Adding standardness to nonstandard models

Richard Kaye

School of Mathematics
University of Birmingham

6th December 2012
Let $M \models \text{PA}$ be nonstandard.

- The set of standard natural numbers $\omega$ is an initial segment.
- We may add a unary predicate $\omega$ for this, and obtain $(M, \omega)$.
- Do structures of the form $(M, \omega)$ have interesting model theory?
Background

- **Henkin–Orey theorem.** Model theory of the omega rule:

\[
\frac{\phi(0), \phi(1), \ldots, \phi(k), \ldots}{\forall x \in \omega \phi(x)}
\]

- Kanovei: *On external Scott algebras in nonstandard models of Peano arithmetic*, JSL 1996. Characterises for \( M \succ \omega \) the algebras

\[
\text{Rep}(M, \omega) = \{ A \subseteq \omega : A \text{ is 0-definable in } (M, \omega) \}\]

General background in models of PA. Many structural properties of $M$ (e.g. strength of $\omega$) can be expressed in a first order way in $(M, \omega)$.


$SSy(M) = \{ A \subseteq \omega : A \text{ is def'd in } (M, \omega) \text{ by } x \in \omega \land \theta(x, a) \}$

Encoding second order systems. $(M, \omega)$ interprets $(\omega, SSy(M))$, a model of at least WKL$_0$. 
Interpretation of second order arithmetic

- Replace number quantifiers $\forall x \ldots$ with $\forall x \in \omega \ldots$.
- Replace set quantifiers $\forall A \ldots u \in A \ldots$ with $\forall a \ldots (a)u \neq 0 \ldots$.
- Everything else stays the same.
- $(M, \omega)$ interprets $(\omega, SS\text{Sy}(M))$, which can be arbitrarily strong.
Definitions

- \( \text{SSy}(M) \) is the set of \( A \subseteq \omega \) coded in \( M \).
  \( \text{Rep}(M) \) is the parameter-free version.
- \( \text{SSy}(M, \omega) \) and \( \text{Rep}(M, \omega) \) are similar, but for the expanded language.
- \( M \) is *full* if \( \text{SSy}(M) = \text{SSy}(M, \omega) \).
- \( M \) is *semi-full* if there is \( \theta(x, v) \) such that
  \[
  \text{SSy}(M, \omega) = \{ A \subseteq \omega : A = \{ x : \theta(x, \omega, a) \} \text{ some } a \in M \}
  \]
- \( M \) is *fully saturated* if \( M \) is full and recursively saturated.
Suppose $\omega$ is strong in $M$, i.e. $(\omega, \text{SSy}(M)) \models ACA_0$. Then there is a definition of truth for $(\omega, +, \cdot, <)$ in $(M, \omega)$.

Application (KKW). Let $M \models PA$ be a nonstandard model, and $\omega$ is strong in $M$. Then $\text{SSy}(M, \omega)$ is closed under the $\omega$th jump map.

Application (KKW). Let $M \models PA$ be a nonstandard model, and $\omega$ is strong in $M$. Let $(K, N)$ be a model of $\text{Th}(M, \omega)$ where $N$ is nonstandard. Then $N$ has a full inductive satisfaction class. In particular, $N$ is recursively saturated.
Suppose $\omega \prec M$. Then there is a definition of truth for $(\omega, +, \cdot, <)$ in $(M, \omega)$.

Application (Kanovei). If $\omega \prec M$ then $\text{Th}(\omega, +, \cdot) \in \text{Rep}(M, \omega)$. 
Since $\text{Th}(M, \omega)$ encodes strong second order arithmetic as well as many model theoretic properties, results can only be relative to what is known about second order arithmetic.

For a given $T$ extending $\text{PA}$ there are continuum-many theories $\text{Th}(M, \omega)$ with $M \models T$.

But, given $T$, there is a canonical $\text{Th}(M, \omega)$ for some $M \models T$. 

If $M_1, M_2$ are $\omega$-saturated then $M_1 \equiv_{\omega_1, \omega} M_2$ hence $(M_1, \omega) \equiv (M_2, \omega)$.

Hence $T^\omega = \text{Th}(M, \omega)$ where $M \models T$ is $\omega$-saturated does not depend on $M$.

