycles $U = \{C_1, \ldots, C_k\}$ and as the corresponding subgraph of $U = C_1 \cup \cdots \cup C_k$ of D. The purpose of this section is to show that any robustly expanding digraph with sufficiently high minimum in- and outdegree contains a factor with few cycles: our main tool is an interesting variation of the rotation-extension technique of Pósa [34]. The first lemma shows that any robustly expanding digraph with large enough minimum in- and outdegree has a factor.

Lemma 5.1. *Let $0 < \mu, \nu \leq \tau < \gamma < 1$ and $n \in \mathbb{N}$. If $D = (V, E)$ is an n-vertex robust (μ, ν, τ)-expander with $\delta^0(D) \geq \gamma n$ then D has a 1-factor.*

Proof. Let $V = \{v_1, \ldots, v_n\}$. Consider the bipartite (undirected) graph G whose vertex set is $X \cup Y$ where $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$ and x_iy_j is an edge of G if and only if $v_iv_j \in E$. Note that D contains a factor if and only if G has a perfect matching, so it is sufficient for us to verify Hall's condition for G. Indeed suppose $S \subseteq X$. If $|S| \leq \tau n$, then $|N_G(S)| = |N^+_D(S)| \geq \gamma n > \tau n \geq |S|$. If $|S| \geq (1-\tau)n$ then since every vertex in Y has degree at least $\gamma n > \tau n$ (since $\delta^-(D) \geq \gamma n$) then $|N_G(S)| = |Y| > |S|$. If $\tau n \leq |S| \leq (1-\tau)n$, then $|N_G(S)| = |N^+_D(S)| \geq |\mathrm{RN}^+_\nu(S)| \geq |S| + \mu n > |S|$. Hence by Hall's Theorem (see e.g. [4]) G has a perfect matching and hence D has a factor. □

We now introduce various notions we shall need. We say F is a *prefactor* of D if F can be obtained from a factor of D by deleting one edge. Thus F' consists of collection of cycles $C_1, \ldots, C_{k-1}$ together with a path P. We interchangeably think of F as the set $F = \{C_1, \ldots, C_{k-1}, P\}$ and as the subgraph $F = C_1 \cup \cdots \cup C_{k-1} \cup P$ of D. If P is a path from a vertex x to a vertex y, we say x is the *origin* of F and y is the *terminus* of F written $x = \mathrm{ori}(F)$ and $y = \mathrm{ter}(F)$ respectively. Every vertex v of D except $\mathrm{ori}(F)$ has a unique inneighbour in F which we denote by $F^-(v)$.

An *extension* of F (in D) is a prefactor F' of D obtained from F as follows. Assuming $F = \{C_1, \ldots, C_{k-1}, P\}$, $x = \mathrm{ori}(F)$ and $y = \mathrm{ter}(F)$, we pick any vertex $z \in N^+_D(y) \setminus \{x\}$:

(i) if $z \in V(P)$ we set $F' = \{C_1, \dots, C_{k-1}, C', P'\}$, where $C' = zPyz$ and $P' = xPz^-$, where $z^- := F^-(z)$ is the predecessor of z on P;
(ii) if $z \in V(C_i)$ for some i then set $F' = \{C_1, \dots, C_{i-1}, C_{i+1}, \dots, C_{k-1}, P'\}$, where $P' = xPyzC_iz^-$ and $z^- := F^-(z)$ is the predecessor of z in C_i.

We say F' is an extension of F along the edge yz. Notice that F and F' differ only in their path and in that one or the other contains an additional cycle. For case (i), we say F' is a *cycle-creating* extension of F and for case (ii) we say F' is a *cycle-destroying* extension of F. Notice also that for any extension F' of F, we have $\mathrm{ori}(F) = \mathrm{ori}(F')$ and that F' is uniquely determined from F by specifying the terminus of F'.

Here is the main step in obtaining a factor with few cycles.

