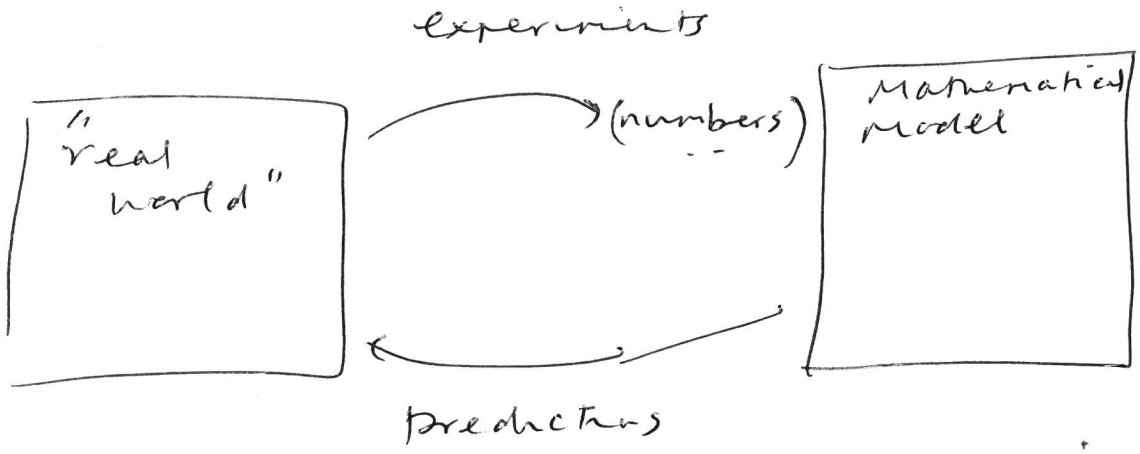
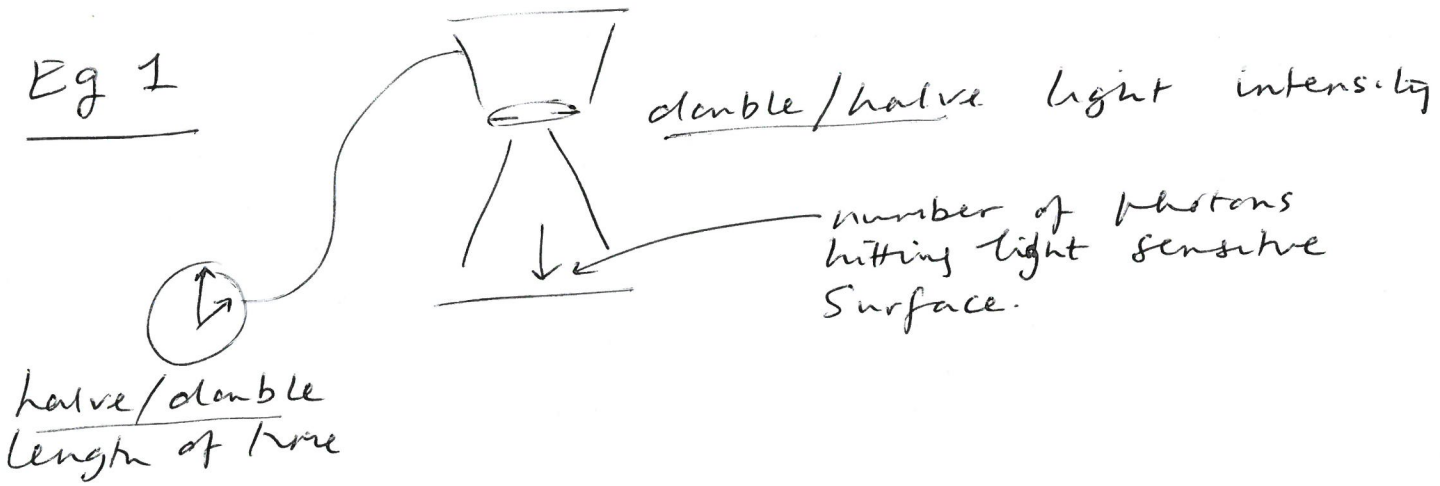
Towers to structuralism

