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AN EDGE-COLORED VERSION OF DIRAC'S THEOREM*

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Abstract. Let G be an edge-colored graph. The minimum color degree $\delta^c(G)$ of G is the largest integer k such that for every vertex v, there are at least k distinct colors on edges incident to v. We say that G is properly colored if no two adjacent edges have the same color. In this paper, we show that every edge-colored graph G with $\delta^c(G) \geq 2|G|/3$ contains a properly colored 2-factor. Furthermore, we show that for any $\varepsilon > 0$ there exists an integer n_0 such that every edge-colored graph G with $|G| = n \geq n_0$ and $\delta^c(G) \geq (2/3 + \varepsilon)n$ contains a properly colored cycle of length ℓ for every $3 \leq \ell \leq n$. This result is best possible in the sense that the statement is false for $\delta^c(G) < 2n/3$.

Key words. proper edge-coloring, 2-factor, Hamiltonian cycle

AMS subject classifications. 05C15, 05C38

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1. Introduction. A classical theorem of Dirac [7] states that every graph G on $n \geq 3$ vertices with minimum degree $\delta(G) \geq n/2$ contains a Hamiltonian cycle. In this paper, we generalize this result to edge-colored graphs.

An *edge-colored graph* is a graph G with an edge coloring c of G. We say that G is *properly colored* if no two adjacent edges of G have the same color. Moreover, G is said to be *rainbow* if all edges have distinct colors. We consider the following analogue of degree for edge-colored graphs G. Given a vertex $v \in V(G)$, the *color degree* $d^c(v)$ is the number of distinct colors of edges incident to v. The *minimum color degree* $\delta^c(G)$ of an edge-colored graph G is the minimum $d^c(v)$ over all vertices v in G.

One elementary result of graph theory states that every graph G with $\delta(G) \geq 2$ contains a cycle. However, for all $k \geq 2$, there exist edge-colored graphs G with $\delta^c(G) \geq k$ that do not contain any properly colored cycles. Grossman and Häggkvist [9] gave a sufficient condition for the existence of properly colored cycles in edge-colored graphs with two colors, which does not depend on $\delta^c(G)$. Later on, Yeo [16] extended the result to edge-colored graphs with any number of colors.

One natural generalization of Dirac's theorem is to determine the minimum color degree threshold for the existence of a rainbow Hamiltonian cycle. However, such thresholds do not exist for all n. Indeed, for every even integer $n > 3$, there exists a properly colored complete graph K_n^c on n vertices using exactly $n - 1$ colors. Note that $\delta^c(K_n^c) = n - 1$, but K_n^c does not contain a rainbow Hamiltonian cycle. In fact, for each $p \geq 2$, there is a properly colored K_{2p}^c that does not contain any rainbow Hamiltonian path (see [13]). Hence, a better question would be to ask for the minimum color degree threshold for the existence of a properly colored Hamiltonian cycle.

The problem of finding properly colored spanning subgraphs in edge-colored complete graphs K_n^c has been investigated by numerous researchers. Bang-Jensen, Gutin,

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and Yeo [4] proved that if K_n^c contains a properly colored 2-factor, then K_n^c also contains a properly colored Hamiltonian path. A graph G is said to be a 1-*path-cycle* if G is a vertex-disjoint union of at most one path and a number of cycles. Feng et al. [8] showed that K_n^c contains a properly colored Hamiltonian path if and only if it contains a spanning properly colored 1-path-cycle. We define $\Delta_{\text{mon}}(K_n^c)$ to be the maximum number of edges of the same color incident to the same vertex. In other words, $\Delta_{\text{mon}}(K_n^c) = \max \Delta(H)$ over all monochromatic subgraphs $H \subseteq K_n^c$, where an edge-colored graph G is *monochromatic* if all edges have the same color. Notice that $\Delta_{\text{mon}}(K_n^c) + \delta^c(K_n^c) \leq n$. Bollobás and Erdős [5] proved that if $\Delta_{\text{mon}}(K_n^c) \leq n/69$, then K_n^c contains a properly colored Hamiltonian cycle. They further conjectured that $\Delta_{\text{mon}}(K_n^c) < \lfloor n/2 \rfloor$ suffices. Their result was subsequently improved by Chen and Daykin [6], Shearer [15] and Alon and Gutin [1]. In [12], the author showed that for any $\varepsilon > 0$, every K_n^c with $\Delta_{\text{mon}}(K_n^c) < (1/2 - \varepsilon)n$ contains a properly colored Hamiltonian cycle, provided n is large enough. Therefore, the conjecture of Bollobás and Erdős is true asymptotically. For a survey regarding properly colored subgraphs in edge-colored graphs, we recommend Chapter 16 of [3].

Let G be an edge-colored graph (not necessarily complete). Li and Wang [10] proved that G contains a properly colored path of length $2\delta^c(G)$ or a properly colored cycle of length at least $2\delta^c(G)/3$. In [11], the author improved this result by showing that G contains a properly colored path of length $2\delta^c(G)$ or a properly colored cycle of length at least $\delta^c(G)+1$. In the same paper, the author showed that every connected edge-colored graph G contains a properly colored Hamiltonian cycle or a properly colored path of length at least $6\delta^c(G)/5-1$. Furthermore, the author also conjectured the following.

CONJECTURE 1 (see [11]). *Every connected edge-colored graph G contains a properly colored Hamiltonian cycle or a properly colored path of length $\lfloor 3\delta^c(G)/2 \rfloor$.*

If the conjecture is true, then every edge-colored graph G with $\delta^c(G) \geq 2|G|/3$ contains a properly colored Hamiltonian cycle. In this paper, we prove the following weaker result that $\delta^c(G) \geq 2|G|/3$ implies the existence of a properly colored 2-factor in G.

THEOREM 1.1. *Every edge-colored graph G with $\delta^c(G) \geq 2|G|/3$ contains a properly colored 2-factor.*

We also show that if $\delta^c(G) \geq (2/3 + \varepsilon)|G|$ and $|G|$ is large enough, then G does indeed contain a properly colored Hamiltonian cycle. Furthermore, G contains a properly colored cycle of length ℓ for all $3 \leq \ell \leq |G|$.

THEOREM 1.2. *For any $\varepsilon > 0$, there exists an integer n_0 such that every edge-colored graph G with $\delta^c(G) \geq (2/3 + \varepsilon)|G|$ and $|G| \geq n_0$ contains a properly colored cycle of length ℓ for all $3 \leq \ell \leq |G|$.*

Given a subgraph H in G, we write $G - H$ for the (edge-colored) subgraph obtained from G by deleting all edges in H. For an edge-colored graph G, let $\delta_1^c(G)$ be the minimum $\delta(G - H)$ over all monochromatic subgraphs H in G. Note that $\delta_1^c(G) \geq \delta^c(G) - 1$. Note that for an edge-colored complete graph K_n^c, we have $\delta_1^c(K_n^c) + \Delta_{\text{mon}}(K_n^c) = n - 1$. We prove the following stronger statement, which implies Theorem 1.2.

THEOREM 1.3. *For any $\varepsilon > 0$, there exists an integer n_0 such that every edge-colored graph G with $\delta_1^c(G) \geq (2/3 + \varepsilon)|G|$ and $|G| \geq n_0$ contains a properly colored cycle of length ℓ for all $3 \leq \ell \leq |G|$.*

We now outline the proof of Theorem 1.3 for the case when $\ell = |G|$, i.e., the existence of a properly colored Hamiltonian cycle. The proof adapts the absorption technique introduced by Rödl, Ruciński, and Szemerédi [14], which was used to tackle Hamiltonicity problems in hypergraphs. The proof is divided into two main steps. In the first step, we find (by Lemma 4.1) a small "absorbing cycle" C in G. The absorbing cycle C has the property that given any small number of vertex-disjoint properly colored paths $P_1, P_2, \dots, P_k$ in G with $V(C) \cap V(P_i) = \emptyset$ for each $i \leq k$, there exists a properly colored cycle C' in G with $V(C') = V(C) \cup \bigcup_{1 \leq i \leq k} V(P_i)$. Thus, we have reduced the problem to covering the vertex set $V(G) \backslash V(C)$ with a small number of vertex-disjoint properly colored paths. We remove the vertices of C from G and let G' be the resulting graph. Since C is small, we may assume that $\delta_1^c(G') \geq (2/3+\varepsilon')|G'|$ for some small $\varepsilon' > 0$. Then we find a properly colored 2-factor in G' using Lemma 3.1 such that every cycle has length at least $\varepsilon'|G'|/2$. (Although Theorem 1.1 also implies that G' contains a properly colored 2-factor, there is no bound on the lengths of the cycles.) Hence $V(G')$ can be covered by at most $2/\varepsilon'$ vertex-disjoint properly colored paths $P_1, P_2, \dots, P_k$. By the "absorbing" property of C, there is a properly colored cycle C' with $V(C') = V(C) \cup \bigcup_{1 \leq i \leq k} V(P_i) = V(G)$. Therefore, C' is a properly colored Hamiltonian cycle as required.

The paper is organized as follows. In the next section, we set up some basic notation and give the extremal example for Conjecture 1. Section 3 is devoted to finding properly colored 2-factors and contains the proof of Theorem 1.1. The absorbing cycle is constructed in section 4. Finally, Theorem 1.3 is proved in section 5.

2. Notation and extremal example. Throughout this paper, unless stated otherwise, G will be assumed to be an edge-colored graph with edge-coloring c. For an edge xy, we denote its color by $c(xy)$. For $v \in V(G)$, we denote by $N_G(v)$ the neighborhood of v in G. If the graph G is clear from the context, we omit the subscript. Let $U \subseteq V(G)$. We write $G[U]$ for the (edge-colored) subgraph of G induced by U. We also write $G \backslash U$ for the subgraph obtained from G by deleting all vertices in U, i.e., $G \backslash U = G[V(G) \backslash U]$. For edge-disjoint (edge-colored) graphs G and H, we denote by $G+H$ the union of G and H. Further, we write $G-H+H'$ to mean $(G-H)+H'$.

Let $U, W \subseteq V(G)$ not necessarily disjoint. Whenever we define an auxiliary bipartite graph H with vertex classes U and W, we mean H has vertex classes U' and W', where U' is a copy of U and W' is a copy of W. Hence, U and W are considered to be disjoint in H. Given an edge uw in H, we say $u \in U$ and $w \in W$ to mean $u \in U'$ and $w \in W'$.