Lemma 5.2. *Let $n \in \mathbb{N}$ and $\mu, \nu, \tau, \gamma, \xi \in (0,1)$ satisfying $\mu, \nu \leq \tau$, $\gamma > 2\tau + \xi$, $\xi < \frac{1}{4}\mu\nu$ and $n > 32\mu^{-2}\nu^{-1}$. Suppose $D = (V, E)$ is an n-vertex robust (μ, ν, τ)-expander with $\delta^0(D) \geq \gamma n$ and suppose that for each prefactor F of D, we have an associated set $B(F) \subseteq V$ of 'forbidden' vertices satisfying $\mathrm{ori}(F) \in B(F)$ and $|B(F)| \leq \xi n$. Fix any prefactor F^* of D. Then for all but at most τn vertices $y \in V$, there exists a sequence of prefactors $F_0 = F^*, F_1, \dots, F_t$ where $y = \mathrm{ter}(F_t)$ and for each $i = 1, \dots, t$ we have that F_i is an extension of F_{i-1} and $\mathrm{ter}(F_i) \notin B(F_{i-1})$.*

Proof. Let $x = \mathrm{ori}(F_0) = \mathrm{ori}(F^*)$. For each $r \in \mathbb{N}$, we define S_r to be the set of vertices that are *reachable* from F_0 by a sequence of at most r successive extensions while avoiding forbidden sets. More precisely, $y \in S_r$ if and only if there exists a sequence $F^* = F_0, F_1, \dots, F_{r'}$ with $r' \leq r$ such that $y = \mathrm{ter}(F_{r'})$, and for all $i = 1, \dots, r'$, F_i is an extension of F_{i-1} and $\mathrm{ter}(F_i) \notin B(F_{i-1})$. For each $y \in S_r$, we set $F_y^{(r)} := F_{r'}$ (if there are many choices of $F_{r'}$, we pick one arbitrarily). In particular $y \in \mathrm{ter}(F_y^{(r)})$.

In order to prove the lemma, it is sufficient to show that $|S_t| \geq (1-\tau)n$ for some t. Let us begin by noting that $|S_1| \geq (\gamma - \xi)n - 1 \geq 2\tau n - 1 \geq \tau n$, where the last two inequalities follow by our choice of parameters and n. To see the first inequality note that each distinct outneighbour w of $\mathrm{ter}(F_0)$ (except possibly x) gives an extension of F_0 with a distinct terminus $w^- := F_0^-(w)$, and each such w^- is in S_1 unless $w^- \in B(F_0)$.

We shall show that S_{r+1} contains most vertices in $\{F_0^-(w) : w \in \mathrm{RN}_\nu^+(S_r)\}$. Fix $r \geq 1$. For each $w \in \mathrm{RN}_\nu^+(S_r) \setminus \{x\}$, we say w is *good* if there exists $v \in S_r$ such that $w \in N^+(v)$, $F_v^{(r)-}(w) = F_0^-(w)$, and $F_0^-(w) \notin B(F_v^{(r)})$. Otherwise we say w is *bad*. Note that if $w \in \mathrm{RN}_\nu^+(S_r) \setminus \{x\}$ is good, then $F_0^-(w) \in S_{r+1}$. Indeed, let $F_0, \dots, F_{r'} = F_v^{(r)}$ be a sequence of extensions that show $v \in S_r$. Then extending $F_{r'} = F_v^{(r)}$ along the edge vw gives an extension F' whose terminus is $F_v^{(r)-}(w) = F_0^-(w) \notin B(F_v^{(r)})$. Thus the sequence $F_0, \dots, F_{r'}, F'$ shows that $F_0^-(w) \in S_{r+1}$.