Mathematical modelling

This is a rather "familiar" but often misinterpreted idea of (forgotten) science



Eg 1



Failure of this model (at very low light levels & typically higher temperatures) is called reciprocity failure

Eg 2 Kant famously regarded Mathematics as synthetic a priori

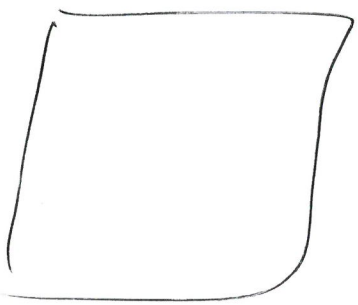
synthetic: judgements about the "real world"

a priori: argued or proved from Logic

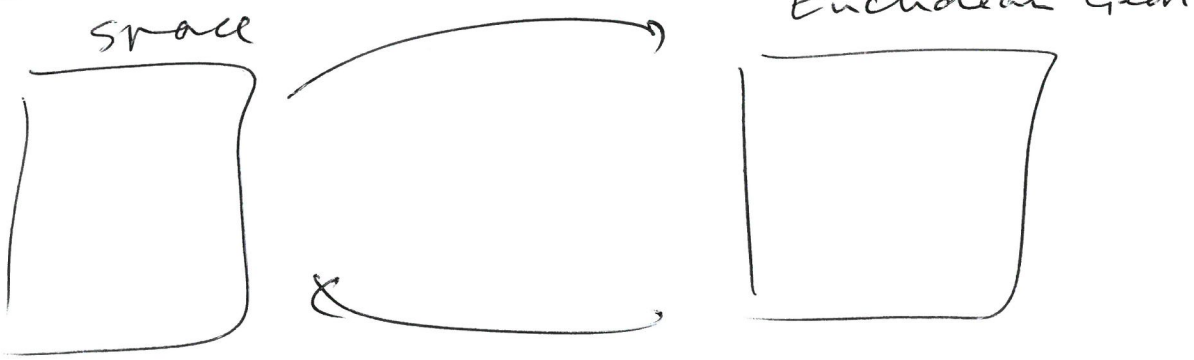
Why?

Ignoring numbers for now, consider geometry. Kant regarded Euclidean geometry not as a model for space but as space itself.

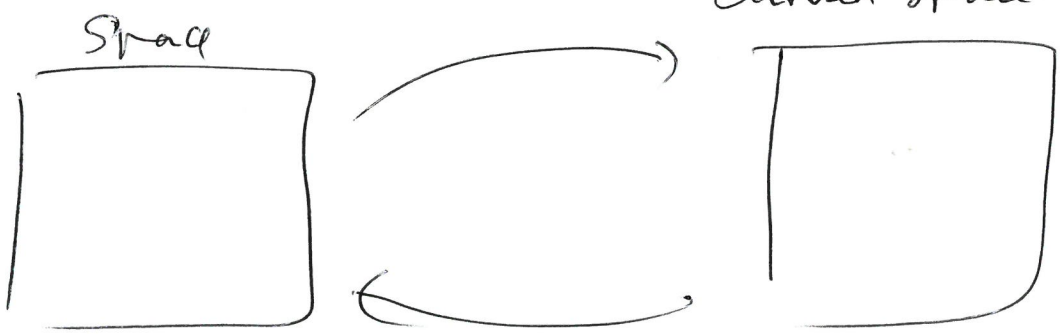
Kant:



Post Kant:



Post Einstein:



Gowers approaches abstraction from this point of view ("various meanings of the word "abstract" p16, & most of chapter 2)

"abstracting away" simplifying a model by removing (inessential) features.