In this paper, every path will be assumed to be directed. Hence, the paths $v_1 v_2 \cdots v_\ell$ and $v_\ell v_{\ell-1} \cdots v_1$ are considered to be different for $\ell \geq 2$. We also allow a single vertex to be a (trivial) path. Given vertex-disjoint paths $P_1, \dots, P_s$, we define the path $P_1 \cdots P_s$ to be the concatenation of $P_1, \dots, P_s$. For example, if $P = v_1 \cdots v_\ell$ and $Q = w_1 \cdots w_{\ell'}$ are vertex-disjoint paths and x is a vertex not in $V(P) \cup V(Q)$, then PxQ denotes the path $v_1 \cdots v_\ell x w_1 \cdots w_{\ell'}$.

Let H be a union of vertex-disjoint directed cycles. For each $y \in V(H)$, let C_y be the cycle in H that contains y. Denote by y_+ and y_- the successor and ancestor of y in C_y, respectively. Write y_{--} for $(y_-)_-$ and y_{++} for $(y_+)_+$. For distinct vertices y, z in a cycle C in H, define yC^+z and yC^-z to be the paths $yy_+ \cdots z_-z$ and $yy_- \cdots z_+z$ in C, respectively. If $y = z$, then set $yC^+y = y = yC^-y$.

In [11], the author gave a construction to show that Conjecture 1 is best possible for infinitely many values of $|G|$ and $\delta^c(G)$. The same construction also shows that Theorem 1.1 is best possible. We include it here for completeness.

PROPOSITION 2.1. *For $n, \delta \in \mathbb{N}$ with $\delta < 2n/3$, there exists a connected edge-colored graph G on n vertices with $\delta^c(G) = \delta$ which does not contain a union of* *vertex-disjoint properly colored cycles spanning more than $3\delta^c(G)/2$ vertices. In particular, G does not contain a properly colored 2-factor or a properly colored Hamiltonian cycle. Moreover, all properly colored paths in G have length at most $3\delta^c(G)/2$.*

Proof. Let $n, \delta \in \mathbb{N}$ with $3\delta < 2n$. Let $X = \{x_1, x_2, \ldots, x_\delta\}$ and Y be vertex sets with $|Y| = n - \delta$ and $X \cap Y = \emptyset$. Define G to be the edge-colored graph on the vertex set $X \cup Y$ as follows. Let $G[X]$ be a rainbow complete graph and let $G[Y]$ be a set of independent vertices. For each $1 \leq i \leq \delta$, add an edge of new color c_i between x_i and y for each $y \in Y$. By our construction, $d^c(v) = |X| = \delta$ for all $v \in V(G)$ and so $\delta^c(G) = \delta$.

Let C be a properly colored cycle in G. Arbitrarily orient C into a directed cycle. Note that every vertex $y \in V(C) \cap Y$ must be immediately followed by two consecutive vertices in X, so $|X \cap V(C)| \geq 2|Y \cap V(C)|$. Therefore, if $\mathcal{C}$ is a collection of vertex-disjoint properly colored cycles in G, then $\mathcal{C}$ spans at most $|X| + |X|/2 = 3\delta/2$ vertices. The "moreover" statement is proved by a similar argument. $\square$

3. Properly colored 2-factors. First we prove Theorem 1.1; that is, every edge-colored graph with $\delta^c(G) \geq 2|G|/3$ contains a properly colored 2-factor. We now present a sketch of the proof. Suppose that G is an edge-colored graph on n vertices. Let H be a properly colored subgraph in G consisting of vertex-disjoint cycles such that $|H|$ is maximal. If $V(H) = V(G)$, then we are done. Hence we may assume that there exists $x \in V(G) \backslash V(H)$. Define a *color neighborhood* $N^c(x)$ of x in G to be a maximal subset of neighbors of x in G such that $c(xy) \neq c(xz)$ for all distinct $y, z \in N^c(x)$. (The choice of $N^c(x)$ will be specified later.) Assume that we are in the ideal case that $N^c(x) \cap V(H) = \emptyset$. If yz is an edge and $c(xy) \neq c(yz) \neq c(xz)$ for some distinct $y, z \in N^c(x)$, then $xyzx$ is a properly colored triangle and so $H + xyzx$ contradicts the maximality of $|H|$. Hence, we may assume that for every $y, z \in N^c(x)$, yz is not an edge, or $c(zy) = c(xy)$, or $c(zy) = c(xz)$. By an averaging argument, there exists a vertex $y_0 \in N^c(x)$ such that the number of $z \in N^c(x) \backslash \{y_0\}$ such that either $y_0 z \notin E(G)$ or $c(y_0 z) = c(xy_0)$ is at least $(|N^c(x)| - 1)/2 \geq (\delta^c(G) - 1)/2$. Recall that there are at least $\delta^c(G) - 1$ neighbors v of y_0 with $c(y_0 v) \neq c(xy_0)$. This means that $n = |V(G)| \geq (\delta^c(G) - 1)/2 + (\delta^c(G) - 1) + |\{x, y_0\}| > 3\delta^c(G)/2$ and so $\delta^c(G) < 2n/3$.

Proof of Theorem 1.1. Let G be an edge-colored graph on n vertices with edge-coloring c. Set $\delta = \delta^c(G)$. Suppose that G does not contain a properly colored 2-factor. We will show that $\delta < 2n/3$. This is trivial if $n \leq 2$, so we may assume that $n \geq 3$. By deleting edges in G, we may assume that G is edge-minimal, that is, any additional edge deletion would lead to a decrease in $d^c(v)$ for some vertex v in G. Suppose that G' is a monochromatic subgraph of G, which is not isomorphic to a disjoint union of stars. Then there exists an edge uv in G' with $d_{G'}(u), d_{G'}(v) \geq 2$. Set $G'' = G - uv$. Note that $d^c_{G''}(x) = d^c_G(x)$ for all vertices $x \in V(G)$. This contradicts the edge-minimality of G. Therefore, every monochromatic subgraph G' in G is a disjoint union of stars. Let H be a properly colored subgraph in G consisting of vertex-disjoint cycles such that $|H|$ is maximal. Note that $|H| < n$. Arbitrarily orient each cycle in H into a directed cycle.

Fix $x \in V(G) \backslash V(H)$. We will choose $N^c(x)$ to be a maximal subset of neighbors of x in G such that $c(xy_1) \neq c(xy_2)$ for all distinct $y_1, y_2 \in N^c(x)$. Note that $|N^c(x)| = d^c(x)$. Obtain $N^c(x)$ as follows. If there are at least two vertices y_1 and y_2 such that $c(xy_1) = c(xy_2)$, then we choose $y \in \{y_1, y_2\}$ to be in $N^c(x)$ according to the following order of preferences (if there are still at least two choices for y, pick one arbitrarily):

(a) $y \notin V(H)$;
(b) $y \in V(H)$ and $c(xy) = c(yy_+)$;
(c) $y \in V(H)$ and $c(xy) = c(yy_-)$;
(d) $y \in V(H)$ and $xy_+ \notin E(G)$;
(e) $y \in V(H)$ and $c(xy) \neq c(xy_+)$;
(f) $y \in V(H)$ and $c(xy) = c(xy_+)$.

Note that if $y \in N^c(x)$ satisfies property (f), then $c(xy') = c(xy)$ for every vertex y' in C_y. Define $N^c_{(a)}(x)$ to be the set of vertices in $N^c(x)$ chosen due to (a), and define $N^c_{(b)}(x), \dots, N^c_{(f)}(x)$ similarly. Next, set

$$\begin{aligned} W &= N^c_{(a)}(x),\\ R' &= \{y_+ : y \in N^c_{(b)}(x) \cup N^c_{(d)}(x) \cup N^c_{(e)}(x) \cup N^c_{(f)}(x)\},\\ S' &= \{y_- : y \in N^c_{(c)}(x)\},\\ R &= R' \backslash S', \qquad S = S' \backslash R', \qquad T = R' \cap S'. \end{aligned}$$

We have

$$|R| + |S| + 2|T| + |W| = |N^c(x)| \geq \delta. \tag{3.1}$$

Note that $R' \cup S' \subseteq V(H)$. If $y \in R'$, then $y_- \in N^c_{(b)}(x) \cup N^c_{(d)}(x) \cup N^c_{(e)}(x) \cup N^c_{(f)}(x) \subseteq N^c(x)$. For $y_- \in N^c_{(b)}(x)$, we have $c(y_-y_{--}) \neq c(y_-y) = c(xy_-)$ since C_{y_-} is a properly colored cycle. Also, for $y_- \in N^c_{(d)}(x) \cup N^c_{(e)}(x) \cup N^c_{(f)}(x)$, we have $c(xy_-) \neq c(y_-y_{--})$ since $y_- \notin N^c_{(c)}(x)$. Thus, together with the definitions of S' and $N^c_{(c)}(x)$, we deduce that

(i) if $y \in R'$, then $y_- \in N^c(x)$ and $c(y_-y_{--}) \neq c(xy_-)$, and
(ii) if $y \in S'$, then $y_+ \in N^c(x)$ and $c(y_+y_{++}) \neq c(y_+y) = c(xy_+)$.

Let F be the edge-colored subgraph of G induced by $W \cup R \cup S \cup T$. For $y \in V(F)$, define the vertex color $c(y)$ to be

$$c(y) = \begin{cases} c(xy) & \text{if } y \in W,\\ c(yy_+) & \text{if } y \in R,\\ c(yy_-) & \text{if } y \in S,\\ c_0 & \text{if } y \in T, \end{cases}$$

where c_0 is a new color that does not appear in G. The following claim concerns the colors of the edges of F.

Claim 1. *If $yz \in E(F)$, then $c(yz) = c(y)$ or $c(yz) = c(z)$. In particular, T is an independent set in G.*

Proof of Claim 1. Suppose the claim is false, so there exists an edge $yz \in E(F)$ with $c(yz) \neq c(y)$ and $c(yz) \neq c(z)$. We will show that there is a properly colored subgraph H' in G consisting of vertex-disjoint cycles with $V(H') = V(H) \cup \{x, y, z\}$, contradicting the maximality of $|H|$.

If $y, z \in W \subseteq N^c(x)$, then $c(xy) \neq c(xz)$. Since $c(xy) = c(y) \neq c(yz) \neq c(z) = c(xz)$, then $xyzx$ is a properly edge-colored triangle in G. Hence $H' = H + xyzx$ contradicts the maximality of $|H|$.