More generally, for $\bar{a} \in N \models \text{PA}$, let $t_p^\omega(\bar{a})$ be the canonical completion of $t_p(\bar{a})$ to $\mathcal{L}_A, \omega, \bar{a}$.
\( N \) is \( \omega \)-elementary if \( (N, \bar{a}, \omega) \models \text{tp}^\omega(\bar{a}) \) for all \( \bar{a} \).

Equivalently, \( N \) is \( \omega \)-elementary if \( (N, \omega) \prec (M, \omega) \) for some \( \omega \)-saturated \( M \).

Countable \( \omega \)-elementary models exist by the Löwenheim–Skolem Theorem.
Kaye–Engström: A model $M$ is *transplendent* if it has expansions to any coded $T + p↑$ that is consistent with $\text{Th}(M)$ in an $\omega$-saturated model.

- Transplendent models of $\text{PA}$ are $\omega$-elementary.
- If $N$ is $\omega$-elementary then $N$ is full.
- If $N$ is $\omega$-elementary then $(\omega, \text{SSy}(N)) \prec (\omega, \mathcal{P}(\omega))$.
- Are $\omega$-elementary models of $\text{PA}$ *transplendent*?
This section makes some progress on $SSy(M, \omega)$ by looking at interpretations between first order arithmetic with $\omega$ and second order arithmetic.

- $(M, \omega)$ interprets $(\omega, SSy(M))$
- In fact, if $M$ is semi-full, $(M, \omega)$ interprets $(\omega, SSy(M, \omega))$

\[ \forall A \ldots u \in A \ldots \text{is replaced by } \forall a \ldots \theta(u, a, \omega) \ldots \]

- Corollary: if $M$ is semi-full then $(\omega, SSy(M, \omega)) \models CA_0$. 

Richard Kaye

Adding standardness
For \( k \in \omega \) we define a family of partial translations \( \tau^k \) from \( \mathcal{L}^\text{cut}_A \) to \( \mathcal{L}_{II} \) describing properties on \((M, \omega)\) in terms of \((\omega, \text{SSy}(M))\).

- \( A^\bar{a}_k \) is the set \( \Sigma^k_{TP}(\bar{a}) \) of all Gödel numbers of \( \Sigma^k \) formulas true of \( \bar{a} \) in \( M \).
- \( (\psi(\bar{n}, \bar{a}))^{\tau^k} \) is the formula

\[
\psi(\text{clterm}(\bar{n}), x_1, \ldots, x_n) \in A^\bar{a}_k,
\]

defined when \( k \) is sufficiently large.

- \( (\forall b \ \psi(\bar{n}, \bar{a}, b))^{\tau^k} \) is

\[
\forall A^\bar{a},b_k \ (A^\bar{a},b_k \text{ extends } A^\bar{a}_k \rightarrow (\psi(\bar{n}, \bar{a}, b))^{\tau^k}),
\]

where this is defined.
Interpreting \((M, \omega)\), continued

- The interpretation commutes with usual boolean connectives, etc.
- Nonstandard models of PA are weakly saturated, and all \(\Sigma_k\), \(\Pi_k\) types are coded.
- Nevertheless, it seems that the interpretation is ‘local’: sufficiently large \(k\) should be chosen for the formula in question.
Full models

- Theorem: $M \models \text{PA}$ is full if and only if $(\omega, \text{SSy}(M)) \models \text{CA}_0$.
- Proof: one direction has been done; for the other, translate $\theta(x, a, \omega)$ defining $A \in \text{SSy}(M, \omega)$ into second order logic and apply comprehension.
- There are full models $N \models \text{PA}$ (indeed, recursively saturated ones) for which $(\omega, \text{SSy}(N))$ is not a $\beta$-model.
Model theoretic consequences

- \((M, \omega) \prec (K, \omega)\) iff \(M \prec K\) and \((\omega, \text{SSy}(M)) \prec (\omega, \text{SSy}(K))\)
- \(M\) is \(\omega\)-elementary iff \((\omega, \text{SSy}(M)) \prec (\omega, \mathcal{P}(\omega))\)
- \(M \prec_e K\) implies \(\text{SSy}(M, \omega) = \text{SSy}(K, \omega)\)
- \(M \prec_{cf} K\) does not preserve \(\text{SSy}(-, \omega)\). In fact if \(M \models \text{PA}\) is countable and \((\omega, \text{SSy}(M)) \prec (\omega, \mathcal{K})\) for some uncountable \(\mathcal{K}\) then there is \(M \prec_{cf} K\) with \((M, \omega) \prec (K, \omega)\) and \(\text{SSy}(M, \omega) \subsetneq \text{SSy}(K, \omega)\)
Classification of Scott algebras