"abstract objects" objects that are not concrete or tangible

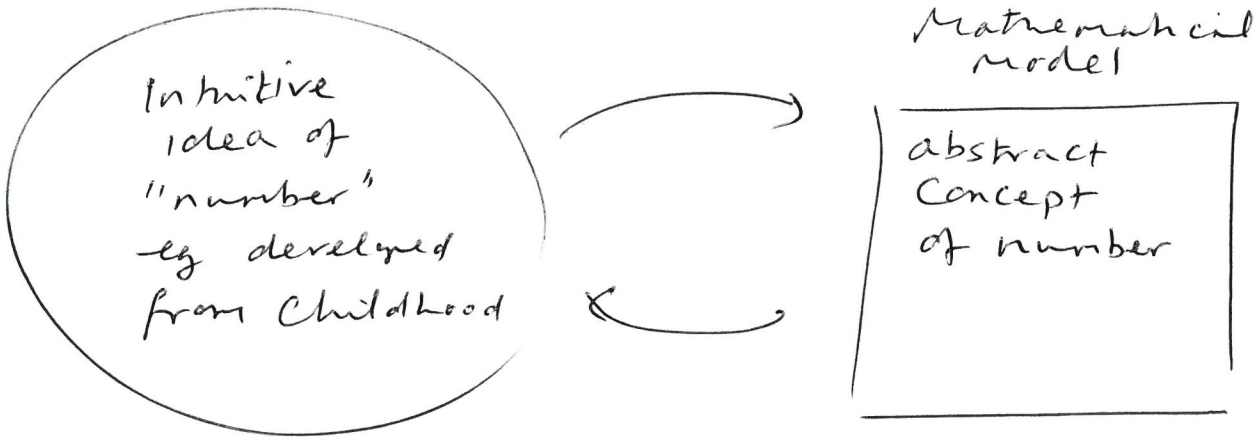
[Gowers p16. But what does this mean...?]

"No need to understand what the objects represent" eg two (sheep, cows molecules...)

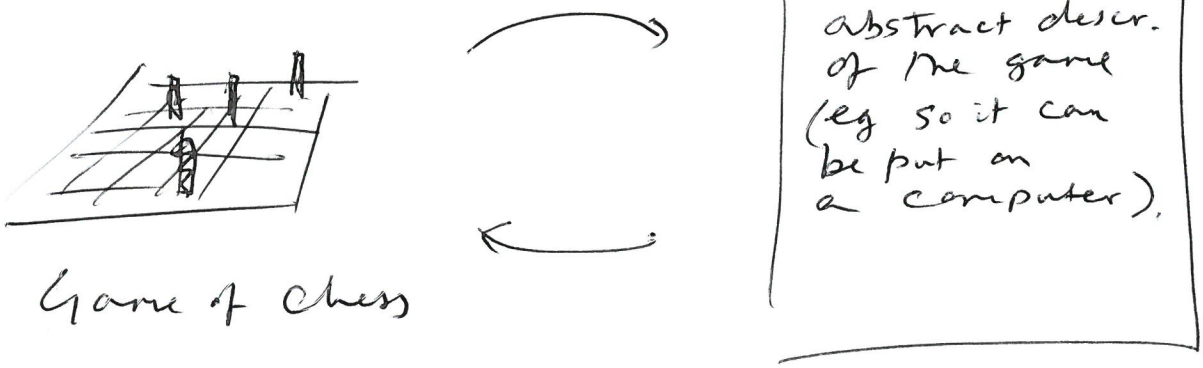
The first is methodologically essential in maths. The other two raise issues of ontology. Does representing something suffice for existence? What if you don't know what it represents? What if you don't know (have a tangible grasp of) the object doing the representation?

* DIAGRAM *

(Some of this is my gloss on Gowers).



Gowers' analogy



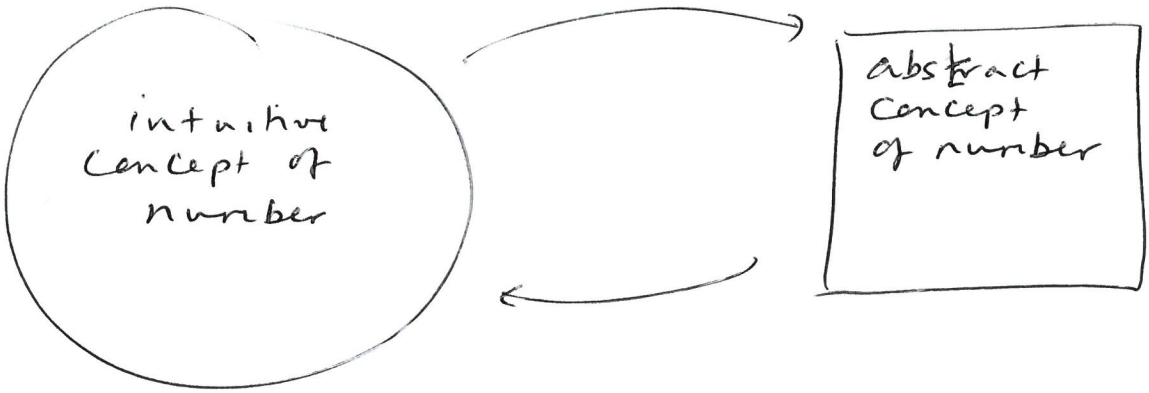
Questions • Does the black king exist?