If $y \in W$ and $z \in R$, then $c(xy) = c(y) \neq c(yz) \neq c(z) = c(zz_+)$. By (i) we have $z_- \in N^c(x)$ and $c(xz_-) \neq c(z_-z_{--})$. Moreover, we have $c(xz_-) \neq c(xy)$ since $y, z_- \in N^c(x)$. Therefore, $C' = xyzC^+_z z_-x$ is a properly colored cycle. We obtain

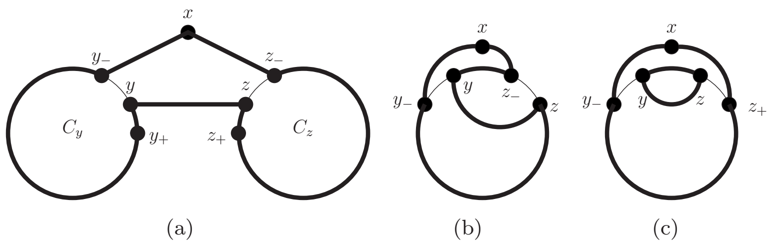


FIG. 3.1. *Properly colored cycles used in the proof of Claim* 1.

a contradiction by setting $H' = H - C_z + C'$. Similarly, if $y \in W$ and $z \in S$, then we obtain a contradiction by considering $C'' = xyzC_z^- z_+ x$ instead of C'. If $y \in W$ and $z \in T$, then $c(xy) = c(y) \neq c(yz)$. Since C_z is a properly colored cycle, we have $c(yz) \neq c(zz_+)$ or $c(yz) \neq c(zz_-)$. Recall that $T = R' \cap S'$, so $c(xz_-) \neq c(z_- z_{--})$ and $c(xz_+) \neq c(z_+ z_{++})$ by (i) and (ii). Also, $y, z_-, z_+ \in N^c(x)$, so $c(xz_-) \neq c(xy) \neq c(xz_+)$. Hence, $H - C_z + C'$ or $H - C_z + C''$ would imply a contradiction.

Therefore, we may assume that $y, z \in R' \cup S' \subseteq V(H)$. In order to prove the claim, it is enough to show that there exist at most two vertex-disjoint properly colored cycles C', C'' spanning $\{x\} \cup V(C_y) \cup V(C_z)$ (which would then imply that $H' = (H - C_y - C_z) + C' + C''$ is a union of properly colored cycles with $|H'| = |H| + 1$, a contradiction).

Suppose that C_y and C_z are distinct. If $y, z \in R$, then by (i) we have

$$y_-, z_- \in N^c(x), \qquad c(y_- y_{--}) \neq c(xy_-), \qquad c(xz_-) \neq c(z_- z_{--}). \tag{3.2}$$

Since $y_-, z_- \in N^c(x)$, we have $c(xy_-) \neq c(xz_-)$. Note that $c(y) \neq c(yz) \neq c(z)$, so

$$c(yy_+) \neq c(yz) \neq c(zz_+). \tag{3.3}$$

Hence, $C' = xy_- C_y^- yzC_z^+ z_- x$ (see Figure 3.1(a)) is a properly colored cycle with vertex set $V(C_y + C_z) \cup \{x\}$. Notice that C' is a properly colored cycle if both (3.2) and (3.3) hold. If $y, z \in R'$, then (3.2) holds by (i). Therefore, we may assume without loss of generality that $y \in S$ or $y \in T$ with $c(yy_+) = c(yz)$. Since $T \subseteq S'$ and the cycle C_y is properly colored, we have $y \in S'$ and $c(yy_-) \neq c(yy_+) = c(yz)$. By (ii) and reversing the orientation of C_y, we have $y_- \in N^c(x)$, $c(y_- y_{--}) \neq c(xy_-)$ and $c(yy_+) \neq c(yz)$. Similarly, by reversing the orientation of C_z if necessary, we may assume that both (3.2) and (3.3) hold and so we derive a contradiction.

Finally, suppose that $C = C_y = C_z$. If $y, z \in R$, then $c(yy_+) = c(y) \neq c(yz) \neq c(z) = c(zz_+)$. In particular, $y_+ \neq z$ and $z_+ \neq y$. By (i), $xy_- C^- zyC^+ z_- x$ (see Figure 3.1(b)) is a properly colored cycle with vertex set $V(C) \cup \{x\}$, so we are done. If $y \in R$ and $z \in S$, then $y_- \neq z$; otherwise $xyC^+ z$ is a properly colored cycle by (i) and (ii). However, both $xy_- C^- z_+ x$ and $yC^+ zy$ (see Figure 3.1(c)) are properly colored cycles spanning $\{x\} \cup V(C)$. If $y, z \in S$, then by reversing the orientation of the cycle C we conclude that $y, z \in R$ and so we are done. If $y \in T$ or $z \in T$, then we apply the following "blindness" argument. Suppose that $y \in T$. Since the cycle C is properly colored, $c(yz) \neq c(yy_+)$ or $c(yz) \neq c(yy_-)$. We treat y to be in R if $c(yz) \neq c(yy_+)$ and y to be in S otherwise. We apply a similar treatment when $z \in T$. Hence, we are back in the case when $y, z \in R \cup S$. This completes the proof of Claim 1. ∎

CLAIM 2. *For each vertex $y \in V(F)$ there exists a vertex $u = u(y) \notin V(F)$ such that either $yu \notin E(G)$ or $c(yu) = c(y)$.*

Proof of Claim 2. If $y \in W$, then $c(y) = c(xy)$ and so we are done by setting $u(y) = x$.

If $y \in R'$ with $y_- \in N^c_{(\mathrm{d})}(x)$, then xy is not an edge. Set $u(y) = x$ for $y \in R$ with $y_- \in N^c_{(\mathrm{d})}(x)$.

If $y \in R'$ with $y_- \in N^c_{(\mathrm{f})}(x)$, then $c(xy_-) = c(xz)$ for $z \in V(C_{y_-})$ by the remark after the definition of $N^c(x)$. Hence, $V(C_y) \cap N^c(x) = \{y_-\}$. This implies that $V(C_y) \cap V(F) = \{y\}$ and so $y_+ \notin V(F)$. Set $u(y) = y_+$ for $y \in R$ with $y_- \in N^c_{(f)}(x)$.

Suppose that $y \in R'$ with $y_- \notin N^c_{(\mathrm{d})}(x) \cup N^c_{(\mathrm{f})}(x)$ (i.e., $y_- \in N^c_{(\mathrm{b})}(x) \cup N^c_{(\mathrm{e})}(x)$). We may assume that $xy \in E(G)$ or else we set $u(y) = x$. If $y_- \in N^c_{(\mathrm{e})}(x)$, then $c(xy) \neq c(xy_-)$ by definition. If $y_- \in N^c_{(\mathrm{b})}(x)$, then $c(xy_-) = c(yy_-)$. Recall that every monochromatic subgraph of G is a disjoint union of stars. Hence, $c(xy) \neq c(xy_-)$. In summary, we have $c(xy) \neq c(xy_-)$ for $y \in R'$ with $y_- \notin N^c_{(\mathrm{d})}(x) \cup N^c_{(\mathrm{f})}(x)$. If $c(xy) \neq c(yy_+)$, then (i) implies that $xyC^+_y y_- x$ is a properly colored cycle and so we can enlarge H, a contradiction. Hence,

$$c(xy) = c(yy_+) \text{ for all } y \in R' \cap N_G(x) \text{ with } y_- \notin N^c_{(\mathrm{d})}(x) \cup N^c_{(\mathrm{f})}(x). \tag{3.4}$$

Set $u(y) = x$ for all $y \in R \cap N_G(x)$ with $y_- \notin N^c_{(\mathrm{d})}(x) \cup N^c_{(\mathrm{f})}(x)$.

Suppose that $y \in S'$, so (ii) implies that $c(y_+y_{++}) \neq c(xy_+)$. We may assume that $xy \in E(G)$ or else we set $u(y) = x$. Recall that every monochromatic subgraph of G is a disjoint union of stars. Hence, $c(xy) \neq c(xy_+)$. If $c(xy) \neq c(yy_-)$, then $xy_+C^+_y y_- x$ is a properly colored cycle and so we can enlarge H, a contradiction. Therefore

$$c(xy) = c(yy_-) \text{ for all } y \in S' \cap N_G(x). \tag{3.5}$$

Hence, we set $u(y) = x$ for all $y \in S \cap N_G(x)$.

Finally, suppose that $y \in T = R' \cap S'$. We may further assume that $y_- \notin N^c_{(\mathrm{d})}(x) \cup N^c_{(\mathrm{f})}(x)$ and $xy \in E(G)$. Since $y \in R' \cap N_G(x)$ and $y_- \notin N^c_{(\mathrm{d})}(x) \cup N^c_{(\mathrm{f})}(x)$, we have $c(xy) = c(yy_+)$ by (3.4). On the other hand, since $y \in S' \cap N_G(x)$, we have $c(xy) = c(yy_-)$ by (3.5). Hence, $c(yy_+) = c(yy_-)$, contradicting the fact that the cycle C_y is properly colored. Therefore, $xy \notin E(G)$ for all $y \in T$, so we can set $u(y) = x$ for $y \in T$. This completes the proof of Claim 2. ∎

If $V(F) = T$, then $|T| \geq \delta/2$ by (3.1). By Claim 1, T is an independent set in G. Claim 2 implies that for each $y \in T$ there exists a vertex $u(y) \notin V(F) \cup N(y)$ as the color $c(y) = c_0$ does not appear in G. Therefore

$$\delta \leq d^c(y) \leq |N_G(y)| \leq n - |T| - |\{u(y)\}| \leq n - \delta/2 - 1,$$

which gives $\delta < 2n/3$ as required. Thus, we may assume $V(F) \neq T$. We are going to show that there exists a vertex $y_0 \in R \cup S \cup W$ such that

$$|\{z \in V(F)\backslash\{y_0\} : y_0 z \notin E(G) \text{ or } c(y_0 z) = c(y_0)\}| \geq (\delta - 1)/2. \tag{3.6}$$

Define F' to be the directed graph on $V(F)$ such that there is a directed edge from y to z if $yz \notin E(F)$ or $c(yz) = c(z)$. Thus, in proving (3.6), it suffices to show that there exists a vertex $y_0 \in R \cup S \cup W$ with indegree at least $(\delta - 1)/2$ in F'. By Claim 1, the base graph of F' is complete. Moreover, $\overrightarrow{yz}$ exists for all $y \in T$ and $z \in V(F)$ as the color $c(y) = c_0$ does not appear in G. Hence, the number of directed edges into

$R \cup S \cup W$ is at least $\binom{|R\cup S\cup W|}{2} + |T||R \cup S \cup W|$. By an averaging argument, there exists $y_0 \in R \cup S \cup W$ with indegree

$$d^-(y_0) \geq (|R| + |S| + |W| - 1)/2 + |T| \overset{(3.1)}{\geq} (\delta - 1)/2,$$

so (3.6) holds. Recall that the color degree $d^c(y_0) \geq \delta^c(G) = \delta$, so y_0 meets at least δ edges of distinct colors. In particular, there are at least $\delta - 1$ neighbors z' of y_0 with $c(y_0z') \neq c(y_0)$. Hence, there are at most $n-\delta$ vertices z in $V(G)\backslash\{y_0\}$ such that either $y_0z \notin E(G)$ or $c(y_0z) = c(y_0)$. By Claim 2, there exists $u = u(y_0) \in V(G)\backslash V(F)$ such that $y_0u \notin E(G)$ or $c(y_0u) = c(y_0)$. Together with (3.6), we have

$$\begin{aligned} n - \delta &\geq |\{z \in V(F)\backslash\{y_0\} : y_0z \notin E \text{ or } c(y_0z) = c(y_0)\}| + |\{u(y_0)\}| \\ &\geq (\delta - 1)/2 + 1. \end{aligned}$$

Thus, $\delta < 2n/3$ as required. This completes the proof of Theorem 1.1. □

The properly colored 2-factor obtained by Theorem 1.1 may contain $|G|/3$ cycles. We would like to minimize the number of cycles in a properly colored 2-factor. In the next lemma, we show that this can be achieved by assuming a slightly larger $\delta_1^c(G)$. Recall that $\delta_1^c(G)$ is the minimum $\delta(G - H)$ over all monochromatic subgraphs H in G. So $\delta_1^c(G) \geq \delta^c(G) - 1$.

