Right-orderability of groups

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This short paper sets out necessary and sufficient conditions for a group to be right-orderable. In fact I give conditions for an partial order $\leqslant$ on a group G to extend to a linear order respected by multiplication on the right. The condition is a sort of mini completeness/soundness theorem and this may also prove to be instructive as an exercise in logic, as an illustration of the usual soundness and completeness theorems for logic, and application of some of the ideas from this area. It puts some interesting light on the classical application of first-order logic that a partial order can always be extended to a linear order—an application which can in fact be proved directly and quite easily using Zorn's lemma.

Terminology. An *order* is a partial order relation on a set G. Here, the set G is fixed. Thus we may say without ambiguity that the order $\leqslant$ *extends* to a linear order $\leqslant^*$, meaning $\leqslant^*$ is a linear order on G and $\leqslant \subseteq \leqslant^*$. (I am thinking of relations on G as subsets of G^2 in the usual way.) Usually, the set G here will be a group. The order $\leqslant$ on G is a *right-order* if $x \leqslant y \Rightarrow xz \leqslant yz$ for all $x, y, z \in G$. It is a *right linear-order* if in addition it is a linear order.

A mini-proof system. I will present necessary and sufficient conditions for $\leqslant$ on G to extend to a right linear-order on G by introducing a little proof-system.

In this system, the well-formed statements are falsity, $\perp$, and those of the form $a \leqslant b$ for $a, b \in G$. (There are no connectives or variables.) If I write $ab \leqslant cd$, this is the statement $x \leqslant y$ where $x \in G$ is the element ab and y is cd. (Well-formed statements do not contain multiplications and each statement of the form $a \leqslant b$ is only allowed to have the two parameters a, b from G. We may however indicate certain equations true in G by annotating the inequalities with equations—see below for examples. These annotations are not part of the system.) The system is a natural-deduction system where assumptions may be made and later discharged. I indicate assumptions and their scope by vertical lines down the left-hand side of the proof. (This is a convenient alternative to presenting a natural deduction proof as a tree.) Only statements in the current scope or 'above' the current scope (i.e., when the proof is regarded as a tree) may be used at any stage.

Here are the six derivation rules G0–G5 for a given group G and an order $\leqslant$ on G.

G0 You may write down $a \leqslant a$ whenever $a \in G$.

G1 You may write down $a \leqslant b$ whenever this is true in G.

G2 From $a \leqslant b$ and $b \leqslant c$ you may deduce $a \leqslant c$.

G3 From $a \leqslant b$ you may deduce $ac \leqslant bc$.

G4 From $a \leqslant b$ and $b \leqslant a$ for $a \neq b$ in G, you may deduce $\perp$.

G5 (Discharging assumptions.)

$\dots$	(1)	
$\quad a \leqslant b$	(2)	assumption
$\quad \dots$	(3)	
$\quad \perp$	(4)	
$b \leqslant a$	(5)	G5

If there is a derivation of this sort in which the last statement θ does not depend on any assumption (i.e., is not to the right of any vertical line) then we write $(G, \leqslant) \vdash \theta$.

Examples. (A) Suppose that $x, y \in G$, $x \neq y$ and $a, b \in G$ with $xa \leqslant y$, $xb \leqslant y$ and $ab = 1$.

$\quad y \leqslant x$	(1)	assumption
$\quad xa \leqslant y$	(2)	G1
$\quad xb \leqslant y$	(3)	G1
$\quad yb \leqslant xb$	(4)	G3
$\quad yb \leqslant y$	(5)	G2
$\quad x \leqslant yb$	(6)	G3 (from 2, as $ab = 1$)
$\quad x \leqslant y$	(7)	G2
$\quad \perp$	(8)	G4 (from 1,7)
$x \leqslant y$	(9)	G5

(B) Suppose that $x, y, a, b \in G$ and $xa = yb \neq xb = ya$.