- If all chess sets in the world spontaneously self-disintegrated would "chess" still exist?
- What do you do if the wind blows over the white king?

(Some question in tennis: What if the wind blows the ball so it lands outside the court?)

Feynman: "mathematics is not science."

(I used to agree with this 100% but now I am not so sure...)



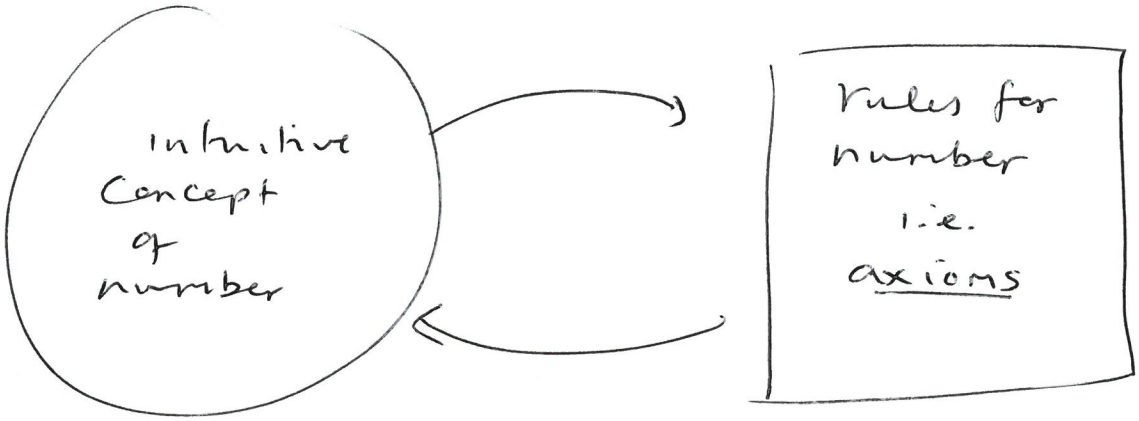
experiments

- look at 3 sheep in a field
  count them.
  count them again
- Look at 3 ^{235}U atoms
  count them.
  count them again.
- count 21 objects in several different ways (7 groups of 3, 3 groups of 7, one after another...)

However mathematics is unlike science in the sense that almost all (all?) the activity is "inside" the abstract world (analytic, apriori)

* COMMENT/Q *

following directly from the chess example



It is unfortunate that "Sets of axioms" are (implicitly) Models in Gowers.

Also,
Gowers takes the "popular" view that axioms are cumulative. (add new axioms for new kinds of numbers. No need to throw any away.) We know that he knows this is a simplification!

Eg

Commutativity of + ×
Associativity of + ×
Distributivity of × over +.
|
additive inverses
|
multiplicative inverses
(for nonzero numbers)
|
Square roots, algebraic eqns, limits ...

Then:

Example Commutativity of $\times$ & other axioms

$\Downarrow$

4-dim. division alg over $\mathbb{R}$
(quaternions, or $\mathbb{H}$)