Since the function $w \mapsto F_0^-(w)$ is injective, each $w \in \mathrm{RN}_\nu^+(S_r) \setminus \{x\}$ that is good corresponds to a distinct vertex of S_{r+1}. Thus, assuming $|S_r| \leq (1-\tau)n$, we have

$$|S_{r+1}| \geq |\mathrm{RN}^+_\nu(S_r)| - b - 1 \geq |S_r| + \mu n - b - 1,$$

where b is the number of bad vertices, which we now bound from above.

Let

$$A := \{(v,w) :\ v \in S_r,\ w \in \mathrm{RN}^+_\nu(S_r) \cap N^+(v),\ w \in B(F_v^{(r)})\}$$
$$B := \{(v,w) :\ v \in S_r,\ w \in \mathrm{RN}^+_\nu(S_r) \cap N^+(v),\ F_v^{(r)-}(w) \neq F_0^-(w)\}.$$

We have that $|A \cup B| \geq b\nu n$. To see this note that each bad vertex $w \in \mathrm{RN}^+_\nu(S_r)$ has at least νn inneighbours $v \in S_r$, and each such pair (v,w) belongs to $A \cup B$. On the other hand, we have $|A| \leq |S_r| \sum_{v \in S_r} |B(F_v^{(r)})| \leq |S_r| \xi n$ and $|B| \leq |S_r| r$. The first inequality is clear while second inequality follows from the following claim:

Claim: For each $v \in S_r$, there are at most r vertices w for which $F_v^{(r)-}(w) \neq F_0^-(w)$.

Proof. (of Claim) If F' is any extension of F then there is exactly one vertex w for which $F'^-(w) \neq F^-(w)$. Therefore if F' is obtained from F by a sequence of at most r successive extensions, then there are at most r vertices w for which $F'^-(w) \neq F^-(w)$, and the claim follows. □

Thus we have that $b\nu n \leq (\xi n + r)|S_r|$, whence $b \leq \nu^{-1}\xi n + \nu^{-1} r$. For each $r \leq 4\mu^{-1}$, if $|S_r| \leq (1-\tau)n$ then we have

$$|S_{r+1}| \geq |S_r| + \mu n - b - 1 \geq |S_r| + \mu n - \nu^{-1}\xi n - 4\mu^{-1}\nu^{-1} - 1 \geq |S_r| + \frac{1}{2}\mu n,$$

where the last inequality follows[9] by our choice of parameters and n. Thus for some $t \leq 2\mu^{-1}$, we have $|S_t| \geq (1-\tau)n$, as required. □

We give one piece of notation before proving the existence of factors with few cycles in robustly expanding digraphs. If P and Q are paths in a directed graph D, we write $Q \subseteq P$ if Q is an initial segment of P, i.e. P and Q have the same initial vertex and $P[V(Q)] = Q$. If $Q \subseteq P$ but $Q \neq P$, we write $Q \subset P$.

Theorem 5.3. *Let $n \in \mathbb{N}$ and $\mu, \nu, \tau, \gamma, \xi \in (0,1)$ satisfying $\mu, \nu \leq \tau$, $\gamma > 2\tau + \xi$, $\xi < \frac{1}{4}\mu\nu$, and $n > 32\mu^{-2}\nu^{-1}$. If $D = (V,E)$ is an n-vertex robust (μ,ν,τ)-expander with $\delta^0(D) \geq \gamma n$ then there exists a factor U^* of D which consists of at most $2\xi^{-1}$ cycles.*

Proof. By Lemma 5.1, D contains a factor U_0. Suppose U is any factor in which all cycles have length at least s for some $s < \frac{1}{2}\xi n$ and where exactly $\ell \geq 1$ cycles have length s. We claim that, using Lemma 5.2, we can obtain a factor U' from U in which all cycles have length at least s and at most $\ell - 1$ cycles have length s. Applying this claim iteratively, we eventually obtain a factor U^* of D in which every cycle has length at least $\frac{1}{2}\xi n$ and so this factor has at most $2\xi^{-1}$ cycles, proving the theorem.

