The Arithmetic of Cuts in Models of Arithmetic

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Abstract

We present a number of results on the structure of initial segments models of Peano arithmetic with the arithmetic operations of addition, subtraction, multiplication, logarithm and a notion of derivative on cuts. Each of these operations is defined in two dual ways, often with quite different results, and we attempt to systemise the issues and show how various calculations may be carried out.

The main issues arise because the notion of 'infinitesimal' is relative to the object being considered and not absolute, and this work constitutes a case study. It also presents some important background on using cuts as numbers, often assumed but not studied in detail in models of arithmetic. It also provides an analysis of the kinds of duality that play a part in these considerations.

1 Introduction and preliminaries

Dedekind's construction of the real numbers shows that there is a arithmetic structure with addition and multiplication (and more) on the set of proper cuts of $\mathbb{Q}$. For a nonstandard model of arithmetic, the set of its cuts is the most important structural feature and a great deal of work in nonstandard models (for example Friedman [2], Paris and Kirby [7]) involves their cuts. It is common knowledge that these cuts also have arithmetic structure, and some rudimentary form of this structure is sometmes used to name cuts, but I am unaware of any paper where this structure is worked out fully.

In fact, the arithmetic structure of such cuts is an 'elementary' but surprisingly tricky topic in models of Peano arithmetic: there are a number of possible definitions of addition and multiplication (for example) on initial segments of such models, and one doesn't get a simple algebraic structure such as an ordered field or even another nonstandard model of Peano arithmetic. When working on a related problem and trying to apply such operations the author quickly discovered that calculations using these operations are rather more tricky than was expected. This paper contains an attempt to systemise the issues, present the main unifying ideas in a helpful way, and show how the various calculations may be carried out.

To motivate the work presented here more thoroughly, it has already been noted that initial segments and cuts are key to understanding many structural properties of models of arithmetic. Therefore a model should be understood in conjunction with its Dedekind completion. Indeed, in many ways this is essential in work with uncountable models. Friedman showed that nonstandard models are isomorphic to many of their initial segements, and the different ways this happens can give important structural information. Paris and Kirby showed that indicators give rise to initial segments, and conversely that the existence of initial segments with particular properties can tell one about first order formulas true in the model.

Another application, one that may be considered part of an on-going research programme, concerns using initial segments to count external sets. In other words, it tries to explain the importance of initial segments by mapping external subsets of the model to initial segments. If $M \vDash \mathrm{PA}$ and $X \subseteq M$ is a bounded subset, but otherwise arbitrary, we may approximate X by internal (M-finite) sets $x \in M$ such that (the extension of) x is a subset of X. How closely X can be approximated in this way becomes of central interest. We define the *lower cardinality of* X, $\underline{\mathrm{card}}\, X$ as the supremum of all cardinalities $\mathrm{card}\, x \in M$ taken over such $x \subseteq X$. Then $\underline{\mathrm{card}}\, X$ is in general a cut of M. Similarly we may approximate X by internal sets $x' \in M$ such that x' is a superset of X. The *upper cardinality of* X, $\underline{\mathrm{card}}\, X$ as the infimum of all $\mathrm{card}\, x' \in M$ taken over such $x' \supseteq X$. In many cases these cuts agree and $\underline{\mathrm{card}}\, X = \overline{\mathrm{card}}\, X$, and the structural properties of X are related to the size of this cut. This is what is happening in a number of important results, including the characterisation of closed normal subgroups of $\mathrm{Aut}\, M$ when M is countable and recursively saturated [4, 5], the external normal subgroup structure of internal groups such as $\mathrm{Sym}(n)$ [1] and forthcoming work by Kaye and Reading [8, 6].

In all such cases, it is essential to understand the arithmetic structure of such initial segments. For example, Reading has considered the size of transversals and the index $[G{:}H]$ of externally defined subgroups $H \leqslant G$ of an internal group $g \in M$. Of course one expects $[G{:}H] = \mathrm{card}(G)/\,\mathrm{card}(H)$ and therefore one needs the appropriate notion of division on initial segments. The current paper is intended to be supplementary to the more exciting work on applications of the cardinality-as-cut idea, and we have found that a detailed understanding of this arithmetic is helpful to understand the strengths and limitations of this idea. For further work in these directions, the reader is directed to Reading's thesis at Birmingham university and planned future papers by him and the present author.

To sum up the findings of the present paper: arithmetic of cuts behaves formally in a similar way to that of arithmetic of cuts in the rationals. However, it is more complicated in that there are usually two different limiting processes to calculate each operation, corresponding to upper and lower cardinality above, and in many cases these operations give different answers. Often the distinction between these two operations can be understood in terms of the derivative ∂I of a cut, or one of its variations $\partial^n I$, to be defined below. This is worked through in some detail for the additive structure below. The derivative bahaves in some respects like a derivative in classical analysis, with some further complications. In the main, these complications arise from the fact that unlike $\mathbb{Q}$ or $\mathbb{R}$ where the notion of 'infinitesimal' $\mathrm{d}x$ (and in particular the size of such values) is rather uniform across the whole structure, but the notion of 'infinitesimal' $\mathrm{d}i \in \partial I$ varies with the cut I under consideration.

The rest of this section will contain some preliminaries, setting the scene and stating a few convenient but important conventions that will aid our work here considerably. Some of these conditions are slightly different to the normal ones found in the literature, so some attention is required.