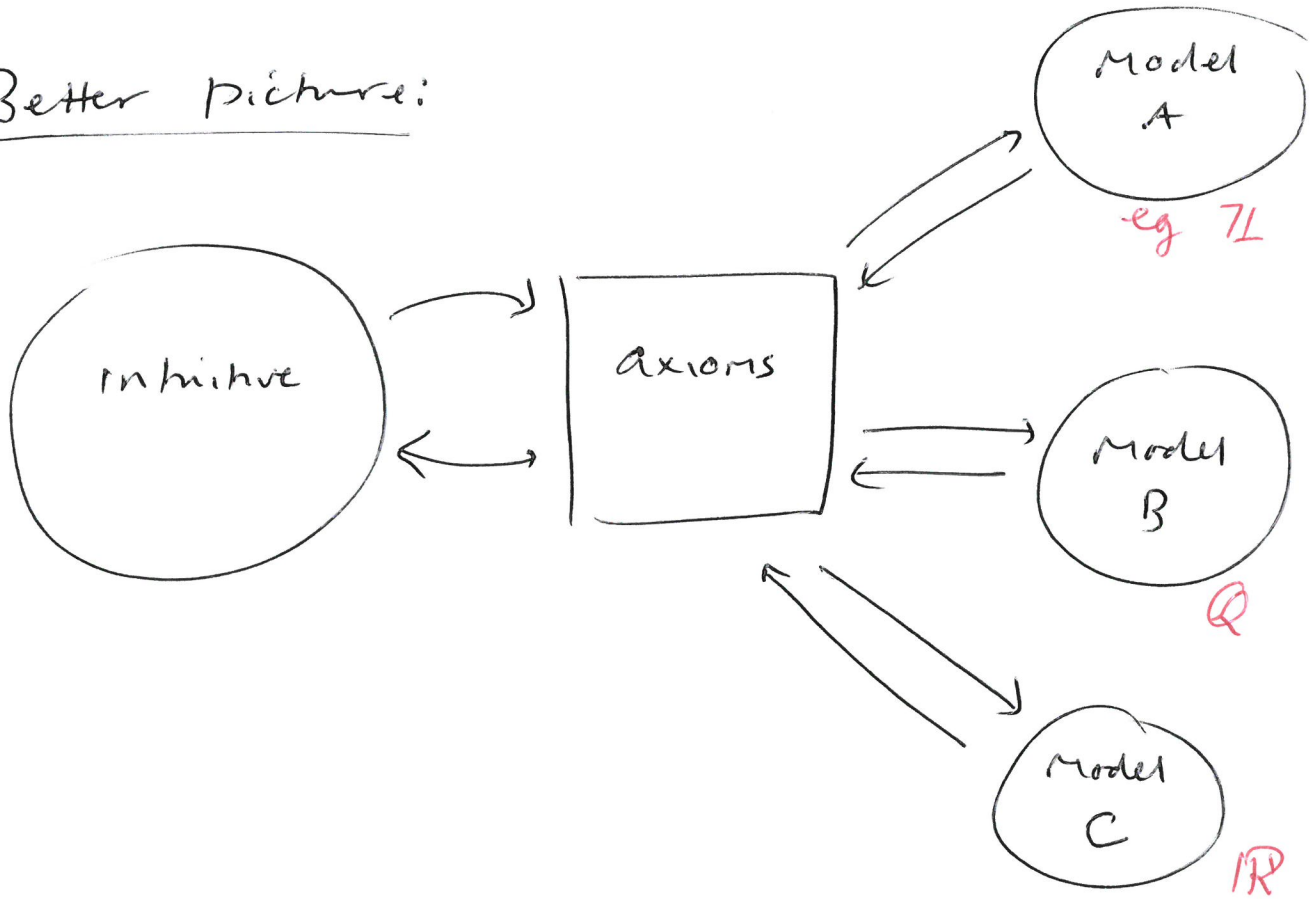
Example discreteness axiom for $\mathbb{Z}$
$\forall x,y\,(x > y \to x \geq y+1)$ + others

$\Downarrow$

multiplicative inverses

Example There is an object ∞ such that $\infty + x = \infty$ and $\infty \times x = \infty$ for all x.

Better picture:



Structuralism: The axioms (and their consistency, construed in one or other way) are what is important for mathematical existence (ontology)

Gowers: it's not what numbers are that matters, but what they do. "a mathematical object is what it does" (p18)