It remains to prove the claim. Suppose $U = \{C_1, \dots, C_k\}$ where $C_1, \dots, C_k$ are the cycles of U in increasing order of length with $|C_1| = s < \frac{1}{2}\xi n$. Delete any edge of C_1 to form a path P and let $F_0 = \{C_2, \dots, C_k, P\}$ be the resulting prefactor of D, and let x be its origin.

For each prefactor F of D, let $B(F)$ denote the set of the first and last $\frac{1}{2}\xi n$ vertices on the path in F (if the path has at most ξn vertices then $B(F)$ is the set of all vertices on the path). Note that for the prefactor F_0, $|P| = |C_1| < \frac{1}{2}\xi n$ and so $B(F_0) = V(P)$. By Lemma 5.2, for at least $(1-\tau)n$ vertices $y \in V$, there exists a sequence of extensions $F_0, F_1, \dots, F_t$ such that F_i is an extension of F_{i-1}, $\mathrm{ter}(F_i) \notin B(F_{i-1})$, and $\mathrm{ter}(F_t) = y$. Since $|N^-(x) \setminus B(F_0)| \geq \gamma n - \xi n > \tau n$, we can choose y to be in $N^-(x) \setminus B(F_0)$.

Writing P_i for the path in the prefactor F_i, by our choice of $B(\cdot)$, it is straightforward to show by induction that $P = P_0 \subset P_i$ for all $i = 1, \dots, t$. Indeed, since $B(F_0) = V(P)$, F_1 must be a cycle-destroying extension of F_0, and so $P = P_0 \subset P_1$. Suppose $P \subset P_{i-1}$ for some $i > 1$ and let P'_{i-1} be the subpath of P_{i-1} consisting of the first $\frac{1}{2}\xi n$ vertices; in particular $P \subset P'_{i-1}$. If F_i is a cycle-creating extension of F_{i-1}, then since $V(P'_{i-1}) \subseteq B(F_{i-1})$, we must have $P_i \supseteq P'_{i-1} \supset P$. If F is a cycle-destroying extension of F_i the $P_i \supset P_{i-1} \supset P_0$.

Our choice of $B(\cdot)$ also ensures that if F_i is a cycle-creating extension of F_{i-1}, then the new cycle has length at least $\frac{1}{2}\xi n$.

Let $F_t = \{C'_1, \dots, C'_{k'}, P_t\}$, where $C'_1 \dots, C'_{k'}$ are cycles and we know P_t is a path from x to y of length more than $|P| = |C_1|$. Since $y \in N^-_D(x)$, we can turn P_t into a cycle C^* and form a factor $U' = \{C'_1 \dots, C'_{k'}, C^*\}$ of D. We have $|C^*| = |P_t| > |C_1| = s$.

Every cycle in U' that was created in the sequence of extensions $F_0, \dots, F_t$ has length at least $\frac{1}{2}\xi n > s$ and $|C^*| > |C_1| = s$. Every other cycle of U' was also a cycle of U. Hence every cycle in U' has length at least s and the number of cycles of length exactly s has been reduced by at least one. This proves the claim and the theorem. □

6. Hamiltonicity

We now combine Theorem 4.11, Corollary 4.7 and Theorem 5.3 to give the following result from which we deduce Theorem 1.3.

Theorem 6.1. *Let $0 < \mu, \nu \leq \tau \leq \gamma/16 < 1/16$ and let $n \in \mathbb{N}$. Set $T := \max(\mu^{-1}, \nu^{-1})$. Assueme*

$$n > 3 \cdot 10^7 \gamma^{-2} \mu^{-3} \nu^{-1} T \log(\gamma^{-2} \mu^{-2} \nu^{-1} T).$$

If D is an n-vertex robust (μ, ν, τ)-expander with $\delta^0(D) \geq \gamma n$, then for any $\min(\mu, \nu)n/2 \leq \ell \leq n$ and any $v \in V(D)$, D contains a cycle of length ℓ containing v.