Throughout, M will be a nonstandard model of Peano Arithmetic (PA). For background on models of PA see Kaye [3]. Very little general theory is needed here, however. The most important result we need is the principle of *overspill.*

It is important for the arithmetic structure of M and its cuts that we consider negative elements and 'negative cuts'. (Definitions will be given shortly.)

Definition 1.1. The structure $Z = Z(M)$ is the full ring associated with M. It is defined as 0 together with two copies of $\{x \in M : x > 0\}$, one copy for positive elements and one for negative elements, and with $+, \cdot, <$ defined appropriately. Z is a discretely ordered ring.

Lower case letters $a, b, c, \ldots, x, y, z, \ldots$ will always range over elements of Z and upper case letters $A, B, C, \ldots, X, Y, Z, \ldots$ range over subsets of Z.

Definition 1.2. An initial segment $I \subseteq_{\mathrm{e}} Z$ is a subset of M which is downward-closed:

$$\forall x \in Z \ \forall y \in I \ (x < y \to x \in I).$$

We identify the elements $a \in Z$ with the initial segment $\{x \in Z : x \leqslant a\}$. We denote the set of initial segements of Z by $\mathrm{Dcl}(M)$, $\overline{Z}(M)$ or simply $\overline{Z}$, and refer to it as the *Dedekind completion* of M or Z.

Note that we make no assumption that an initial segment is proper nor that it is closed under any function such as successor. We use $-\infty$ to denote the initial segment $\varnothing$ and ∞ to denote Z. Note too the essential convention that a number a is identified with the set of elements of Z *less than or equal* to it. This is necessary for the formulas below to work in all cases, and to avoid considerable extra complexity.

Variables $I, J, K, \ldots$ range over initial segments of Z, so $I, J, K, \ldots$ might be, but need not be, elements of Z itself.

Definition 1.3. The ordering relation $\leqslant$ on $\overline{Z}$ is $\subseteq$. We write $I < J$ to mean I is a subset of J and I, J are not equal.

Proposition 1.4. The ordering $\leqslant$ on $\overline{Z}$ is a linear order on $\overline{Z}$. As ordered sets, we have $M \subseteq Z \subseteq \overline{Z}$, i.e. the ordering $\leqslant$ on $\overline{Z}$ extends that on Z.

The proof is a straightforward check using the definitions. (The same applies to a number of other propositions given in this paper without proofs, and in all such cases the reader should be able to supply necessary proofs easily enough.)

To aid calculations and to make this material manageable, it is essential at all times to bear in mind various dualities. Chief amongst these is the duality between $\leqslant$ and $\geqslant$, and between 'initial segments' and 'end segments'. We might say a subset I is an end segment of Z if I is closed upwards under $\geqslant$. The dual of an initial segment is an end segment which is almost always its set theoretic complement. To reduce the amount of terminology we shall redefine 'complement' slightly to make the identification exact.

Definition 1.5. The *complement* of an initial segment I is

$$I^c = \cup\{z \in Z : \forall w \in I \ z \geqslant w\}.$$

The complement of an end segment E is

$$E^c = \{z \in Z : \forall w \in I \ z \leqslant w\}.$$

In other words the complement of an initial segment I is its normal set-theoretic complement but with $\max I$ added in case this maximum exists, i.e. in case $I = a = \max I$ for some $a \in Z$. The complement of an end segment is dual to this.

Given an initial segment $I \subseteq M$ we usually use lower case $i, i_1, i_2, \ldots$ to range over elements of I, i.e. elements of Z less than or equal to I. We use $i', i'_1, i'_2, \ldots$ to range over elements of I^c, i.e. elements of Z greater than or equal to I.

We use $X \setminus Y$ for set difference, and this is not to be confused with $X - Y$ as defined below. We usually make no cardinality assumption on M, and often PA can be weakened to one of its subtheories—but this will not be made explicit here.

Definition 1.6. A *cut* of M is a (not necessarily proper) $I \in \overline{Z} \setminus Z$, i.e., is an initial segment that is not an element of Z.

Definition 1.7. If $X \subseteq \overline{Z}$ we set

$$\inf X = \{x \in Z : \forall y \in X \ x \leqslant y\}$$

and

$$\sup X = \{x \in Z : \exists y \in X \ x \leqslant y\}.$$

Thus $\inf Z = -\infty$, $\inf \varnothing = \infty$, $\inf M = 0$ and $\sup Z = \sup M = \infty$, $\sup \varnothing = -\infty$. Also $\inf a = \sup a = a$ for all $a \in M$.

2 Additive structure

The goal of this section is to define and understand the binary operation of addition on initial segments, in particular on cuts. We start though with the unary operation of $-$.

Definition 2.1. If $I \in \overline{Z}$ we define

$$-I = \{x : I \leqslant -x\} = \{-x : x \in I^c\}.$$

Proposition 2.2. The unary $-$ operation has the following properties.

(a) $- - I = I$ for each $I \in \overline{Z}$.

(b) The $-$ operation applied to $a \in Z$ gives $-a$. In particular $-0 = 0$.

(c) The $-$ operation applied to Z and $\varnothing$ gives $\varnothing$ and Z respectively, justifying the notation $-\infty$ for $\varnothing$.

