Adding standardness to nonstandard models

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Let $M \models \text{PA}$ be nonstandard.

The set of standard natural numbers $\omega$ is an initial segment.

We may add a unary predicate $\omega$ for this, and obtain $(M, \omega)$.

Do structures of the form $(M, \omega)$ have interesting model theory?
**Background**

- **Henkin–Orey theorem.** Model theory of the omega rule:

\[
\begin{align*}
\phi(0), \phi(1), \ldots, \phi(k), \ldots \\
\forall x \in \omega \phi(x)
\end{align*}
\]

- Kanovei: *On external Scott algebras in nonstandard models of Peano arithmetic*, JSL 1996. Characterises for \( M \prec \omega \) the algebras

\[
\text{Rep}(M, \omega) = \{ A \subseteq \omega : A \text{ is 0-definable in } (M, \omega) \}
\]

General background in models of $\mathbb{PA}$. Many structural properties of $M$ (e.g. strength of $\omega$) can be expressed in a first order way in $(M, \omega)$.


$\text{SSy}(M) = \{ A \subseteq \omega : A \text{ is def'd in } (M, \omega) \text{ by } x \in \omega \land \theta(x, a) \}$

Encoding second order systems. $(M, \omega)$ interprets $(\omega, \text{SSy}(M))$, a model of at least $\text{WKL}_0$. 
Interpretation of second order arithmetic

- Replace number quantifiers $\forall x \ldots$ with $\forall x \in \omega \ldots$.
- Replace set quantifiers $\forall A \ldots u \in A \ldots$ with $\forall a \ldots (a)_u \neq 0 \ldots$.

- Everything else stays the same.
- $(M, \omega)$ interprets $(\omega, \text{SSy}(M))$, which can be arbitrarily strong.
Definitions

- \( \text{SSy}(M) \) is the set of \( A \subseteq \omega \) coded in \( M \).
- \( \text{Rep}(M) \) is the parameter-free version.
- \( \text{SSy}(M, \omega) \) and \( \text{Rep}(M, \omega) \) are similar, but for the expanded language.
- \( M \) is full if \( \text{SSy}(M) = \text{SSy}(M, \omega) \).
- \( M \) is semi-full if there is \( \theta(x, v) \) such that
  \[
  \text{SSy}(M, \omega) = \{ A \subseteq \omega : A = \{ x : \theta(x, a) \} \text{ some } a \in M \}
  \]
- \( M \) is fully saturated if \( M \) is full and recursively saturated.
Truth 1

Let $M \models PA$ and $K \subseteq M$, possibly but not necessarily definable in $(M, \omega)$. A *certificate* of truth in $K$ is an $M$-finite set of triples $\langle \phi, a, t \rangle$ such that

- $\langle \phi, a, t \rangle \in c \Rightarrow \langle \phi, a, 1 - t \rangle \notin c$
- $\langle \neg \phi, a, t \rangle \in c \Rightarrow \langle \phi, a, 1 - t \rangle \in c$
- $\langle \phi \land \psi, a, 1 \rangle \in c \Rightarrow \langle \phi, a, 1 \rangle \in c$ and $\langle \psi, a, 1 \rangle \in c$
- $\langle \phi \land \psi, a, 0 \rangle \in c \Rightarrow \langle \phi, a, 0 \rangle \in c$ or $\langle \psi, a, 0 \rangle \in c$
- $\langle \forall x_i \phi, a, 1 \rangle \in c \Rightarrow \forall b \in K \langle \phi, a[b/i], 1 \rangle \in c$
- $\langle \forall x_i \phi, a, 0 \rangle \in c \Rightarrow \exists b \in K \langle \phi, a[b/i], 0 \rangle \in c$
- $\langle \phi, a, 1 \rangle \in c \Rightarrow K \models \phi[a]$, when $\phi$ is atomic
- $\langle \phi, a, 0 \rangle \in c \Rightarrow K \not\models \phi[a]$, when $\phi$ is atomic
c is \( K\)-complete if it is uniform in \( K \): if some formula is decided by \( c \) then all \( K \)-substitution instances of it are decided too.