Proof. Let $\xi := \mu\nu/32$ and $d := \lceil 2\xi^{-1} \rceil$. We begin by applying Theorem 4.11 to D to find a d-absorber S, where

$$|V(S)| \leq 1600\mu^{-2}\gamma^{-2}(d\log(d\gamma^{-2}) + \log n).$$

One can check that the conditions on the parameters and n are met[10].

Set $D' := D - V(S)$. By our choice[11] of n, we have $|V(S)| < \min(\mu,\nu)n/2$ and so by Proposition 3.3 D' is a robust $(\frac{1}{2}\mu, \frac{1}{2}\nu, \frac{32}{31}\tau)$-expander with $\delta^0(D') > \frac{31}{32}\gamma n$. By our choice of ξ and n, we can apply Theorem 5.3[12] to D' to obtain a factor in D' with at most $2\xi^{-1} \leq d$ cycles. By removing one edge from each of the cycles let $P_1, \dots, P_r$ be the resulting paths with $r \leq d$. Consider any $\min(\mu,\nu)n/2 \leq \ell \leq n$ and any $v \in V(D)$. Note D' contains vertex-disjoint paths $P'_1, \dots, P'_{r'}$ such that $r' \leq d$ and $|P'_1| + \ldots + |P'_{r'}| = \ell - |V(S)|$ and $v \in V(S) \cup \bigcup_{i \in r'} V(P'_i)$ (by removing appropriate vertices of $P_1, \dots, P_r$ if necessary). Applying Corollary 4.7, to these paths and the d-absorber S, we obtain a cycle C of length ℓ in D with $v \in V(C)$. □

Finally we can prove Theorem 1.3.

Proof. (of Theorem 1.3) Given that D is an n-vertex robust (ν,τ)-outexpander with $\delta^0(D) \geq \gamma n$, by Proposition 3.4, D is a robust (μ', ν', τ)-expander where $\mu' = \nu' = \nu^2/2$ (our choice of parameters ensures the conditions of Proposition 3.4 are met). By our choice[13] of n we can apply Theorem 6.1 to D to obtain a Hamilton cycle in D. □

We deduce Corollary 1.5 from Theorem 1.3, but first we need a lemma from [27].

Lemma 6.2. *Let $n \in \mathbb{N}$ and $\nu, \tau, \varepsilon \in (0,1)$ satisfy $\nu \leq \frac{1}{8}\tau^2$ and $\tau \leq \frac{1}{2}\varepsilon$. If D is an oriented graph on n vertices with $\delta^+(D) + \delta^-(D) + \delta(D) \geq 3n/2 + \varepsilon n$ then G is a robust (ν,τ)-outexpander.*

The explicit dependence between the parameters was not given in [27], but we have computed them and included them in the statement above.

Proof. [Proof of Corollary 1.5] Given an n-vertex oriented graph D with $\delta^0(D) \geq 3n/8 + \varepsilon n$, we have that $\delta(D) + \delta^+(D) + \delta^-(D) \geq 3n/2 + 4\varepsilon n$. So by Lemma 6.2, D is a robust $(\varepsilon^2/2, 2\varepsilon)$-outexpander. Finally, we apply Theorem 1.3 to obtain a Hamilton cycle. □

Proof. [Proof of Theorem 1.6] Let $\tau \leq \gamma/4$. By [29, Lemma 13], D is a robust (τ^2, τ)-outexpander and $\delta^0(D) \geq \gamma n$. Finally, we apply Theorem 1.3 to obtain a Hamilton cycle. □

7. Concluding remarks and an open problem

Let $D \sim \mathcal{D}(n,p)$ be the random digraph obtained by including each possible directed with probability p and making these choices independently. Note that if $p \geq (1+o(1))\log n/n$, then w.h.p. $D \sim \mathcal{D}(n,p)$ contain a Hamiltonian cycle. It is easy to show that if $p \geq (1+o(1))\log n/n$ then w.h.p. D is a robust outexpander. Theorem 1.3 only implies that if $p \geq C\sqrt[12]{\log n/n}$, then w.h.p. $D \sim \mathcal{D}(n,p)$ is Hamiltonian. Thus we would like to know whether Theorem 1.3 still holds when $\nu \geq (1+o(1))\log n/n$.