Lemma 3.1. *For every integer $k \geq 1$, every edge-colored graph G with $|G| = n$ and $\delta_1^c(G) \geq 2n/3 + k$ contains a properly colored 2-factor in which every cycle has length at least $k/2$. In particular, G can be covered by at most $\lfloor 2n/k \rfloor$ vertex-disjoint properly colored paths.*

The idea of the proof is rather simple. Recall that a 1-path-cycle is a vertex-disjoint union of at most one path P and a number of cycles. We consider a properly colored 1-path-cycle H in G such that every cycle has length at least $k/2$ and $|H|$ is maximal. If we suppose that $V(H) = V(G)$, then we may assume that H contains a path P as a component or else we are done. Our aim is to show that there exists a properly colored 2-factor H' in $G[V(P)]$ such that each cycle has length at least $k/2$. Therefore $H - P + H'$ is the desired properly colored 2-factor.

Proof of Lemma 3.1. Suppose the contrary, and let G be a counterexample with $|G| = n$ and $\delta_1^c(G) \geq 2n/3 + k$. Let H be a properly colored 1-path-cycle in G such that every cycle has length at least $k/2$ and $|H|$ is maximal. If $V(H) = V(G)$, then H contains a path P as a component or else we are done. If $V(H) \neq V(G)$, then we may assume H contains a path (maybe consisting of a single vertex). Let $P = z_1z_2 \cdots z_\ell$ be the path in H. We further assume that H is chosen such that ℓ is maximal (subject to $|H|$ maximal). Suppose that $\ell = 1$ and so $P = z_1$. Then $N(z_1) \subseteq V(H)$ follows by the maximality of $|H|$. Let $y \in N(z_1)$, and orient C_y into a directed cycle such that $c(yy_+) \neq c(z_1y)$. Then $(H - C_y - z_1) + z_1yC_y^+y_-$ is a properly colored 1-path-cycle on vertex set $V(H)$ with path $z_1yC_y^+y_-$, contradicting the maximality of ℓ. Therefore, $\ell \geq 2$.

Let $N_1 = \{x \in N_G(z_1) : c(z_1x) \neq c(z_1z_2)\}$ and $N_\ell = \{x \in N_G(z_\ell) : c(z_\ell x) \neq c(z_\ell z_{\ell-1})\}$. So

$$|N_1|, |N_\ell| \geq \delta_1^c(G) \geq 2n/3 + k. \tag{3.7}$$

By the maximality of $|H|$, we have $N_1, N_\ell \subseteq V(H)$. If $y \in N_1\backslash V(P)$, then orient C_y into a directed cycle such that $c(yy_+) \neq c(z_1y)$. Then $H - C_y - P + y_-C_y^-yP$

contradicts the maximality of ℓ. Therefore, $N_1 \subseteq V(P)$ and similarly $N_\ell \subseteq V(P)$. Moreover,

$$\ell = |V(P)| \geq |N_1| \overset{(3.7)}{\geq} 2n/3 + k.$$

If $z_\ell \in N_1$ and $z_1 \in N_\ell$, then $C = z_1 \cdots z_\ell z_1$ is a properly colored cycle of length at least $\ell \geq k$. Hence, $H - P + C$ consists of vertex-disjoint properly colored cycles each of length at least $k/2$. Note that $H - P + C$ spans $V(H)$. If $V(H) = V(G)$, then we are done. If $V(H) \neq V(G)$, then together with a vertex in $V(G)\backslash V(H)$ we obtain a properly colored 1-path-cycle contradicting the maximality of $|H|$. Therefore, we have $z_1 \notin N_\ell$ or $z_\ell \notin N_1$. Let

$$N'_q = N_q \backslash (\{z_1, z_2, \ldots, z_{\lceil k/2 \rceil}\} \cup \{z_\ell, z_{\ell-1}, \ldots, z_{\ell - \lceil k/2 \rceil + 1}\}) \tag{3.8}$$

for $q \in \{1, \ell\}$. Since $z_1 \notin N_1$ and $z_\ell \notin N_\ell$, by (3.7), we have

$$|N'_1|, |N'_\ell| \geq 2n/3 + k - (2\lceil k/2 \rceil - 1) \geq 2n/3. \tag{3.9}$$

For a given $x = z_i \in V(P)$, we write x_+ for z_{i+1} if $i < \ell$ and x_- for z_{i-1} if $i > 1$. Set $c_-(x) = c(xx_-)$ for $x \neq z_1$ and $c_+(x) = c(xx_+)$ for $x \neq z_\ell$. For distinct $x, y \in V(P)$, xPy denotes the subpath of P from x to y. If $x = y$, set $xPx = x$. For $q \in \{1, \ell\}$, define

$$\begin{aligned}
R_q &= \{x \in V(P)\backslash\{z_1\} : x_- \in N'_q \text{ and } c(x_- z_q) = c_+(x_-)\},\\
S'_q &= \{x \in V(P)\backslash\{z_\ell\} : x_+ \in N'_q \text{ and } c(x_+ z_q) = c_-(x_+)\},\\
S''_q &= \{x \in V(P)\backslash\{z_\ell\} : x_+ \in N'_q \text{ and } c_+(x_+) \neq c(x_+ z_q) \neq c_-(x_+)\},\\
S_q &= S'_q \cup S''_q.
\end{aligned}$$

Note that $S'_q \cap S''_q = \emptyset$. Since P is properly colored, for each $x \in N_1$, we have $c(xz_1) \neq c_+(x)$ or $c(xz_1) \neq c_-(x)$. Therefore, the three sets $\{x_- : x \in R_1\}$, $\{x_+ : x \in S'_1\}$, $\{x_+ : x \in S''_1\}$ are pairwise disjoint. Moreover, the union of these three sets is precisely N'_1. A similar statement also holds for N'_ℓ. Hence, $|R_q| + |S_q| = |N'_q|$ for $q \in \{1, \ell\}$. Note that if $z_\ell \in R_1 \cup S_1$, then $z_\ell \in R_1 \backslash S_1$. Similarly if $z_1 \in R_\ell \cup S_\ell$, then $z_1 \in S_\ell \backslash R_\ell$. For $q \in \{1, \ell\}$, we set

$$W_q = (R_q \cup S_q)\backslash\{z_1, z_\ell\}) \cup \{z_q\}, \qquad T_q = R_q \cap S_q, \qquad U_q = W_q \backslash T_q.$$

Note that $z_q \notin R_q \cup S_q$ and $z_1, z_\ell \notin R_q \cap S_q$. Hence, $z_q \in U_q$ and $|W_q| + |T_q| \geq |N'_q|$. Moreover, for $q \in \{1, \ell\}$,

$$2|W_q| = |W_q| + |T_q| + |U_q| \geq |N'_q| + |U_q| \overset{(3.9)}{\geq} |U_q| + 2n/3. \tag{3.10}$$

Now define an auxiliary directed bipartite graph F as follows. The vertex classes of F are W_1 and W_ℓ. (Recall that even though W_1 and W_ℓ might not be disjoint, we treat W_1 and W_ℓ to be so in F.) For $\{q, q'\} = \{1, \ell\}$ there is a directed edge in F from $x \in W_q$ to $y \in W_{q'}$ if and only if $xy \in E(G)$ and one of the following statements holds:

(i) $x \in R_q \backslash (T_q \cup \{z_1, z_\ell\})$ and $c(xy) \neq c_+(x)$.
(ii) $x \in S_q \backslash (T_q \cup \{z_1, z_\ell\})$ and $c(xy) \neq c_-(x)$.
(iii) $x \in T_q$.

(iv) $q = 1$, $x = z_1$ and $c(xy) \neq c(z_1z_2)$.
(v) $q = \ell$, $x = z_\ell$ and $c(xy) \neq c(z_\ell z_{\ell-1})$.

We are going to show that there exists a vertex $u \in U_1 \cup U_\ell$ such that u is in at least k directed 2-cycles in F. If $x \in R_1 \backslash T_1$, then $\overrightarrow{xy}$ exists for $y \in W_\ell$ if and only if $c(xy) \neq c_+(x)$. Hence, the outdegree of x in F is

$$\begin{aligned} d^+_F(x) &\geq |\{z \in N_G(x) : c(xz) \neq c_+(x)\}| + |W_\ell| - n \\ &\geq \delta^c_1(G) + |W_\ell| - n \\ &\overset{(3.10)}{\geq} (2n/3 + k) + (|U_\ell| + 2n/3)/2 - n = (|U_\ell| + 2k)/2. \end{aligned}$$

A similar statement holds for $x \in (S_1 \backslash T_1) \cup \{z_1\} = U_1 \backslash R_1$. In summary, for all $x \in U_1$,

$$d^+_F(x) \geq (|U_\ell| + 2k)/2.$$

Hence, there are at least $|U_1|(|U_\ell|+2k)/2$ directed edges from U_1 to W_ℓ, and similarly there are at least $|U_\ell|(|U_1| + 2k)/2$ directed edges from U_ℓ to W_1. Note that if $\overrightarrow{xy}$ is a directed edge in F with $x \in U_1$ and $y \in T_\ell$, then $xy \in E(G)$ and so $\{x, y\}$ forms a directed 2-cycle in F by (iii). Hence, if $\overrightarrow{xy}$ is a directed edge in F not contained in a directed 2-cycle with $x \in U_1$ and $y \in W_\ell$, then $y \in U_\ell$. A similar statement holds for $\overrightarrow{xy}$ with $x \in U_\ell$ and $y \in W_1$. Therefore, at most $|U_1||U_\ell|$ directed edges from $U_1 \cup U_\ell$ are not contained in a directed 2-cycle in F. The number of directed edges $\overrightarrow{xy}$ in F such that $x \in U_1 \cup U_\ell$ and $\overrightarrow{xy}$ is contained in a directed 2-cycle is at least

$$\frac{|U_1|(|U_\ell| + 2k)}{2} + \frac{|U_\ell|(|U_1| + 2k)}{2} - |U_1||U_\ell| = k(|U_1| + |U_\ell|).$$

Hence, by an averaging argument, there exists a vertex $u \in U_1 \cup U_\ell$ that is in at least k directed 2-cycles in F. Assume that $u \in U_1$. There are at least k vertices $w \in W_2 \backslash u$ such that uw is a directed 2-cycle in F. Pick one such $w \in W_2$ such that the subpath uPw in G has length at least $k/2$. Similarly, if $u \in U_\ell$, then there is a vertex $w \in W_1$ such that uw is a directed 2-cycle in F and the subpath uPw in G has length at least $k/2$. Without loss of generality, we may assume that $u \in W_1$ and $w \in W_\ell$.