Proposition 2.3. For $I, J \in \overline{Z}$, we have

$$I \leqslant J \;\Leftrightarrow\; -J \leqslant -I.$$

Definition 2.4. If $I \in \overline{Z}$ we define

$$|I| = I \text{ if } I \geqslant 0,\ |I| = -I \text{ otherwise.}$$

We now define addition on $\overline{Z}$. The various conventions chosen earlier have been set up specifically to make this definition as simple as possible and so that it extends the usual addition on Z.

Definition 2.5. For $I, J \subseteq_{\mathrm{e}} M$ we let

$$I + J = \{i + j : i \leqslant I \text{ and } j \leqslant J\}.$$

It is immediate that $I + J$ is an initial segment if both I, J are, and that it gives the correct answer for $a + b$, $0 + I$, etc.

Note that if I is empty *or* J is empty there is no pair of elements $(i, j) \in Z \times Z$ such that $i \leqslant I$ *and* $j \leqslant J$. (This is the only reasonable logical convention here, though does need some care.) Thus $-\infty + I = -\infty$ for all I, including the case $I = Z$. Hence we have the simple but important result that $-\infty + \infty = -\infty$. (The apparent asymmetry between ∞ and $-\infty$ here will be resolved shortly.) The other basic properties of $+$ are straightforward.

Proposition 2.6. For $I, J, K \in \overline{Z}$, we have:

(a) $I + (J + K) = (I + J) + K$;

(b) $I + J = J + I$;

(c) $I \leqslant J$ implies $I + K \leqslant J + K$;

(d) $I + 0 = I$.

(e) $I + -\infty = -\infty$ and $J + \infty = \infty$ for $J > -\infty$.

It is not true in general that $I < J$ implies $I + K < J + K$. Counterexamples are easy to find, including but not restricted to $K = \pm\infty$. See Proposition 2.30 below.

The dual of $+$ is the following operation.

Definition 2.7. For $I, J \subseteq_{\mathrm{e}} M$, let

$$I \oplus J = -((-I) + (-J)).$$

The reader can check that $I \oplus J = \inf\{i' + j' : i' \geqslant I, j' \geqslant J\}$. Once again, this addition agrees with addition on $a, b \in Z$.

We will investigate soon when it is the case that $I + J = I \oplus J$. One inclusion is, however, always true.

Proposition 2.8. For $I, J \in \overline{Z}$, we have $I + J \leqslant I \oplus J$.

By duality, $\oplus$ has all of the basic properties enjoyed by $+$.

Proposition 2.9. For $I, J, K \in \overline{Z}$, we have:

(a) $I \oplus (J \oplus K) = (I \oplus J) \oplus K$;

(b) $I \oplus J = J \oplus I$;

(c) $I \leqslant J$ implies $I \oplus K \leqslant J \oplus K$;

(d) $I \oplus 0 = I$.

(e) $I \oplus \infty = \infty$ and $J \oplus -\infty = -\infty$ for $J < \infty$.

Proof. Straightforward. For example, $I \oplus J = -((-I) + (-J)) = -((-J) + (-I)) = J \oplus I$. □

The next proposition records a particular case where addition of cuts is easy.

Proposition 2.10. Let $I, J \in \overline{Z}$ be nonnegative and closed under $+$. Then $I + J = I \oplus J = \max(I, J)$.

The additive structure of $\overline{Z}$ is understood best in terms of another operation on cuts, defined next.

Definition 2.11. Given $I \in \overline{Z}$ we define the *derivative of* I as

$$\partial I = \{x \in Z : \forall y \in Z \ (y \leqslant I \Rightarrow x + y \leqslant I)\} \in \overline{Z}.$$

We use $\mathrm{d}i$, $\mathrm{d}i_1$ etc. as variables ranging over ∂I, and $\mathrm{d}i'$ for variables over elements of ∂I^c.

Thus ∂I is an initial segment of Z. We will see shortly that it measures the size of the 'boundary' between I and I^c.

Note that $x \in \partial I$ can be defined by any of the equivalent statements

$$\forall y \in Z \ (y \leqslant I \Rightarrow x + y \leqslant I),$$
$$\forall y \in Z \ (y < I \Rightarrow x + y < I),$$
$$\forall y \in Z \ (y \geqslant I \Rightarrow x + y \geqslant I)$$

or

$$\forall y \in Z \ (y > I \Rightarrow x + y > I).$$

Example 2.12. $\partial a = 0$ for each $a \in Z$. Also, $\partial\infty = \partial(-\infty) = \infty$.

Proposition 2.13. Given $I \in \overline{Z}$, ∂I is non-negative and closed under addition. In fact $\partial I = I$ whenever $I \geqslant 0$ is closed under addition.

It is immediate from the definition that $I + \partial I = I$ for all $I \in \overline{Z}$. It is not true in general that $I \oplus \partial I = I$.

The notion of ∂I is the 'fuzziness' or 'uncertainty' we know I by. For each $k \in \partial I$ and $i \leqslant I \leqslant i'$ we have $i + k \leqslant I \leqslant i' - k$. Moreover,

Proposition 2.14. For $I \in \overline{Z}$ and $\mathrm{d}i' > \partial I$ there is $i \in I$ with $i \leqslant I < i + \mathrm{d}i'$.