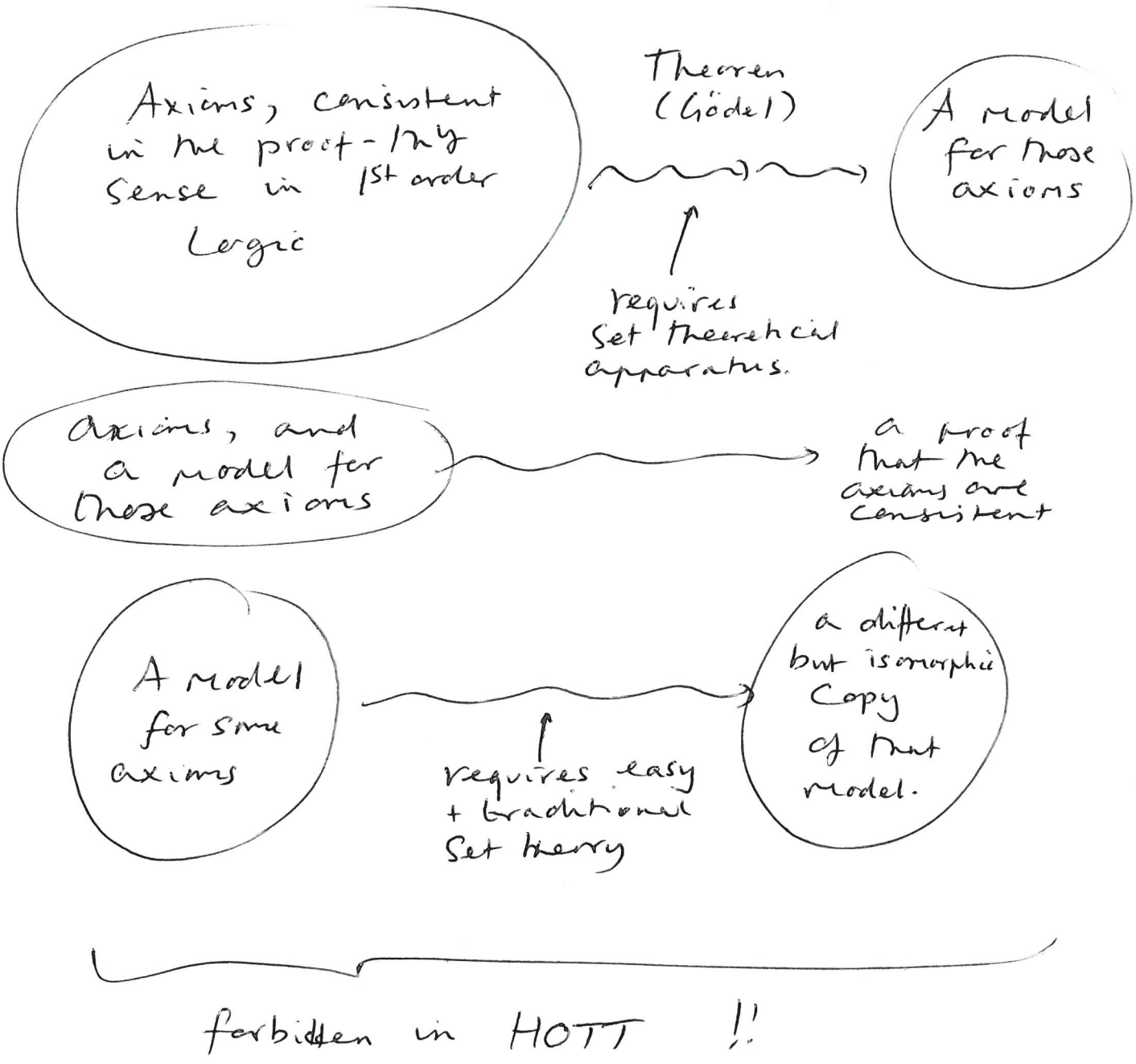
— DISCUSSION —

practical mathematics: if I use the axioms I don't need to worry if it's sheep or golden eggs or something else I am counting.

Computer view: the axioms are the API (and the contract they satisfy) If I restrict to this API, I can change the implementation later on.

Generalisation: Axioms generalise. Allow theorems to be applied to other situations (Yes, but sometimes this is a red herring. Generalisation is not the objective but rather a happy byproduct.)

Aspects of Structuralism



$\mathbb{R}$ is defined as,

1) The set of Dedekind cuts on $\mathbb{Q}$

bounded $A \subseteq \mathbb{Q}$ closed under $<$ with no greatest element

with $+, \times$ defined appropriately

2) The set of equivalence classes of Cauchy sequences over $\mathbb{Q}$.