If $u \in T_1 = R_1 \cap S_1$, then $u \in R_1 \backslash \{z_1, z_\ell\}$ with $c(uw) \neq c_+(u)$, or $u \in S_1 \backslash \{z_1, z_\ell\}$ with $c(uw) \neq c_-(u)$, as P is a properly colored path and $u \notin \{z_1, z_\ell\}$. Together with (i), (ii), and (iv), we may assume that at least one of the following statements holds:
(a'_1) $u \in R_1$ and $c(uw) \neq c_+(u)$.
(a'_2) $u \in S_1$ and $c(uw) \neq c_-(u)$.
(a'_3) $u = z_1$ and $c(z_1w) = c(uw) \neq c(z_1z_2)$.

If $u \in R_1$, then the definition of R_1 implies that $u_- \in N'_1$ and $c(z_1u_-) = c_+(u_-) \neq c_-(u_-)$. Since $u_- \in N'_1$, $c(z_1z_2) \neq c(z_1u_-)$. A similar statement also holds for $u \in S_1$, so at least one of the following statements holds:
(a_1) $u \in R_1$, $c(uw) \neq c_+(u)$ and $c(z_1z_2) \neq c(z_1u_-) \neq c_-(u_-)$.
(a_2) $u \in S_1$, $c(uw) \neq c_-(u)$ and $c(z_1z_2) \neq c(z_1u_+) \neq c_+(u_+)$.
(a_3) $u = z_1$ and $c(z_1w) \neq c(z_1z_2)$.

By symmetry, we can deduce that at least one of the following statements holds for w:
(b_1) $w \in R_\ell$, $c(uw) \neq c_+(w)$ and $c(z_\ell z_{\ell-1}) \neq c(z_\ell w_-) \neq c_-(w_-)$.
(b_2) $w \in S_\ell$, $c(uw) \neq c_-(w)$ and $c(z_\ell z_{\ell-1}) \neq c(z_\ell w_+) \neq c_+(w_+)$.
(b_3) $w = z_\ell$ and $c(z_\ell u) \neq c(z_{\ell-1}z_\ell)$.

We now claim that

(†) there exists a properly colored 2-factor H' in $G[V(P)]$ such that each cycle has length at least $k/2$.

If (†) holds, then $H'' = H - P + H'$ is a disjoint union of properly colored cycles each of length at least $k/2$ on vertex set $V(H)$. If $V(H'') = V(G)$, then H'' is the desired properly colored 2-factor. If $z \in V(G)\backslash V(H'')$, then $H'' \cup \{z\}$ is a properly colored 1-path-cycle contradicting the maximality of $|H| = |H''|$. Therefore, in order to complete the proof, it suffices to prove (†) for each combination of (a_i) and (b_j) for $1 \leq i, j \leq 3$. We say that $u < w$ if $u = z_i$ and $w = z_j$ with $i < j$. We would like to point out that Cases 2–6 are proved by similar arguments used in Case 1. Since the explicit structures of H' (see Figure 3.2) are different, we include their proofs for completeness.

Case 0. (a_3) *and* (b_3) *hold.* Since $u = z_1$ and $w = z_\ell$, (a_3) implies that $z_\ell \in N_1$ and (b_3) implies that $z_1 \in N_\ell$. Recall that $z_1 \notin N_\ell$ or $z_\ell \notin N_1$. Hence, we have a contradiction.

Case 1. (a_1) *and* (b_2) *hold.* We have the following three statements:

$$c_+(u) \neq c(uw) \neq c_-(w). \tag{3.11}$$

$$c(z_1z_2) \neq c(z_1u_-) \neq c_-(u_-). \tag{3.12}$$

$$c(z_\ell z_{\ell-1}) \neq c(z_\ell w_+) \neq c_+(w_+). \tag{3.13}$$

If $u_- = w$, then we set $H' = z_1Pu_-z_1 + w_+Pz_\ell w_+$ (see Figure 3.2(a)). Note that H' is properly colored and a union of two cycles on vertex set $V(H)$. By (a_1) and (3.8), we have $u_- \in N_1' \subseteq N_1\backslash\{z_1, \ldots, z_{\lceil k/2 \rceil}\}$. Thus, $z_1Pu_-z_1$ is a cycle of length at least $k/2$. By a similar argument, we deduce that $w_+Pz_\ell w_+$ is also a cycle of length at least $k/2$ as $w_+ \in N_\ell'$ by (b_2). Hence (†) holds.

Next suppose that $u_- > w$. If $u_- = w_+$ and $c(z_\ell w_+) = c(z_1u_-)$, then

$$c(z_\ell w_+) = c(z_1u_-) = c_+(u_-) = c_+(w_+) \neq c_-(w_+),$$

where the second equality is due to the fact that $u \in R_1$. Recall from (3.13) that $c(z_\ell w_+) \neq c_+(w_+)$. This contradicts the fact that $w \in S_\ell$. Therefore, if $u_- = w_+$, then $c(z_1u_-) \neq c(z_\ell w_+)$. Together with (3.11)–(3.13), we conclude that $H' = z_1u_-Pw_+z_\ell PuwPz_1$ (see Figure 3.2(b)) is a properly colored cycle with vertex set $V(P)$, so (†) holds.

If $u < w$, then $z_1Pu_-z_1$, $uPwu$, and $w_+Pz_\ell w_+$ (see Figure 3.2(c)) are properly colored cycles by (3.11)–(3.13). We will show that each is a cycle of length at least $k/2$ (which then implies (†)). By (a_1) and (3.8), we have $u_- \in N_1' \subseteq N_1\backslash\{z_1, \ldots, z_{\lceil k/2 \rceil}\}$. Thus, $z_1Pu_-z_1$ is a cycle of length at least $k/2$. Similarly, $w_+Pz_\ell w_+$ is also a cycle of length at least $k/2$ as $w_+ \in N_\ell'$ by (b_2) and (3.8). Since u and w are chosen such that the subpath uPw has length at least $k/2$, $uPwu$ has length at least $k/2$. Therefore, (†) holds for (a_1) and (b_2).

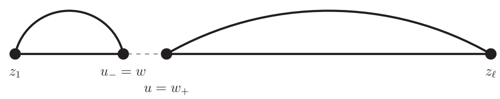
Case 2. (a_1) *and* (b_1) *hold.* We have the following three statements:

$$c_+(u) \neq c(uw) \neq c_+(w). \tag{3.14}$$

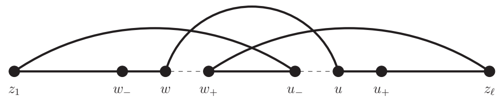
$$c(z_1z_2) \neq c(z_1u_-) \neq c_-(u_-). \tag{3.15}$$

$$c(z_\ell z_{\ell-1}) \neq c(z_\ell w_-) \neq c_-(w_-). \tag{3.16}$$

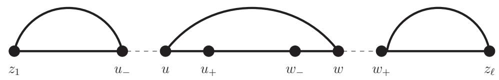
If $u < w$, then we set $H' = z_1Pu_-z_1 + uPw_-z_\ell Pwu$ (see Figure 3.2(d)). Note that H' is properly colored. By (a_1) and (3.8), we deduce that $z_1Pu_-z_1$ is a cycle of length at least $k/2$. Recall our choices of u and w that uPw has length at least $k/2$. This implies that $uPw_-z_\ell Pwu$ is also a cycle of length at least $k/2$. Hence (†) holds.



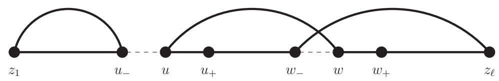
(a) $u \in R_1$ and $w = S_\ell$ with $u_- = w$



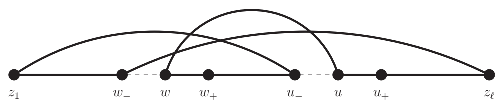
(b) $u \in R_1$ and $w \in S_\ell$ with $u_- > w$



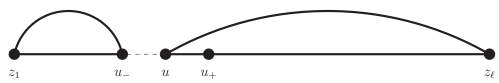
(c) $u \in R_1$ and $w \in S_\ell$ with $u < w$



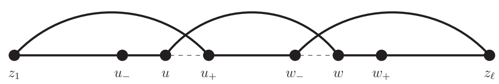
(d) $u \in R_1$ and $w \in R_\ell$ with $u < w$



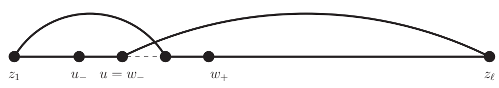
(e) $u \in R_1$ and $w \in R_\ell$ with $u > w$



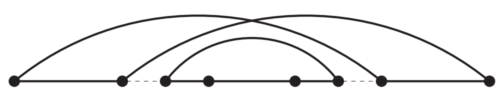
(f) $u \in R_1$ and $w = z_\ell$



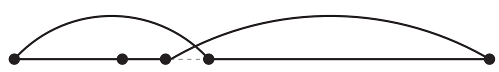
(g) $u \in S_1$ and $w = R_\ell$ with $u < w_-$



(h) $u \in S_1$ and $w = R_\ell$ with $u = w_-$



(i) $u \in S_1$ and $w = R_\ell$ with $u > w$



(j) $u \in S_1$ and $w = z_\ell$

FIG. 3.2. *Structures of H'.*

Next suppose that $u > w$. By (3.14), we deduce that $w_+ \neq u$ and so $w < u_-$. Hence, wPu_- is a path with length at least one. Together with (3.14)–(3.16), we conclude that $H' = z_1Pw_-z_\ell PuwPu_-z_1$ (see Figure 3.2(e)) is a properly colored cycle with vertex set $V(P)$, so (†) holds. Therefore, (†) holds for (a$_1$) and (b$_1$).