Proof. Since $k' > \partial I$ there is $i \leqslant I$ with $i + k' \nleqslant I$. □

Example 2.15. Given a non-negative cut $K \in \overline{Z}$ closed under addition and $x \in M$ we may define the cuts

$$x + K = \sup\{x + k : k \leqslant K\}$$

and

$$x - K = \sup\{x - k' : k' > K\}.$$

These have derivative equal to K itself.

Definition 2.16. For nonnegative cut $K \in \overline{Z}$ closed under $+$, I is a *downwards K-limit* if $I = a - K$ for some $a \in M$. I is a *upwards K-limit* if $I = a + K$ for some $a \in M$.

Proposition 2.17. Let $K \in \overline{Z}$ be a nonnegative cut. Then no cut I is both an upwards K-limit and a downwards K-limit.

Proof. Suppose $x, y \in M$ with

$$I = \sup\{x + k : k \in K\} = \inf\{y - k : k \in K\}.$$

Then, for all $k \in K$, $x + k < y - k$ so by overspill there is $k' > K$ with $x + k' < y - k'$, hence $I < x + k' < y - k' < I$. □

Example 2.18. To find cuts with a given derivative K that are neither of the form $a + K$ nor $a - K$ for some a, we may use a tree argument. We assume for simplicity that M is countable and show that there are continuum many cuts with derivative K but only countably many upwards and downwards K-limits. One inductively takes $u_i, v_i \in M$ with $u_i + K < v_i$ and shows using Proposition 2.17 that there is always w with $u_i + K < w$, $w + K < v_i$. Then setting u_{i+1}, v_{i+1} to be either u_i, w or w, v_i we continue and all the limits of these sequences can be arranged to all have derivative K.

Example 2.19. Given a nonnegative cut K closed under $+$, and $a, b \in M$, we can easily calculate

$$\begin{aligned}
(a + K) + (b + K) &= (a + K) \oplus (b + K) = (a + b) + K \\
(a + K) + (b - K) &= (a + b) - K \\
(a + K) \oplus (b - K) &= (a + b) + K \\
(a - K) + (b - K) &= (a - K) \oplus (b - K) = (a + b) - K
\end{aligned}$$

In particular, $(a + K) + (b - K) < (a + K) \oplus (b - K)$.

Example 2.20. Example 2.19 is not the only way one can get $I + J < I \oplus J$. We can give other examples by a tree argument similar to that in Example 2.18. Suppose M is countable, and x, K are given. Inductively suppose we have u_i, v_i, r_i, s_i with

$$\begin{aligned}
&u_i + K < v_i \\
&r_i + K < s_i \\
&u_i + r_i + K < x \\
&u_i + r_i + K < x \\
&v_i + s_i > x + K.
\end{aligned}$$

Then we can split the contruction into two, making a binary tree as follows. Choose $u_i < w < v_i$ and $r_i < t < s_i$ with $w + t = x$. If we can perform this choice with $u_i + K < w$, $w + K < v_i$ and $r_i + K < t$, $t + K < s_i$ we then take $u_{i+1}, v_{i+1}, r_{i+1}, s_{i+1}$ to be either u_i, w, t, s_i or w, v_i, r_i, t and continue, arraning that the limits of the u_i, v_is is a cut I with derivative K and the limits of the r_i, s_is is a cut J with derivative K and $I + J = x - K < x + K = I \oplus J$.

There is always $u_i < w < v_i$ and $r_i < t < s_i$ with $w + t = x$ because $u_i + r_i < x < v_i + s_i$. For any such w we have either $u_i + K < w$ or $w + K < v_i$

as $u_i + K < v_i$. Suppose that $u_i + K < w$, other cases being similar, but that $v_i - w \in K$. As $r_i + K < t$ or $t + K < s_i$ and $v_i + s_i > x + K$ we must have $t + K < s_i$. Suppose that $t - r_i \in K$. Then for each $k \in K$ we have

$$\exists w', t' \ (w' + t' = x \wedge u_i + k < w' < v_i - k \wedge r_i + k < t' < s_i - k)$$

for we may just take $w' = w - k$, $t' = t + k$. By overspill there is w', t' with $w' + t' = x \wedge u_i + k' < w' < v_i - k' \wedge r_i + k' < t' < s_i - k'$ for some $k' > K$ and we are done.

The derivative operation on $\overline{Z}$ is a kind of valuation, as the next proposition shows.

Proposition 2.21. For cuts I, J,

$$\partial(-I) = \partial I$$

and

$$\partial(I + J) = \partial(I \oplus J) = \partial I + \partial J = \partial I \oplus \partial J = \max(\partial I, \partial J).$$

Proof. The first is an easy check.

For the second, note that ∂I and ∂J are nonnegative and closed under $+$ so Proposition 2.10 applies. Without loss of generality assume $K = \partial J \geqslant \partial I$. If $i \in I$, $j \in J$ and $k \in K$ then $(i+j)+k = i+(j+k) \in I+J$ showing $\partial(I+J) \geqslant \partial J$. Similarly if $i' > I$, $j' > J$ and $k \in K$ then $i' + j' - k = i' + (j' - k) > I \oplus J$ so $\partial(I \oplus J) \geqslant \partial J$.