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Notes

[1]It suffices to show $\sqrt{n} > 2m$. So (crudely) sufficient that $\sqrt{n} \geq 50d\gamma^{-2}\log(24d\gamma^{-2}) + 96\gamma^{-2}\log n$. By Proposition 3.7 this holds if $\sqrt{n} \geq 288\gamma^{-2}(\log(96\gamma^{-2})+1)+150d\gamma^{-2}\log(24d\gamma^{-2})$ and for this to hold it is sufficient (assuming $d \geq 3$) that $\sqrt{n} \geq 300d\gamma^{-2}\log(100d\gamma^{-2})$ i.e. $n > 10^5 d^2\gamma^{-4}\log^2(100d\gamma^{-2})$.

[2]Note $n^2m^d\exp(-\gamma^2 m/8) \leq 1/2$ holds if $e^m \geq 2m^{8d\gamma^{-2}}n^{16\gamma^{-2}}$, which holds by Proposition 3.7 if $m \geq 24d\gamma^{-2}(\log(8d\gamma^{-2})+1)+48\gamma^{-2}\log 2n$, which holds if $m \geq 24d\gamma^{-2}\log(24d\gamma^{-2})+ 48\gamma^{-2}\log n$ as $d \geq 3$.

[3]The choice of n implies $\mu^{-1} \leq \frac{1}{2}\min(\mu,\nu)n \leq \frac{1}{16}\gamma n \leq \frac{1}{16}n$.

[4]Note that

$$n-(t+2) \geq n-\mu^{-1} \geq 56\mu^{-2}\max(\mu^{-1},\nu^{-1}) \geq 4(12\mu^{-1}+11)\mu^{-1}\max(\mu^{-1},\nu^{-1})$$

Thus conditions of Lemma 3.6 hold (with $r = t/2 \leq \mu^{-1}/2$ and μ,ν,τ,γ,n replaced by $\frac{1}{2}\mu$, $\frac{1}{2}\nu$, $\frac{16}{15}\tau$, $\frac{15}{16}\gamma, n-(t+2)$).

[5]We check the conditions of Lemma 4.9 with μ,ν,τ,γ,n replaced by $\frac{1}{2}\mu$, $\frac{1}{2}\nu$, $\frac{32}{31}\tau$, $\frac{31}{32}\gamma, |D_i|$. Note that $|D_i| \geq n - 12k\mu^{-2} \geq 459\mu^{-2}\max(\mu^{-1},\nu^{-1}) \geq 57(\mu/2)^{-2}\max((\mu/2)^{-1},(\nu/2)^{-1})$.

[6] We need to check that $n > 460m\mu^{-2}T$, which holds if

$$n \geq 460\left(25d\gamma^{-2}\log(24d\gamma^{-2}) + 48\gamma^{-2}\log n\right)\mu^{-2}T.$$

Proposition 3.7 implies the inequality above holds if

$$n \geq 10^5\gamma^{-2}\mu^{-2}T\log(70000\gamma^{-2}\mu^{-2}T) + 35000d\gamma^{-2}\mu^{-2}T\log(24d\gamma^{-2})$$

Since $d \geq 3$ and $\mu, \nu < 1/16$, the inequality above holds if $n \geq 10^5 d\gamma^{-2}\mu^{-2}T\log(1500d\gamma^{-2}T)$, as required.

[7] We need $n \geq 24\mu^{-2}T$, which is true by the previous note.