Case 3. (a$_1$) *and* (b$_3$) *hold*. Since $w = z_\ell$, we have

$$c_+(u) \neq c(uz_\ell) \neq c_-(z_{\ell-1}z_\ell) \text{ and } c(z_1z_2) \neq c(z_1u_-) \neq c_-(u_-). \tag{3.17}$$

Note that both $z_1Pu_-z_1$ and $uPz_\ell u$ (see Figure 3.2(f)) are properly colored cycles. By (3.8), both cycles have length at least $k/2$. Therefore, (†) holds for (a$_1$) and (b$_3$) by setting $H' = z_1Pu_-z_1 + uPz_\ell u$.

Case 4. (a$_2$) *and* (b$_1$) *hold*. We have the following three statements:

$$c_-(u) \neq c(uw) \neq c_+(w). \tag{3.18}$$

$$c(z_1z_2) \neq c(z_1u_+) \neq c_+(u_+). \tag{3.19}$$

$$c(z_\ell z_{\ell-1}) \neq c(z_\ell w_-) \neq c_-(w_-). \tag{3.20}$$

First suppose that $u < w_-$. If $u_+ = w_-$ and $c(z_\ell w_-) = c(z_1u_+)$, then

$$c(z_1u_+) = c(z_\ell w_-) = c_+(w_-) = c_+(u_+) \neq c_-(u_+),$$

where the second inequality is due to the fact that $w \in R_\ell$. This contradicts the fact that $u \in S_1$ as $c(z_1u_+) \neq c_+(u_+)$ by (3.19). Therefore, if $u_+ = w_-$, then $c(z_1u_-) \neq c(z_\ell w_+)$. Together with (3.18)–(3.20), we conclude that $H' = z_1PuwPz_\ell w_-Pu_+z_1$ (see Figure 3.2(g)) is a properly colored cycle with vertex set $V(P)$, so (†) holds.

If $u = w_-$, then $H' = z_1Pw_-z_\ell Pu_+z_1$ (see Figure 3.2(h)) is a properly colored cycle with vertex set $V(P)$ by (3.19) and (3.20). Thus (†) holds.

Suppose that $u > w$. By (3.18), we deduce that $w_+ \neq u$ and so $w < u_-$. Hence, $wPuw$ is a properly colored cycle. Moroever, it has length at least $k/2$ by our choices of u and w. By (3.19) and (3.20), we conclude that $z_1Pw_-z_\ell Pu_+z_1$ is a properly colored cycle. It has length at least $k/2$ by (3.8). Hence (†) holds by setting $H' = wPuw + z_1Pw_-z_\ell Pu_+z_1$ (see Figure 3.2(i)). Therefore, (†) holds for (a$_2$) and (b$_1$).

Case 5. (a$_2$) *and* (b$_3$) *hold*. Since $w = z_\ell$, we have

$$c_-(u) \neq c(uz_\ell) \neq c_-(z_{\ell-1}z_\ell) \text{ and } c(z_1z_2) \neq c(z_1u_+) \neq c_+(u_+).$$

Note that $z_1Puz_\ell Pu_+z_1$ (see Figure 3.2(j)) is a properly colored cycle with vertex set $V(P)$ and so (†) holds.

Case 6. (a$_i$) *and* (b$_j$) *hold for* $(i,j) \in \{(3,2),(2,2),(3,1)\}$. Let $P' = z'_1z'_2\cdots z'_\ell$ be the properly colored path obtained by reversing P. Hence, $z'_i = z_{\ell-i+1}$ for all $1 \leq i \leq \ell$. Given $x' = z'_i \in V(P')$ with $i < \ell$, write $x_{+'}$ to mean z'_{i+1} and $c_{+'}(x) = c(xx_{+'})$. Similarly, given $x' = z'_i \in V(P')$ with $i > 1$, we write $x_{-'} = z'_{i-1}$ and $c_{-'}(x) = c(xx_{-'})$. (Hence, these primed notations are defined relative to P'.) Recall that P' is the reversed P, so $c_{+'}(x) = c_-(x)$ and $c_{-'}(x) = c_+(x)$. Further, set $u' = w$ and $w' = u$.

Now suppose that (a$_3$) and (b$_2$) hold. Under the primed notation, we have

$$c_{+'}(u') \neq c(u'z'_\ell) \neq c_-(z'_{\ell-1}z'_\ell) \text{ and } c(z'_1z'_2) \neq c(z'_1u'_{-'}) \neq c_{-'}(u'_{-'}).$$

More importantly, if we ignore the primes, then we obtain (3.17). Therefore, in hindsight, we are in the case when (a$_1$) and (b$_3$) hold with respect to P', i.e., we are in Case 3. Hence, (†) holds by setting $H' = z'_1P'u'_{-'}z'_1 + u'P'z'_\ell u'$.

Similarly, we deduce that the case when (a_2) and (b_2) hold with respect to P corresponds to the case when (a_1) and (b_1) hold with respect to P'. Also, (a_3) and (b_1) (with respect to P) corresponds to (a_2) and (b_3) (with respect to P'). Therefore, we are in Case 1 and Case 5, respectively. The proof of the lemma is completed. ∎

4. Absorbing cycle. The aim of this section is to prove the following lemma, that is, to show that there exists a small absorbing cycle.

LEMMA 4.1 (absorbing cycle lemma). *Let $0 < \varepsilon < 2^{-9}3^{-1}$. Then there exists an integer n_0 such that whenever $n \geq n_0$ the following holds. Suppose that G is an edge-colored graph of order n with $\delta_1^c(G) \geq (2/3 + \varepsilon)n$. Then there exists a properly colored cycle C of length at most $2\varepsilon n/3$ such that for all $k \leq (8\varepsilon^2 n)/243$ and for any collections $P_1, \dots, P_k$ of vertex-disjoint properly colored paths in $G \backslash V(C)$, there exists a properly colored cycle with vertex set $V(C) \cup \bigcup_{1 \leq i \leq k} V(P_i)$.*

Given a vertex x, we say that a path P is an *absorbing path for x* if the following conditions hold:

(i) $P = z_1 z_2 z_3 z_4$ is a properly colored path of length 3;
(ii) $x \notin V(P)$;
(iii) $z_1 z_2 x z_3 z_4$ is a properly colored path.

Next we define an absorbing path for two disjoint edges. Given two vertex-disjoint edges x_1x_2, y_1y_2, we say that a path P is an *absorbing path for* $(x_1, x_2; y_1, y_2)$ if the following conditions hold:

(i) $P = z_1 z_2 z_3 z_4$ is a properly colored path of length 3;
(ii) $V(P) \cap \{x_1, x_2, y_1, y_2\} = \emptyset$;
(iii) both $z_1 z_2 x_1 x_2$ and $y_1 y_2 z_3 z_4$ are properly colored paths of length 3.

Note that the ordering of $(x_1, x_2; y_1, y_2)$ is important. Given a vertex x, let $\mathcal{L}(x)$ be the set of absorbing paths for x. Similarly, given two vertex-disjoint edges x_1x_2, y_1y_2, let $\mathcal{L}(x_1, x_2; y_1, y_2)$ be the set of absorbing paths for $(x_1, x_2; y_1, y_2)$. The following simple proposition follows immediately from the definition of an absorbing path for $(x_1, x_2; y_1, y_2)$.

PROPOSITION 4.2. *Let $P' = x_1 x_2 \cdots x_{\ell-1} x_\ell$ be a properly colored path with $\ell \geq 4$. Let $P = z_1 z_2 z_3 z_4$ be an absorbing path for $(x_1, x_2; x_{\ell-1}, x_\ell)$ with $V(P) \cap V(P') = \emptyset$. Then $z_1 z_2 x_1 x_2 \cdots x_{\ell-1} x_\ell z_3 z_4$ is a properly colored path.*

Lemma 4.1 will be proved as follows. Suppose that $\delta_1^c(G) \geq (2/3 + \varepsilon)n$. In the next lemma, we show that every $\mathcal{L}(x)$ and every $\mathcal{L}(x_1, x_2; y_2, y_1)$ are large. By a simple probabilistic argument, Lemma 4.5 shows that there exists a small family $\mathcal{F}'$ of vertex-disjoint properly colored paths (each of length 3) such that $\mathcal{F}'$ contains a linear number of members of $\mathcal{L}(x)$ for all $x \in V(G)$ and a similar statement holds for $\mathcal{L}(x_1, x_2; y_1, y_2)$. Finally, we join all paths in $\mathcal{F}'$ into one short properly colored cycle C using Lemma 4.6. Moreover, C satisfies the desired property in Lemma 4.1.

LEMMA 4.3. *Let $0 < \varepsilon < 1/2$ and let $n \geq 4/\varepsilon^2$ be an integer. Suppose that G is an edge-colored graph on n vertices with $\delta_1^c(G) \geq (1/2+\varepsilon)n$. Then $|\mathcal{L}(x)| \geq \varepsilon^3 n^4$ for every $x \in V(G)$ and $|\mathcal{L}(x_1, x_2; y_1, y_2)| \geq \varepsilon^3 n^4$ for any distinct vertices $x_1, x_2, y_1, y_2 \in V(G)$ with x_1x_2, $y_1y_2 \in E(G)$.*

Proof. Fix a vertex $x \in V(G)$. Choose a vertex $z_2 \in N(x)$. Next pick another vertex $z_3 \in N(z_2) \cap N(x)$ such that $c(xz_2) \neq c(xz_3)$. Notice that the number of such z_3 is at least $\delta_1^c(G) + \delta(G) - n \geq 2\varepsilon n$. Since $\delta_1^c(G) \geq (1/2 + \varepsilon)n$, $\Delta(H) \leq (1/2 - \varepsilon)n$ for all monochromatic subgraphs H in G. Hence, the number of $z_1 \in N(z_2) \backslash \{x, z_3\}$ such that $c(z_2z_3) \neq c(z_1z_2) \neq c(z_2x)$ is at least $\delta_1^c(G) - (1/2 - \varepsilon)n - 2 \geq \varepsilon n$. Fix one such z_1. Similarly, there are at least εn choices for $z_4 \in N(z_3) \backslash \{x, z_1, z_2\}$ such that $c(z_2z_3) \neq c(z_3z_4) \neq c(xz_3)$. Notice that $z_1 z_2 z_3 z_4$ is an absorbing path for x.

Recall that every path is assumed to be directed. Therefore, there are at least $\delta_1^c(G) \times 2\varepsilon n \times \varepsilon n \times \varepsilon n \geq \varepsilon^3 n^4$ many absorbing paths for x.