3) The set of <u>almost linear transformations</u>

($f: \mathbb{Z} \longrightarrow \mathbb{Z}$ s.t.

$$\{ f(n+m) - f(n) - f(m) : n, m \in \mathbb{Z} \}$$

is finite.) <u>up to equivalence</u>

(<u>Lin</u>ear would imply this set is $\{0\}$

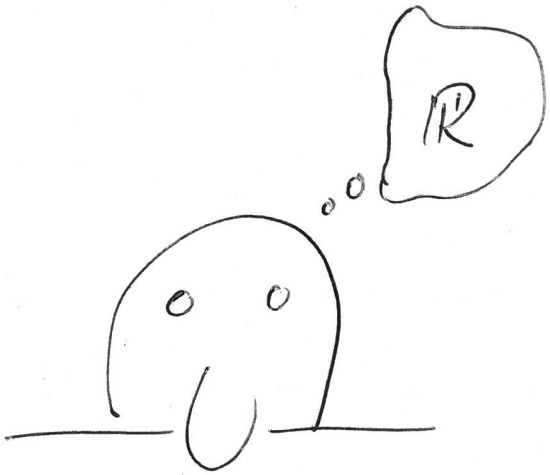
$f \sim g$ if $\{ f(n) - g(n) : n \in \mathbb{Z} \}$ is finite

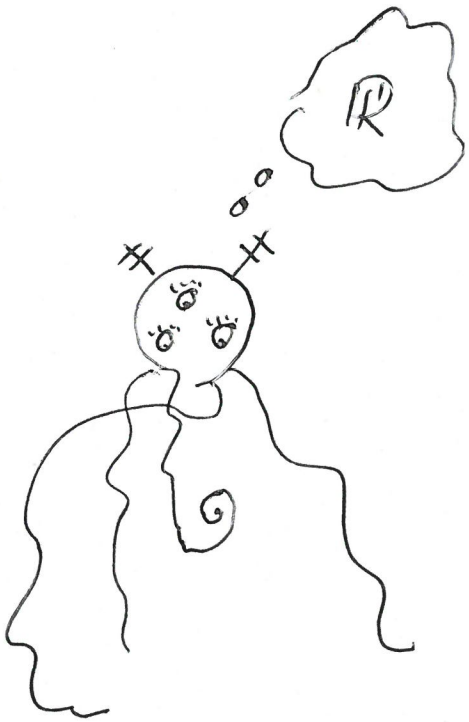
<u>Example</u> $\beta \in \mathbb{R}$

$f(n) = [\beta n]$

$[\beta n] + [\beta m] > \beta(n+m) - 2$ so

$f(n+m) - f(n) - f(m) \in \{0, 1\}$.





Axioms allow discussion!

(especially if the axioms are

Canonical

(any two models are isomorphic)

eg $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}$

Canonically Canonical

(any two models are isomorphic and there is only one possible isomorphism)

eg $\mathbb{R}$, $\mathbb{Z}$ but not $\mathbb{C}$.

Flavours of structuralism

Axioms / theories / rules

"talks about" "describes"

Structures : the concept of a domain (domains) of objects with operations (what the objects do)

"describes"

Models : The actual mathematical 'thing' that instantiates a structure

Which comes first ?

A. Models

Structures arise as abstractions or equivalence classes of models

Rules are linguistic statements describing models

B. Structures

Models exist as (secondary) instantiations of structures.

C. Models & structures co-exist and are "created" at the same time

Axioms (etc) are objects in certain structures or models and the relation of axioms to structures is described in larger enclosing structures.

D. Structures & models are essentially the same as two (isomorphic) models for the same structure behave in the same way.

E. Structures & models are the same as any two isomorphic models are identical (Univalence axiom, HoTT)

F. Structures are mathematical models of intuitive concepts (so we have an intuitive idea of "Hilbert space" and not just as a construction in a formal theory)

Q: Do <u>incomplete</u> structures or models exist?

G. Structures and models are convenient fictions for the (necessary) control of assumptions (axioms, API, ...) and "hiding" of unnecessary / irrelevant features

H. Theories and axioms come first and structures & models arise as ideal objects realising consistent theories.