- If $M \models PA$ is nonstandard and $\mathcal{X} = \text{SSy}(M)$ then $\text{SSy}(M, \omega) = \text{Def}(\omega, \mathcal{X})$, the set of sets $A \subseteq \omega$ definable in $\text{Def}(\omega, \mathcal{X})$ (with parameters).

- Some similar information is obtained for $\text{Rep}(M, \omega)$ but the definition of $A \in \text{Rep}(M, \omega)$ in $\text{Def}(\omega, \mathcal{X})$ requires a parameter from $\mathcal{X}$.

- Note that the comprehension scheme ($\text{CA}_0$) says that $\mathcal{X} = \text{Def}(\omega, \mathcal{X})$, but this is not true for all $\mathcal{X}$. 

Richard Kaye

Adding standardness
Question

Characterise the countable Scott sets $\mathcal{K}$ that arise as $\text{SSy}(M,\omega)$ for some nonstandard model $M \models \text{PA}$.

Necessary condition: $\mathcal{K}$ is closed under jump.

Theorem

Let $T \supseteq \text{PA}$ be arithmetic and $\mathcal{K} \subseteq \mathcal{P}(\omega)$ be the set of arithmetic sets. Then the prime model $P_T \models T$ has $\text{SSy}(M,\omega) = \mathcal{K}$.

Corollary: In general, truth on $\omega$ is not definable in $(M,\omega)$ when $M$ is not a model of true arithmetic.
Question

Characterise the countable Scott sets $\mathcal{K}$ that arise as $\text{SSy}(M, \omega)$ for some nonstandard model $M \models \text{Th}(\mathbb{N})$.

Necessary conditions: $\mathcal{K}$ contains $0^{(\omega)}$ and is closed under jump. By a relativisation of the previous result,

Theorem

Let $T \supseteq \text{Th}(\mathbb{N})$ be arithmetic in some $A \supseteq 0^{(\omega)}$ and $\mathcal{K} \subseteq \mathcal{P}(\omega)$ be the set of sets arithmetic in $A$. Then there is a model $M = \text{cl}_M(a) \models T$ with $\text{SSy}(M, \omega) = \mathcal{K}$.
Conjectures

<table>
<thead>
<tr>
<th>Conjecture</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A countable Scott set $\mathcal{K}$ is $\text{SSy}(M, \omega)$ for some nonstandard model $M \models \text{PA}$ iff $\mathcal{K}$ is closed under jump.</td>
<td></td>
</tr>
<tr>
<td>A countable Scott set $\mathcal{K}$ is $\text{SSy}(M, \omega)$ for some nonstandard model $M \models \text{Th}(\mathbb{N})$ iff $\mathcal{K}$ contains $0^{(\omega)}$ and is closed under jump.</td>
<td></td>
</tr>
</tbody>
</table>

Kanovei proved the last one above for $\text{Rep}(M, \omega)$. 
When $\omega$ is strong

**Question**

When is a countable Scott set $\mathcal{X} \in \text{SSy}(M, \omega)$ for some nonstandard model $M \models \text{PA}$ in which $\omega$ is strong?

Necessary condition (KKW): closure under $a \mapsto a^{(\omega)}$.

**Conjecture**

A countable Scott set $\mathcal{X}$ is $\text{SSy}(M, \omega)$ for some nonstandard model $M \models \text{Th}(\mathbb{N})$ in which $\omega$ is strong iff $\mathcal{X}$ is closed under $a \mapsto a^{(\omega)}$. 
Questions on ‘semi-full’

Recall, $M$ is semi-full if there is $\theta(x, \nu)$ such that

$$SSy(M, \omega) = \{A \subseteq \omega : A = \{x : \theta(x, \omega, a)\} \text{ some } a \in M\}$$

**Question**

Do there exist countable models $M \models PA$ which are semi-full but not full?

**Question**

Is every $M \models PA$ with $(\omega, SSy(M, \omega)) \models CA_0$ semi-full?