Next, fix vertex-disjoint edges x_1x_2 and y_1y_2 in G. Choose a vertex $z_2 \in N(x_1)\backslash\{x_2, y_1, y_2\}$ such that $c(x_1z_2) \neq c(x_1x_2)$. Pick another vertex $z_3 \in (N(z_2) \cap N(y_2))\backslash\{x_1, x_2, y_1, y_2\}$ such that $c(y_2z_3) \neq c(y_1y_2)$. The number of such z_3 is at least $\delta(G) + \delta_1^c(G) - n - 4 \geq 2\varepsilon n - 4$. Recall that $\Delta(H) \leq (1/2 - \varepsilon)n$ for all monochromatic subgraphs H in G. Hence, the number of $z_1 \in N(z_2)\backslash\{x_1, x_2, y_1, y_2, z_3\}$ such that $c(z_2z_3) \neq c(z_1z_2) \neq c(z_2x_1)$ is at least $\delta_1^c(G) - (1/2 - \varepsilon)n - 4 \geq \varepsilon n$. Fix one such z_1. Similarly, there are at least $\delta_1^c(G) - (1/2 - \varepsilon)n - 5 \geq \varepsilon n$ choices for $z_4 \in N(z_3)\backslash\{x_1, x_2, y_1, y_2, z_1, z_2\}$ such that $c(z_2z_3) \neq c(z_3z_4) \neq c(y_2z_3)$. Note that $z_1z_2z_3z_4$ is an absorbing path for $(x_1, x_2; y_1, y_2)$. Therefore, there are at least

$$(\delta_1^c(G) - 4)(2\varepsilon n - 4)\varepsilon^2 n^2 \geq (\varepsilon n^2 + 2n(\varepsilon^2 n - 6\varepsilon - 1) + 16)\varepsilon^2 n^2 \geq \varepsilon^3 n^4$$

many absorbing paths for $(x_1, x_2; y_1, y_2)$. □

Lemma 4.5 is proved by a simple probabilistic argument since each of $\mathcal{L}(x)$ and $\mathcal{L}(x_1, x_2; y_1, y_2)$ is large. We will need the following Chernoff bound for the binomial distribution (see, e.g., [2]). Recall that the binomial random variable with parameters (n, p) is the sum of n independent Bernoulli variables, each taking value 1 with probability p or 0 with probability $1 - p$.

Proposition 4.4. *Suppose that X has the binomial distribution and* $0 < a < 3/2$. *Then* $\mathbb{P}(|X - \mathbb{E}X| \geq a\mathbb{E}X) \leq 2e^{-a^2\mathbb{E}X/3}$.

Lemma 4.5. *Let $0 < \gamma < 1$. Then there exists an integer n_0 such that whenever $n \geq n_0$ the following holds. Let G be an edge-colored graph on n vertices. Suppose that $|\mathcal{L}(x)| \geq \gamma n^4$ for every $x \in V(G)$ and $|\mathcal{L}(x_1, x_2; y_1, y_2)| \geq \gamma n^4$ for all distinct vertices $x_1, x_2, y_1, y_2 \in V(G)$ with x_1x_2, $y_1y_2 \in E(G)$. Then there exists a family $\mathcal{F}'$ of vertex-disjoint properly colored paths each of length* 3*, which satisfies the properties*

$$|\mathcal{F}'| \leq 2^{-6}\gamma n,$$
$$|\mathcal{L}(x) \cap \mathcal{F}'| \geq 2^{-9}\gamma^2 n,$$
$$|\mathcal{L}(x_1, x_2; y_1, y_2) \cap \mathcal{F}'| \geq 2^{-9}\gamma^2 n$$

for all $x \in V(G)$ and for all distinct vertices $x_1, x_2, y_1, y_2 \in V(G)$ with x_1x_2, $y_1y_2 \in E(G)$.

Proof. Choose $n_0 \in \mathbb{N}$ large so that

$$\exp(-\gamma n_0/(3 \times 2^7)) + (n_0 + n_0^4)\exp(-\gamma^2 n_0/(3 \times 2^9)) \leq 1/6. \tag{4.1}$$

Recall that each path is assumed to be directed. So a path $z_1z_2z_3z_4$ will be considered as a 4-tuple (z_1, z_2, z_3, z_4). Choose a family $\mathcal{F}$ of 4-tuples in $V(G)$ by selecting each of the $n!/(n-4)!$ possible 4-tuples independently at random with probability

$$p = 2^{-7}\gamma\frac{(n-4)!}{(n-1)!} \geq 2^{-7}\gamma n^{-3}.$$

Notice that

$$\mathbb{E}|\mathcal{F}| = p\frac{n!}{(n-4)!} = 2^{-7}\gamma n,$$
$$\mathbb{E}|\mathcal{L}(x) \cap \mathcal{F}| = p|\mathcal{L}(x)| \geq 2^{-7}\gamma^2 n,$$
$$\mathbb{E}|\mathcal{L}(x_1, x_2; y_1, y_2) \cap \mathcal{F}| = p|\mathcal{L}(x_1, x_2; y_1, y_2)| \geq 2^{-7}\gamma^2 n$$

for every $x \in V(G)$ and for all distinct $x_1, x_2, y_1, y_2 \in V(G)$ with x_1x_2, $y_1y_2 \in E(G)$. Then by Proposition 4.4, the union bound and (4.1) with probability at least 5/6, the family $\mathcal{F}$ satisfies the properties

$$|\mathcal{F}| \leq 2\mathbb{E}(|\mathcal{F}|) = 2^{-6}\gamma n, \tag{4.2}$$

$$|\mathcal{L}(x) \cap \mathcal{F}| \geq 2^{-1}\mathbb{E}(|\mathcal{L}(x) \cap \mathcal{F}|) \geq 2^{-8}\gamma^2 n, \tag{4.3}$$

$$|\mathcal{L}(x_1, x_2; y_1, y_2) \cap \mathcal{F}| \geq 2^{-1}\mathbb{E}(|\mathcal{L}(x_1, x_2; y_1, y_2) \cap \mathcal{F}|) \geq 2^{-8}\gamma^2 n \tag{4.4}$$

for every $x \in V(G)$ and for all distinct $x_1, x_2, y_1, y_2 \in V(G)$ with x_1x_2, $y_1y_2 \in E(G)$.

We say that two 4-tuples (a_1, a_2, a_3, a_4) and (b_1, b_2, b_3, b_4) are *intersecting* if $a_i = b_j$ for some $1 \leq i, j \leq 4$. Furthermore, we can bound the expected number of pairs of 4-tuples in $\mathcal{F}$ that are intersecting from above by

$$\frac{n!}{(n-4)!} \times 4^2 \times \frac{(n-1)!}{(n-4)!} \times p^2 = 2^{-10}\gamma^2 n.$$

Thus, using Markov's inequality, we derive that with probability at least 1/2,

$$\mathcal{F} \text{ contains at most } 2^{-9}\gamma^2 n \text{ intersecting pairs of 4-tuples.} \tag{4.5}$$

Hence, with positive probability the family $\mathcal{F}$ has all properties stated in (4.2)–(4.5). Remove one 4-tuple in each intersecting pair in such a family $\mathcal{F}$. Further remove those 4-tuples that are not absorbing paths. We get a subfamily $\mathcal{F}'$ consisting of pairwise disjoint 4-tuples, which satisfies

$$|\mathcal{L}(x) \cap \mathcal{F}'| > 2^{-8}\gamma^2 n - 2^{-9}\gamma^2 n = 2^{-9}\gamma^2 n$$

for every $x \in V(G)$ and a similar statement holds for $|\mathcal{L}(x_1, x_2; y_1, y_2) \cap \mathcal{F}'|$. Since each 4-tuple in $\mathcal{F}'$ is an absorbing path, $\mathcal{F}'$ is a set of vertex-disjoint properly colored paths of length 3. ◻

In order to prove Lemma 4.1, it is sufficient to join the paths in $\mathcal{F}'$ given by Lemma 4.5 into a short properly colored cycle C. The next lemma shows that we can join any two disjoint edges into a properly colored path of length 5.

LEMMA 4.6. *Suppose that G is an edge-colored graph of order n with $\delta_1^c(G) \geq 2n/3+1$. Let x_1x_2, y_1y_2 be two edges in G with $x_2 \neq y_2$. Then there exists an edge z_1z_2 with z_1, $z_2 \in V(G) \backslash \{x_1, x_2, y_1, y_2\}$ such that both $x_1x_2z_1z_2$ and $z_1z_2y_1y_2$ are properly colored paths. In particular, if x_1, x_2, y_1, and y_2 are distinct, then $x_1x_2z_1z_2y_1y_2$ is a properly colored path.*

Proof. Let X be the set of vertices $x \in N(x_2) \backslash \{x_1, x_2, y_1, y_2\}$ such that $c(xx_2) \neq c(x_1x_2)$. Similarly, let Y be the set of vertices $y \in N(y_1) \backslash \{x_1, x_2, y_1, y_2\}$ such that $c(yy_1) \neq c(y_1y_2)$. So $|X|, |Y| \geq \delta_1^c(G) - 2 \geq 2n/3 - 1$. Define an auxiliary bipartite directed graph H with vertex classes X and Y such that for $x \in X$ and $y \in Y$

(a) $\overrightarrow{xy} \in E(H)$ if and only if $xy \in E(G)$ and $c(xy) \neq c(xx_2)$;

(b) $\overrightarrow{yx} \in E(H)$ if and only if $xy \in E(G)$ and $c(xy) \neq c(yy_1)$.

(Recall that in H we treat X and Y to be disjoint.) The outdegree of each $x \in X$ in H is

$$\begin{aligned} d_H^+(x) &= |Y \cap \{z \in N_G(x) : c(xz) \neq c(xx_2)\}| \\ &\geq |Y| + \delta_1^c(G) - n \geq (|Y| + 1)/2 \end{aligned}$$

and similarly for each $y \in Y$, $d^+_H(y) \geq (|X|+1)/2$. Hence the number of directed 2-cycles in H is at least

$$\sum_{x\in X} d^+_H(x) + \sum_{y \in Y} d^+_H(y) - |X||Y| \geq \frac{|X|(|Y|+1)}{2} + \frac{|Y|(|X|+1)}{2} - |X||Y|$$
$$= \frac{|X|+|Y|}{2} \geq 2n/3 - 1.$$

Let z_1z_2 be a directed 2-cycle in H with $z_1 \in X$ and $z_2 \in Y$. This implies that $c(x_2z_1) \neq c(z_1z_2) \neq c(z_2y_1)$. Hence, $x_1x_2z_1z_2$ and $z_1z_2y_1y_2$ are properly colored paths. Therefore, the lemma follows. □

We are ready to prove Lemma 4.1. Given a graph family $\mathcal{F}$, we write $V(\mathcal{F}) = \bigcup_{F\in\mathcal{F}} V(F)$.