Conversely, if $k' > K$ let $i \in I$ with $i' = i+k' > I$ and $j \in J$ with $j' = j+k' > J$. Then for $i_1 \in I$ and $j_1 \in J$, $i'+j' = (i+k')+(j+k') = (i+j)+2k' \leqslant i_1+j_1$ implies $i+k' \leqslant i_1$ or $j+k' \leqslant j_1$, which are both impossible. Thus $i'+j' > I+J$, showing $I + J$ is not closed under addition by $2k'$, so $k' > \partial(I + J)$. Similarly, $i' + j' - 2k' \in I + J \leqslant I \oplus J$ but $i' + j' > I \oplus J$ so $I \oplus J$ is not closed under addition by $2k'$ and $k' > \partial(I \oplus J)$. □

Proposition 2.22. If I, J are cuts and $\partial I < \partial J$ then $I + J = I \oplus J$.

Proof. If I, J are as given there are $i_0 \leqslant I$, $k_0 \leqslant \partial J$ with $i'_0 = i_0 + k_0 > I$. Then

$$\begin{aligned} I \oplus J &= \inf\{i' + j' : i' > I, j' > J\} \\ &= \inf\{i_0 + j' : j' > J\} \end{aligned}$$

since the difference between sufficiently small $i' > I$ and i_0 is less than $k_0 \leqslant \partial J$. Thus $I \oplus J = i_0 + J$. Similarly

$$\begin{aligned} I + J &= \sup\{i + j : i \in I, j \in J\} \\ &= \sup\{i_0 + j : j \in J\} \end{aligned}$$

so $I + J = i_0 + J$ also. □

In some cases, $I \oplus \partial I$ can be taken as a canonical cut greater than or equal to I with approximately the same amount of information as I, but which is never a downwards ∂I-limit. The following explains exactly when $I < I \oplus \partial I$.

Proposition 2.23. For a cut $I \in \overline{Z}$,

$$I = I + \partial I \leqslant I \oplus \partial I.$$

Also, $I = I \oplus \partial I$ unless some (equivalently all) $x \in (I \oplus \partial I) \setminus I$ has $I = x - \partial I$ and $I \oplus \partial I = x + \partial I$.

Proof. Let $K = \partial I$. It is easy to see that $I = I + K$, as $i \leqslant i + k < I$ for all $i \in I$, $k \in K$. If $I < I \oplus K$ let x be such that $I < x < I \oplus K$. Then $x - k > I$ for each $k \in K$, and given $i' > I$, $k' > K$ we have $x + k < i' + k'$ as $x < (i' - k) + k'$. This shows that $I \leqslant x - K < x + K \leqslant I \oplus K$. To see that the nonstrict inequalities here are in fact equalities, suppose at least one is strict. Then there are $I < x < x + K < y < I \oplus K$, and as K is closed under addition $k' = [(y-x)/2]$ has $k' > K$ with $x + 2k' \leqslant y$. Now let $i \in I$ with $i' = i + k' > i$. Then $i < I < x < y < I \oplus K < i' + k'$ with $i' + k' - i = 2k'$ which is impossible. □

Proposition 2.24. Given cuts I, J, we have

$$I + J \leqslant I \oplus J \leqslant (I+J) \oplus \partial(I+J) = (I+J) \oplus \max(\partial I, \partial J)$$

with at least one of the inequalities here being equality.

Proof. Let $K = \max(\partial I, \partial J) = \partial(I+J)$. Suppose $x \geqslant I + J$, $k' \geqslant K$. Then there are $i \leqslant I$ and $j \leqslant J$ with $i' = i + [k'/2] \geqslant I$ and $j' = j + [k'/2] \geqslant J$. So $i' + j' = i + j + k' \leqslant x + k'$ hence $I \oplus J \leqslant (I+J) \oplus \partial(I+J)$ and so

$$I + J \leqslant I \oplus J \leqslant (I+J) \oplus \partial(I+J).$$

Either all here are equal or $I + J = a - K$, $(I+J) \oplus \partial(I+J) = a + K$, in which case $I \oplus J$ (having derivative K) must be equal to one of these. □

This describes the cases when $+$ and $\oplus$ differ.

Proposition 2.25. Suppose I, J are cuts. Then $I + J \leqslant I \oplus J$ with inequality holding if and only if $I = a - J$, $J = a - I$, for some $a \in Z$.

Proof. Most of the work has already been done. $I + J \leqslant I \oplus J$ is true generally, and $I + J < I \oplus J$ holds if and only if there is $I + J < a < I \oplus J$. For this a we can check $I = a - J$. For if $i \leqslant I$ putting $x = i - a$ gives $-x = a - i \geqslant J$ (since $a - i = j \in J$ gives $a = i + j \leqslant I + J$) so $i = a + x \in a - J$, and conversely if $-x \geqslant J$ then $a + x \leqslant I$ (for $a + x > I$ implies $a = (a+x) + -x > I \oplus J$) hence $a - J \leqslant I$. $J = a - I$ is similar.

For the converse, assuming $I = a - J$ or $J = a - I$ we may calculate $I + J = a - J + J = a - \partial J$ and $I \oplus J = (a - J) \oplus J = a + \partial J$. □

It may be easier to remember the conclusion to the previous result using the derivative. If $I + J < I \oplus J$ then $\partial I = \partial J$ and $I = a - \partial I$, $J = a + \partial J$ for some a, or *vice versa*.