$x \leqslant y$	(1)	assumption
$\quad xa \leqslant xb$	(2)	assumption
$\quad xb \leqslant yb = xa$	(3)	G3
$\quad xb \leqslant xa$	(4)	G2
$\quad \perp$	(5)	G4
$xb \leqslant xa$	(6)	G5
$xa \leqslant ya = xb$	(7)	G3
$xa \leqslant xb$	(8)	G2
$\perp$	(9)	G4
$y \leqslant x$	(10)	G5
$\quad ya \leqslant yb$	(11)	assumption
$\quad yb \leqslant xb = ya$	(12)	G3
$\quad yb \leqslant ya$	(13)	G2
$\quad \perp$	(14)	G4
$yb \leqslant ya$	(15)	G5
$ya \leqslant xa = yb$	(16)	G3
$ya \leqslant yb$	(17)	G2
$\perp$	(18)	G4

So the group G is not right-orderable. (See the soundness theorem below.) This case (which is well know in the literature as an example of failure of the 'unique product' property) is of interest as the initial conditions apply to any group G, not necessarily a group with an order relation.

Extending an order to a right linear order. In the following results, $(G, \leqslant)$ is a group with a binary relation $\leqslant$. (No axioms for $\leqslant$ are required, though if $\leqslant$ is reflexive then rule $G0$ is just a special case of $G1$. To apply the results below to a pure group G, either use the empty relation for $\leqslant$, or else use the minimal reflexive relation $\leqslant = \{(x, x) : x \in G\}$ and omit the rule $G0$.)

Theorem 1 (Soundness). *(a) If $(G, \leqslant) \vdash x \leqslant y$ then $x \leqslant^* y$ in any right linear-order $\leqslant^*$ extending $\leqslant$.*

(b) If $(G, \leqslant) \vdash \perp$ then the order $\leqslant$ on G cannot be extended to a right linear-order.

Proof. An easy induction on the length of derivations. □

Say that $(G, \leqslant)$ is *consistently right-orderable* (or *consistent* for short) if it is not the case that $(G, \leqslant) \vdash \perp$. It is *inconsistent* if it is not consistent.

Lemma 2. *If $(G, \leqslant)$ is consistent and $\leqslant^*$ is defined by*

$$x \leqslant^* y \;\Leftrightarrow\; (G, \leqslant) \vdash x \leqslant y,$$

then $(G, \leqslant^*)$ is consistent.

Proof. Suppose $(G, \leqslant^*) \vdash x \leqslant y$ and $(G, \leqslant^*) \vdash y \leqslant x$ for some $x \neq y$. Then we may replace all the statements $a \leqslant b$ in these proofs that are given by rule G1 (i.e., because $a \leqslant^* b$ holds) by their proofs in $(G, \leqslant)$. Thus $(G, \leqslant)$ is inconsistent. □

Lemma 3. *If $(G, \leqslant)$ is consistent and $\leqslant^*$ is defined by*

$$x \leqslant^* y \;\Leftrightarrow\; (G, \leqslant) \vdash x \leqslant y,$$

then $\leqslant^*$ is a right (possibly partial) order on G.

Proof. If $\leqslant^*$ is not antisymmetric, then $(G, \leqslant) \vdash x \leqslant y$ and $(G, \leqslant) \vdash y \leqslant x$ for some $x \neq y$, so $(G, \leqslant)$ is inconsistent—contrary to hypothesis. Similarly, transitivity is by G2, and right multiplication by G3. □

Theorem 4 (Completeness). *If $(G, \leqslant)$ is consistent then there is $\leqslant^*$ extending $\leqslant$ such that $\leqslant^*$ is a right linear-order on G.*

Proof. Let X be the set of all relations $\leqslant^*$ extending $\leqslant$ such that $(G, \leqslant^*)$ is consistent. Since a derivation may use at most finitely many applications of rule G1, the union of a chain of elements of X is in X, so by Zorn's Lemma, X has an element $\leqslant^*$ which is maximal with respect to $\subseteq$. By Lemma 2, $x \leqslant^* y \Leftrightarrow (G, \leqslant^*) \vdash x \leqslant y$, and by Lemma 3, $\leqslant^*$ is a right-order on G. It suffices to prove it is linear.

Suppose that $a, b \in G$ and $a \leqslant^* b$ does not hold. We must show $b \leqslant^* a$. To this end, define $\leqslant'$ by $\leqslant' = \{(a, b)\} \cup \leqslant^*$, which properly extends $\leqslant^*$. Hence $\leqslant'$ is inconsistent. It follows that there is a derivation from $(G, \leqslant^*)$ of the form

$a \leqslant b$	(1)	assumption
$\cdots$	(2)	
$\perp$	(3)	
$b \leqslant a$	(4)	G5

i.e., $b \leqslant^* a$, as required. □