Two \( K \)-complete certificates agree on any formula they decide.

‘there is a \( K \)-complete certificate making \( \phi \) true’ would be a definition of truth for \( K \) in \((M, K, \omega)\) provided enough certificates exist.
Suppose $K = \omega$ is strong in $M$ (i.e. $\omega, \text{SSy}(M) \models \text{ACA}_0$). Then there are enough $K$-certificates.

Proof is by induction in the meta-theory.

Application (KKW). Let $M \models \text{PA}$ be a nonstandard model, and $\omega$ is strong in $M$. Let $(K, N)$ be a model of $\text{Th}(M, \omega)$ where $N$ is nonstandard. Then $N$ has a full inductive satisfaction class. In particular, $N$ is recursively saturated.
Elementary substructures

- Suppose $K \prec M$ is not cofinal. Then there are enough $K$-certificates.
- Proof is by induction in the meta-theory.
- Application (Kanovei). If $\omega \prec M$ then $\text{Th}(\omega, +, \cdot) \in \text{Rep}(M, \omega)$. 
Questions

Make reasonable assumptions on $M$ (recursively saturated, $\omega$ is strong, etc.)

- $\text{cl}(a) = \{ x \in M : x \text{ is definable in } M \} \prec M$.
- Does $\text{cl}(a)$ have a definition of truth in $(M, a, \omega)$?
- Is the set $\text{cl}(a)$ 0-definable in $(M, a, \omega)$?
- Is there a single formula in $(M, K, \omega)$ saying that $K \subseteq M$ is an elementary substructure?
Since \( \text{Th}(M, \omega) \) encodes strong second order arithmetic as well as many model theoretic properties, results can only be relative to what is known about second order arithmetic.

For a given \( T \) extending \( \text{PA} \) there are continuum-many theories \( \text{Th}(M, \omega) \) with \( M \models T \).

But, given \( T \), there is a canonical \( \text{Th}(M, \omega) \) for some \( M \models T \).
The canonical completion

- If $M_1, M_2$ are $\omega$-saturated then $M_1 \equiv_{\omega_1, \omega} M_2$ hence $(M_1, \omega) \equiv (M_2, \omega)$.
- Hence $T^\omega = \text{Th}(M, \omega)$ where $M \models T$ is $\omega$-saturated does not depend on $M$.
- More generally, for $\bar{a} \in N \models \text{PA}$, let $tp^\omega(\bar{a})$ be the canonical completion of $tp(\bar{a})$ to $\mathcal{L}_A, \omega, \bar{a}$. 
$\omega$-elementary models

- $N$ is $\omega$-elementary if $(N, \bar{a}, \omega) \models tp^\omega(\bar{a})$ for all $\bar{a}$.
- Equivalently, $N$ is $\omega$-elementary if $(N, \omega) \prec (M, \omega)$ for some $\omega$-saturated $M$.
- Countable $\omega$-elementary models exist by the Löwenheim–Skolem Theorem.
Transplendent models

- Kaye–Engström: A model $M$ is *transplendent* if it has expansions to any coded $T + p \uparrow$ that is consistent with $\text{Th}(M)$ in an $\omega$-saturated model.

- Transplendent models of $\text{PA}$ are $\omega$-elementary.

- If $N$ is $\omega$-elementary then $N$ is full.

- If $N$ is $\omega$-elementary then $(\omega, SSy(N)) \prec (\omega, P(\omega))$.

- *Are $\omega$-elementary models of $\text{PA}$ transplendent?*
This section makes some progress on $SSy(M, \omega)$ by looking at interpretations between first order arithmetic with $\omega$ and second order arithmetic.