Corollary 2.26. If $I, J \in \overline{Z}$ have $I + J < I \oplus J$ then

$$I \oplus J = (I+J) \oplus \partial(I+J) = (I \oplus J) \oplus \partial(I \oplus J).$$

Proof. If $K = \partial(I + J) = \partial(I \oplus J)$ then $(I \oplus J) \oplus K = (I \oplus J) + K = (I \oplus J)$ since $(I \oplus J)$ is not a downwards K-limit. □

It is interesting to see what happens when $I + J = I \oplus J$ and only one part of the Proposition 2.25 holds.

Proposition 2.27. Let I, J, K be cuts with $K = \max(\partial I, \partial J)$ and $I+J = I\oplus J$.

- If $I\oplus J$ is an upward K-limit then there are $i \in I$ and $j \in J$ with $I \leqslant i+K$, $J \leqslant j + K$ and equality holding in at least one of these two cases.
- If $I + J$ is a downward K-limit then there are $i' > I$ and $j' > J$ with $I \leqslant i' - K$, $J \leqslant j' - K$ and equality holding in at least one of these two cases.

Proof. For the first, if $I + J = I \oplus J = x + K$ for some x then $x = i + j$ for some $i \in I$, $j \in J$, and for each $k' > K$ $i + j + k' > I \oplus J$. Therefore for each $k' > K$ there are $i' > I$ and $j' > J$ with $i' + j' = i + j + k'$, or $(i' - i) + (j' - j) = k'$. Thus both $i' - i \leqslant k'$ and $j' - j \leqslant k'$, and hence $I \leqslant i + K$, $J \leqslant j + K$, for if $i_1 \in I = i + k'$ with $k' > K$ then by the above there is $i' > I$ with $i' - i \leqslant i_i - i$ so $i_1 \in I$ is greater than $i' > I$, a contradiction. Similarly for j. At least one of the inequalities $I \leqslant i + K$, $J \leqslant j + K$ must be an equality as K is the derivative of either I or J or both.

The other case is similar. □

For a given positive K closed under addition there are many cuts I with $\partial I = K$. Cuts of the form $a + K$ or $a - K$ are examples. So are $J + K$ and $J \oplus K$ when $\partial J \leqslant K$. Others can be obtained from a tree argument as in Example 2.18. It is natural to ask, given I and $K = \partial I$, whether and how I may be 'simplified' to $J + K$ or $J \oplus K$ with $\partial J < K$. The following two results show that this can only be done in particular circumstances, and if so we may assume without loss that $J = j \in Z$ is not a cut.

Proposition 2.28. If $J+\partial I = I$ with $J \leqslant I$ then either $I = J$ or else $I = j+\partial I$ for some $j \leqslant J$.

Proof. Let $K = \partial I$ and assume that I is not $j + K$ for all $j \leqslant J$. Then given arbitrary $j_0 \leqslant J$ there is $i > j_0 + K$ in I and hence $j_1 \in J$ and $k_1 \in K$ with $j_1 + k_1 \geqslant i$. It follows that $j_1 - j_0 > K$ and hence $\partial J \geqslant K$. Therefore $I = J + \partial I \leqslant J + \partial J = J$. □

Proposition 2.29. If $J\oplus\partial I = I$ with $J \leqslant I$ then either $I = J$ or else $I = i+\partial I$ and $J = i - \partial I$ for all $i \leqslant I$.

Proof. Let $K = \partial I$ and assume that $J < I$ with $i \in I \setminus J$. As $J \oplus K = I$ and $i + K \leqslant I$ we have $j' + k' \geqslant i + k$ for all $j' \geqslant J$, $k' \geqslant K \geqslant k$. Thus $i - j' \leqslant k' - k$ for all such j', k, k' showing that $i - j' \leqslant K$ for all $j' \geqslant J$. It follows that $J = i - K$ and $I = (i - K) \oplus K = i + K$. □

Finally, for this section, we show how to use $\oplus$ and ∂ to determine when $I + K = J + K$ for $I \leqslant J$.

Proposition 2.30. Given $I \leqslant J$ and K in $\overline{Z}$ we have $I + K < J + K$ if and only if one of the following holds.

(a) $I \oplus \partial(K) < J$.

(b) For some $j \leqslant J$ and $k \in K$, $I + \partial(K) = j - \partial(K)$, $I \oplus \partial(K) = j + \partial(K)$ and $K = k + \partial(K)$.

Proof. Suppose $I \leqslant J$ and K are given.

If $I \oplus \partial(K) < J$ we show that $I + K < J + K$. Let $j \leqslant J$ and $i' \geqslant I$, $k' \geqslant \partial(K)$ be such that $i' + k' < j$. Since $k' > \partial K$ there is $k \leqslant K$ with $k + k' > K$. Now $i' + k' + k < j + k \leqslant J + K$ but $i' \geqslant I$ and $k' + k \geqslant K$ so $I + K < J + K$.

Similarly, if $j \leqslant J$ and $k \in K$ with $I+\partial(K) = j-\partial(K)$, $I \oplus \partial(K) = j+\partial(K)$ and $K = k+\partial(K)$ then $I+K = I+k+\partial(K) = k+j-\partial(K)$ does not contain $j+k$, so $I+K < J+K$.