Proof of Lemma 4.1. Let $\gamma = 2^6\varepsilon/9$. Choose $n_0 \in \mathbb{N}$ large so that $n_0 \geq 4/\varepsilon^2$ and Lemma 4.3 holds. By Lemma 4.3, $\mathcal{L}(x) \geq \gamma n^4$ for all $x \in V(G)$ and $\mathcal{L}(x_1, x_2; y_1, y_2) \geq \gamma n^4$ for all distinct vertices $x_1, x_2, y_1, y_2 \in V(G)$ with x_1x_2, $y_1y_2 \in E(G)$. Let $\mathcal{F}'$ be the set of properly colored paths obtained by Lemma 4.5. Therefore, $|\mathcal{F}'| \leq 2^{-6}\gamma n = \varepsilon n/9$,

$$|\mathcal{L}(x) \cap \mathcal{F}'| \geq 2^{-9}\gamma^2 n = \frac{8\varepsilon^2 n}{81} \text{ and } |\mathcal{L}(x_1, x_2; y_1, y_2) \cap \mathcal{F}'| \geq \frac{8\varepsilon^2 n}{81} \tag{4.6}$$

for all $x \in V(G)$ and for all distinct vertices $x_1, x_2, y_1, y_2 \in V(G)$ with x_1x_2, $y_1y_2 \in E(G)$.

We now show that C has the desired property. Let $P_1, \dots, P_{|\mathcal{F}'|}$ be the properly colored paths in $\mathcal{F}'$. For each $1 \leq j \leq |\mathcal{F}'|$, we are going to find an edge $z^j_1z^j_2$ such that $\{z^j_1, z^j_2\} \cap V(\mathcal{F}') = \emptyset$, $P_jz^j_1z^j_2P_{j+1}$ is a properly colored path, where we take $P_{|\mathcal{F}'|+1} = P_1$, and $\{z^j_1, z^j_2\} \cap \{z^{j'}_1, z^{j'}_2\} = \emptyset$ for all $j \neq j'$. Assume that we have already found edges $z^1_1z^1_2, \dots, z^{j-1}_1z^{j-1}_2$ for some $1 \leq j \leq |\mathcal{F}'|$. Let $P_j = v_1v_2v_3v_4$ and $P_{j+1} = v'_1v'_2v'_3v'_4$. Set

$$W_j = \left(V(\mathcal{F}') \cup \bigcup_{1\leq j'<j} \{z^{j'}_1, z^{j'}_2\} \right) \Big\backslash \{v_3, v_4, v'_1, v'_2\}.$$

Note that

$$|W_j| = 4|\mathcal{F}'| + 2(j-1) - 4 < 6|\mathcal{F}'| \leq 2\varepsilon n/3.$$

Define $G_j = G[V(G)\backslash W_j]$, so $\delta^c_1(G_j) \geq 2n/3 + \varepsilon n/3 \geq 2n/3 + 1$. Apply Lemma 4.6 with $G = G_j$, $x_1 = v_3$, $x_2 = v_4$, $y_1 = v'_1$, and $y_2 = v'_2$ to obtain an edge $z^j_1z^j_2$ such that $v_3v_4z^j_1z^j_2v'_1v'_2$ is a properly colored path in G_j. This implies that $P_jz^j_1z^j_2P_{j+1}$ is a properly colored path in G. Therefore, there exist vertex-disjoint edges $z^1_1z^1_2, \dots, z^{|\mathcal{F}'|}_1z^{|\mathcal{F}'|}_2$ as desired. Let C be the properly colored cycle obtained by concatenating $P_1, z^1_1z^1_2, P_2, z^2_1z^2_2, \dots, P_{|\mathcal{F}'|}, z^{|\mathcal{F}'|}_1z^{|\mathcal{F}'|}_2$. Note that $|C| = 6|\mathcal{F}'| \leq 2\varepsilon n/3$.

Suppose that $\mathcal{P}$ is a family of at most $(8\varepsilon^2 n)/243$ vertex-disjoint properly colored paths in $V(G)\backslash V(C)$. Let $\mathcal{P}'$ be the family of vertex-disjoint properly colored paths obtained from $\mathcal{P}$ by breaking up every path $P \in \mathcal{P}$ with $|P| \leq 3$ into isolated vertices. Hence, $\mathcal{P}'$ contains at most $(8\varepsilon^2 n)/81$ paths and, for each path $P \in \mathcal{P}'$, either $|P| = 1$ or $|P| \geq 4$. Now, we assign each $P \in \mathcal{P}'$ to a path $Q \in \mathcal{F}'$ such that $Q \in \mathcal{L}(V(P))$

if $|P| = 1$ and $Q \in \mathcal{L}(x_1, x_2; x_{\ell-1}, x_\ell)$ if $P = x_1 x_2 \cdots x_\ell$ with $\ell \geq 4$. Moreover, no two paths in $\mathcal{P}'$ are assigned to the same $Q \in \mathcal{F}'$. Note that such an assignment exists by (4.6). Apply Proposition 4.2 to each pair (P, Q) and obtain a family $\mathcal{F}''$ of vertex-disjoint paths such that $|\mathcal{F}'| = |\mathcal{F}''|$. Note that

$$V(\mathcal{F}'') = V(\mathcal{F}') \cup V(\mathcal{P}') = V(\mathcal{F}') \cup V(\mathcal{P}).$$

Moreover, there is a one-to-one correspondence between paths $P' \in \mathcal{F}'$ and $P'' \in \mathcal{F}''$ such that the endedges of P' and P'' are the same. Recall that C is a properly colored cycle containing $\mathcal{F}'$. Let C' be the cycle obtained from C by replacing the paths in $\mathcal{F}'$ with paths in $\mathcal{F}''$. Note that C' is properly colored and $V(C') = V(C) \cup V(\mathcal{P})$. This completes the proof of Lemma 4.1. □

5. Proof of Theorem 1.3. First, we prove that G contains a properly colored triangle if $\delta_1^c(G) > 2n/3 - 1$.

PROPOSITION 5.1. *Let G be an edge-colored graph on n vertices with $\delta_1^c(G) > 2n/3 - 1$. Then every vertex in G is contained in a properly colored triangle.*

Proof. Let x be a vertex in G and let c be the edge-coloring on G. Define H to be the directed graph on $N(x)$ with directed edges $\overrightarrow{yz}$ if and only if yz is an edge in $G[N(x)]$ with $c(yz) \neq c(xy) \neq c(xz)$. For $y \in V(H)$, the outdegree of y in H is

$$\begin{aligned} d_H^+(y) &\geq |\{z \in N_G(y) : c(yz) \neq c(xy)\}| + |\{z \in N_G(x) : c(xz) \neq c(xy)\}| - (n-2) \\ &\geq 2\delta_1^c(G) - n + 2. \end{aligned}$$

Note that if yz is a directed 2-cycle in H, then $\{x, y, z\}$ forms a properly colored triangle in G. Hence, we may assume that H does not contain any directed 2-cycles. Moreover, we may assume that if $\overrightarrow{zy}$ is in H, then $c(zy) = c(xy)$. By an averaging argument, there exists a vertex $y \in V(H)$ such that $d_H^-(y) \geq d_H^+(y) \geq 2\delta_1^c(G) - n + 2$. Therefore

$$\begin{aligned} d_G(y) &\geq |\{z \in N_G(y) : c(yz) = c(xy)\}| + |\{z \in N_G(y) : c(yz) \neq c(xy)\}| \\ &\geq d_H^-(y) + \delta_1^c(G) \geq 3\delta_1^c(G) - n + 2. \end{aligned}$$

Since $d_G(y) \leq n - 1$, the inequality above implies that $\delta_1^c(G) \leq 2n/3 - 1$, a contradiction. □

Finally, we prove Theorem 1.3.

Proof of Theorem 1.3. Without loss of generality, we may assume that $\varepsilon < 2^{-9}3^{-1}$. Choose $n_0 \in \mathbb{N}$ large so that $\varepsilon(1 - 2\varepsilon/3)n_0/3 \geq 1$, $n_0 \geq 243\varepsilon^{-3}$, and Lemma 4.1 holds. Let G be an edge-colored graph on n vertices with $\delta_1^c(G) \geq (2/3 + \varepsilon)n$ as stated in Theorem 1.3. By Proposition 5.1, G contains a properly colored triangle.

Suppose that ℓ is an integer with $4 \leq \ell \leq 2\varepsilon n/3$. Since $\delta_1^c(G) \geq (2/3 + \varepsilon)n$, we can greedily construct a properly colored path of length $2n/3$. In particular, G contains a properly colored path $P = x_1 x_2 \cdots x_{\ell-2}$ of length $\ell - 3$. Let G' be the subgraph of G obtained by removing all the vertices $x_3, x_4, \ldots, x_{\ell-4}$. Note that

$$\delta_1^c(G') \geq \delta_1^c(G) - (\ell - 6) \geq (2/3 + \varepsilon/3)n \geq 2|G'|/3 + 1.$$

Hence, Lemma 4.6 implies that there is an edge $z_1 z_2$ in G' such that $x_{\ell-3} x_{\ell-2} z_1 z_2 x_1 x_2$ is a properly colored path. Therefore, $x_1 x_2 \cdots x_{\ell-2} z_1 z_2 x_1$ is a properly colored cycle of length ℓ.

Suppose that $2\varepsilon n/3 < \ell \leq n$. Let C be the properly colored cycle given by Lemma 4.1, so $|C| \leq 2\varepsilon n/3$. Let G'' be the subgraph of G obtained after removing all the vertices of C. Note that

$$\delta_1^c(G'') > \delta_1^c(G) - |C| \geq (2+\varepsilon)n/3 \geq (2+\varepsilon)|G''|/3.$$

Note that $\varepsilon|G''|/3 \geq \varepsilon(1-2\varepsilon/3)n/3 \geq 1$. By Lemma 3.1, G'' can be covered by at most $\lfloor 6\varepsilon^{-1} \rfloor$ vertex-disjoint properly colored paths. That is, we can find vertex-disjoint properly colored paths $P_1, \dots, P_k$ in G'' such that $k \leq 6\varepsilon^{-1} \leq 8\varepsilon^2 n/243$, $V(P_i) \cap V(P_j) = \emptyset$ for all $i \neq j$ and $\bigcup_{1 \leq i \leq k} V(P_i) = V(G'')$. By removing vertices in the paths, we may assume that the paths $P_1, \dots, P_k$ span exactly $\ell - |C|$ vertices. By the property of C guaranteed by Lemma 4.1, there exists a properly colored cycle C' with $V(C') = V(C) \cup \bigcup_{1 \leq i \leq k} V(P_i)$. Note that $|C'| = |C| + \sum_{1 \leq i \leq k} |P_i| = \ell$. Therefore, C' is a properly colored cycle of length ℓ as required. ∎

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