- $(M, \omega)$ interprets $(\omega, SSy(M))$
- In fact, if $M$ is semi-full, $(M, \omega)$ interprets $(\omega, SSy(M, \omega))$

$$\forall A \ldots u \in A \ldots$$ is replaced by $$\forall a \ldots \theta(u, a, \omega) \ldots$$

- Corollary: if $M$ is semi-full then $(\omega, SSy(M, \omega)) \vdash CA_0$. 

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Adding standardness
Interpreting \((M, \omega)\) in \(L_{II}\)

For \(k \in \omega\) we define a family of partial translations \(\tau_k\) from \(L^\text{cut}_A\) to \(L_{II}\) describing properties on \((M, \omega)\) in terms of \((\omega, \text{SSy}(M))\).

- \(A^\bar{a}_k\) is the set \(\Sigma_k - \text{tp}(\bar{a})\) of all Gödel numbers of \(\Sigma_k\) formulas true of \(\bar{a}\) in \(M\).
- \((\psi(\bar{n}, \bar{a}))^{\tau_k}\) is the formula

\[
\psi(\text{clterm}(\bar{n}), x_1, \ldots, x_n) \in A^\bar{a}_k,
\]

defined when \(k\) is sufficiently large.

- \((\forall b \ \psi(\bar{n}, \bar{a}, b))^{\tau_k}\) is

\[
\forall A^\bar{a}, b \ (A^\bar{a}, b \text{ extends } A^\bar{a}_k \rightarrow (\psi(\bar{n}, \bar{a}, b))^{\tau_k}),
\]

where this is defined.
The interpretation commutes with usual boolean connectives, etc.

Nonstandard models of PA are weakly saturated, and all $\Sigma_k$, $\Pi_k$ types are coded.

Nevertheless, it seems that the interpretation is ‘local’: sufficiently large $k$ should be chosen for the formula in question.
Theorem: \( M \models \text{PA} \) is full if and only if \( (\omega, \text{SSy}(M)) \models \text{CA}_0 \).

Proof: one direction has been done; for the other, translate \( \theta(x, a, \omega) \) defining \( A \in \text{SSy}(M, \omega) \) into second order logic and apply comprehension.

There are full models \( N \models \text{PA} \) (indeed, recursively saturated ones) for which \( (\omega, \text{SSy}(N)) \) is not a \( \beta \)-model.
A sufficient condition for \( \text{SSy}(M, \omega) \)

- Theorem: If a countable Scott set \( \mathcal{X} \) has \((\omega, \mathcal{X}) \models \text{CA}_0\) then it is \( \text{SSy}(M, \omega) \) for some \( M \).
- In fact we may take \( M \) to be fully saturated here, . . .
- . . . or alternatively we may take \( M \) to be prime so that \( \mathcal{X} = \text{Rep}(M) = \text{Rep}(M, \omega) \).
Classification of Scott algebras

- If $M \models PA$ is nonstandard and $\mathcal{X} = \text{SSy}(M)$ then $\text{SSy}(M, \omega) = \text{Def}(\omega, \mathcal{X})$, the set of sets $A \subseteq \omega$ definable in $\text{Def}(\omega, \mathcal{X})$ (with parameters).

- Note that the comprehension scheme ($\text{CA}_0$) says that $\mathcal{X} = \text{Def}(\omega, \mathcal{X})$, but this is not true for all $\mathcal{X}$. 

Prime models

- Scott: if \((\omega, \mathcal{X}) \models WKL_0\) there is \(M \models PA\) such that \(\text{Rep}(M) = SSy(M) = \mathcal{X}\).
- There are such \(\mathcal{X}\) such that each \(A \in \mathcal{X}\) is \(\Pi^0_{\infty}\).
- Let \(M \models PA\) be prime such that each \(A \in SSy(M)\) is \(\Pi^0_{\infty}\). Then \(SSy(M, \omega) = \Pi^0_{\infty}\).
- Hence there are models \(M \models PA\) with \((\omega, SSy(M, \omega)) \not\models CA_0\) and \(M\) is not semi-full.
- In general truth on \(\omega\) is not definable in \((M, \omega)\) when \(M\) is not a model of true arithmetic.