Conversely, suppose $I \oplus \partial(K) \geqslant J$ and $I + K < J + K$. We shall show the existence of j, k as given in (b). The proof will give a little more information concerning this exceptional case.

We may assume $\partial I \leqslant \partial K$, since if $\partial I > \partial K$ we would have $I \oplus \partial K = I$ since each for each $i' > I$ there is c' with $\partial I \geqslant c' > \partial(K)$ and $i' - c' > I$. This means that $i' = (i' - c') + c'$ is the sum of an element of I^c and an element of $\partial(K)^c$, so this would show that $J \leqslant I \oplus \partial(K) \leqslant I \leqslant J$.

Next, we show there is $k \in K$ with $K = k + \partial(K)$. Take arbitrary $j \in J$ and $k \in K$. We attempt to show that there is some $i \in I$ and $k_1 \in K$ with $i + k_1 = j + k$. Suppose first that $k + \partial(K) < K$. Then there is $c' > \partial K$ with $k + c' \leqslant K$. Since $c' > \partial I$ there is $i \in I < i + [c'/2]$. Thus $j < i + [c'/2] + [c'/2]$ since $I < i + [c'/2]$, $\partial(K) < [c'/2]$ and $I \oplus \partial(K) \geqslant J$. Therefore $j + k < i + [c'/2] + [c'/2] + k = i + (k + c') \leqslant I + K$. If this works for all j, k we would have $I + K = J + K$. So by our assumption $I + K < J + K$ there is $k \in K = k + \partial(K)$.

If $I + \partial(K) \geqslant J$, then for each $j \leqslant J$ and $k \leqslant K$ there is $i \in I$ and $c \in \partial(K)$ with $i + c = j$ so $j + k = i + c + k \in I + K$ as $c + k \in K$ so $I + K \geqslant J + K$. So we may assume $I + \partial(K) < J$ and in particular $I + \partial(K) < I \oplus \partial(K)$.

So we have $I + \partial(K) < J \leqslant I \oplus \partial(K)$ and $\partial I \leqslant \partial(K)$. Therefore there is $j \in J \setminus I + \partial(K)$ and for any such j we have $I + \partial(K) = j - \partial(K) \leqslant j \leqslant j + \partial(K) = I \oplus \partial(K)$ and $J \leqslant j + \partial(K)$. This is what we were required to prove. □

3 Subtraction and duality

The unary subtraction operation $-I$ has already been defined. It is natural to extend this to a binary subtraction operation as usual. In fact, as well as $I - J$ we also have a dual operation to this.

Definition 3.1. Given cuts $I, J \in \overline{Z}$ with $I \geqslant J$, define

- $I - J = I + (-J) = \{i - j' : i \leqslant I, j' \geqslant J\}$,
- $I \ominus J = I \oplus (-J) = -(J - I) = \inf\{i' - j : i' \geqslant I, j \leqslant J\}$.

The following collects together properties of $-$ and $\ominus$ that follow from previous results about addition.

Proposition 3.2. Let $I, J \in \overline{Z}$. Then:

(a) $I - 0 = I \ominus 0 = I$;

(b) $I - J \leqslant I \ominus J$;

(c) $\partial(I - J) = \partial(I \ominus J) = \max(\partial I, \partial J)$;

(d) $I - J = I \ominus J$ whenever $\partial I \neq \partial J$;

(e) $I - J < I \ominus J$ iff $\partial I = \partial J$ and $I - J = a - \partial I$, $I \ominus J = a + \partial I$ for some (equivalently all) $I - J < a < I \ominus J$.

Proof. By Propositions 2.6, 2.8, 2.21, 2.25 and 2.22. □

Remark 3.3. It might be argued that the definitions of $I - J$ and $I \ominus J$ are not the 'correct' ones as they give rather naïve upper and lower bounds for the difference, and do not give the finer details one expects. In contrast to these definitions, the following might be proposed for $I - J$.

$$\inf\{\sup\{i' - j' : j' \geqslant J\} : i' \geqslant I\}$$
$$\sup\{\inf\{i' - j' : i' \geqslant I\} : j' \geqslant J\}$$
$$\sup\{\inf\{i - j : j \leqslant J\} : i \leqslant I\}$$
$$\inf\{\sup\{i - j : i \leqslant I\} : j \leqslant J\}.$$

However, it is easy to check that each of these alternative definitions give a value between $I - J$ and $I \ominus J$, so by Proposition 3.2(d) are equal to both $I - J$ and $I \ominus J$ when $\partial I \neq \partial J$. Also, if $\partial I = \partial J$ then each of the alternative definitions has the same derivative, as can be checked, and hence by Proposition 3.2(e) are all equal to one of the two values $I - J$ and $I \ominus J$. Therefore it seems that the extra complexity of the alternative definitions is not required in the general theory.

Note that Proposition 3.2(c) above highlights one aspect in which the derivative is *not* like a valuation function. As $\partial(I - J) = \partial(I \ominus J) = \max(\partial I, \partial J)$, the derivative operation is rather well-behaved on the difference even when $\partial I = \partial J$. (In valuation theory, the valuation of a difference of two points with the same value, can be anything below this value.) On the other hand, it is precisely when $\partial I = \partial J$ that we have difficulty evaluating $I - J$ and $I \ominus J$.

One special case for $I - J$ and $I \ominus J$ is when $I = J$. This is given next.