[8] Note that $11 \leq s \leq 4\mu^{-1}m$ and $n > 460m\mu^{-2}T$ by Note 6. Thus, $|D'| \geq n - 12\mu^{-2}m \geq 30\mu^{-2}T \geq 4(6s+11)\mu^{-1}T$.

[9] We note $\nu^{-1}\xi n \leq \frac{1}{4}\mu n$ and $1 \leq 4\mu^{-1}\nu^{-1} \leq \frac{1}{8}\mu n$.

[10] We must check that

$$n > \max\left(10^4 d^2\gamma^{-5}\log^2(100d\gamma^{-2}), 10^5 d\gamma^{-2}\mu^{-2}T\log(1500d\gamma^{-2}T)\right).$$

Note $d := \lceil 2\xi^{-1}\rceil \leq 3\xi^{-1}$ so $d \leq 100\mu^{-1}\nu^{-1}$. Also, $T \geq 16$. Thus it is sufficient

$$n > 10^5 d\gamma^{-2}\mu^{-2}T\log(1500d\gamma^{-2}T)$$

so sufficient to have $n > 10^7\gamma^{-2}\mu^{-3}\nu^{-1}T\log(150000\gamma^{-2}\mu^{-1}\nu^{-1}T)$. Since $\gamma^3\mu^{-1}\nu^{-1}T \geq 16^3$ and, we have $(150000\gamma^{-2}\mu^{-1}\nu^{-1}T) \leq (\gamma^{-2}\mu^{-1}\nu^{-1}T)^3$. So indeed the desired inequality holds.

[11] Need $n > 2T|V(S)|$ so sufficient that $n > 3200\mu^{-2}\gamma^{-2}T(d\log(d\gamma^{-2}) + \log n)$. By Proposition 3.7 this holds if

$$n > 9600\mu^{-2}\gamma^{-2}T\left(\log(3200\mu^{-2}\gamma^{-2}T) + 1\right) + 9600\mu^{-2}\gamma^{-2}Td\log(d\gamma^{-2}).$$

Recall that $d \leq 100\mu^{-1}\nu^{-1}$. The inequality above holds if $n > 10^7\mu^{-3}\nu^{-1}\gamma^{-2}T\log(10^5\mu^{-2}\gamma^{-2}\nu^{-1}T)$, which holds if $n > 3\cdot 10^7\mu^{-3}\nu^{-1}\gamma^{-2}T\log(\mu^{-2}\gamma^{-2}\nu^{-1}T)$.

[12] We check that $\frac{31}{32}\gamma > \frac{64}{31}\tau + \xi$, which holds (using $\gamma > 16\tau$ and $\xi \leq \mu \leq \tau$). We check that $\xi < \frac{1}{4}(\frac{1}{2}\mu)(\frac{1}{2}\nu) = \frac{1}{16}\mu\nu$, which holds. We check $n - |V(S)| > \max(32(\frac{1}{2}\mu)^{-2}(\frac{1}{2}\nu)^{-1}, \tau^{-1})$. Since $|V(S)| \leq \frac{1}{2}\mu n \leq \frac{1}{2}n$, it is sufficient that $n > 512\mu^{-2}\nu^{-1}$. This is clearly implied by our choice of n.

[13] Let $T' := \max((\mu')^{-1}, (\nu')^{-1}) = 2\nu^{-2}$. Note that

$$\begin{aligned}&3\cdot 10^7\gamma^{-2}(\mu')^{-3}(\nu')^{-1}T'\log(\gamma^{-2}(\mu')^{-2}(\nu')^{-1}T')\\ &= 3\cdot 2^5\cdot 10^7\gamma^{-2}\nu^{-10}\log(2^4\gamma^{-2}\nu^{-8}) \leq 2\cdot 10^9\gamma^{-2}\nu^{-10}\log(\gamma^{-2}\nu^{-8}) \leq 2\cdot 10^9\nu^{-12}\log(\nu^{-10}) \leq n.\end{aligned}$$

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