Proposition 3.4. For $I \in \overline{Z}$ we have $I - I = -\partial I$ and $I \ominus I = \partial I$.

Proof. If $k \leqslant -\partial I$ then $-k \geqslant \partial I$ so there is $i \in I$ with $i' = i - k \geqslant I$ hence $k = i - i' \leqslant I - I$. Conversely if $-k \leqslant I - I$ then $-k = i - i'$ for some $i \leqslant I \leqslant i'$ hence $k \leqslant \partial I$. The case for $I \ominus I$ is dual, and $I \ominus I = -(I - I) = \partial I$. □

We also have the following easy identities.

Proposition 3.5. For all $I, J, K \in \overline{Z}$

(a) $I - (J \oplus K) = (I - J) - K$

(b) $I \ominus (J + K) = (I \ominus J) \ominus K$

(c) $(I + J) - K = I + (J - K)$

(d) $(I \oplus J) \ominus K = I \oplus (J \ominus K)$

Proof. For the first, note that $I - (J \oplus K) = I - -(-J + -K) = I + (-J + -K) = (I - J) - K$. The others are similar. □

If $I \oplus \partial I$ was a canonical cut greater than equal to I, with similar properties, the corresponding (dual) downward operation is $I - \partial I$.

Proposition 3.6. Let I be a cut. Then $I \ominus \partial I = I$. Also $I - \partial I \leqslant I$ with equality except in the case when for some (equivalently for all) $I - \partial I < a < I$ we have $I - \partial I = a - \partial I$, $I = a + \partial I$.

Proof. This is the dual of Proposition 2.23. For example, $I \ominus \partial I = -(\partial I - I) = -(\partial(-I) + (-I)) = - - I = I$ and $I - \partial I = -(\partial I \ominus I) = -(\partial(-I) \oplus (-I)) \leqslant - - I = I$. □

Proposition 3.7. Given $I \leqslant J$ and K in $\overline{Z}$ we have $I \oplus K < J \oplus K$ if and only if one of the following holds.

(a) $J - \partial K > I$, or

(b) for some $i' \geqslant I$ and $k' \geqslant K$, $J \ominus \partial K = i + \partial K$, $J - \partial K = i - \partial K$ and $K = k - \partial K$.

Proof. Dual to Proposition 2.30. □

Proposition 3.8. Let $I, J \in \overline{Z}$. Then

$$(I \ominus J) \oplus J = I \oplus \partial J$$

and

$$(I - J) + J = I - \partial J.$$

Proof. Let $K = \partial J$. Then

$$\begin{aligned}(I \ominus J) \oplus J &= \inf\{(i' - j) + j' : i' \geqslant I, j \in J \leqslant j'\}\\ &= \inf\{i' + k' : i' \geqslant I, k' \geqslant K\}\\ &= I \oplus K.\end{aligned}$$

Also,

$$\begin{aligned}(I - J) + J &= \sup\{(i - j') + j : i \leqslant I, j \leqslant J \leqslant j'\}\\ &= \inf\{i - k' : i \leqslant I, k' \geqslant K\}\\ &= I - K\end{aligned}$$

as required. □

Proposition 3.9. Suppose $I, K \in \overline{Z}$ are cuts with $\partial I < \partial K$. Then

$$I - \partial K = I \ominus \partial K < I < I + \partial K = I \oplus \partial K$$

and $I \oplus \partial K = x + \partial K$, $I - \partial K = x - \partial K$ for any x with $(I - \partial K) < x \leqslant (I \oplus \partial K)$.

Proof. For simplicity suppose $K = \partial K$. Obviously $I \leqslant I + K \leqslant I \oplus K$ and $I - K \leqslant I \ominus K$. Let $k \in K$ and $i \in I$ with $i' = i + k > I$. Then $i = i' - k > I \ominus K$ showing $I \ominus K < I$. For these i, i', k, if $i_1 \in I$, $k_1 \in K$ then $i + k > i_1$ so $i_1 + k_1$ is at most $i + (k + k_1)$ showing $I + K = i + K$, and since $I + K$ is an upwards K-limit, $I \oplus K = I + K$. Also, if $k_1' > K$ then $i - k_1' \in I - K$ showing $i - K = I - K = I \ominus K$ as $\partial(I - K) = \partial(I \ominus K) = K$. □

We can also write down duals of Propositions 2.28 and 2.29.

Proposition 3.10. If $J - \partial I = I$ with $J \geqslant I$ then $I = J$ of $J = j + \partial I$, $I = j - \partial I$ for all $j \in J \setminus I$.

Proof. From the assumptions and duality we have $-J \oplus \partial(-I) = -I$ with $-I \geqslant -J$ and the result follows by Proposition 2.29. □

Proposition 3.11. If $J \ominus \partial I = I$, $J \geqslant I$ then $I = J$ or $I = j - \partial I$ some $j \in J$.

Proof. From the assumptions we have $-J + \partial(-I) = -I$ with $-I \geqslant -J$. Now apply Proposition 2.28. □

The following proposition evaluates all terms of the form '(I minus J) plus J' with addition and subtraction being either the normal $+, -$ or the dual form, $\oplus, \ominus$.

Proposition 3.12. Let $I, J \in \overline{Z}$ and let $K = \max(\partial I, \partial J)$.

- If $K = \partial J > \partial I$ we have $I - J = I \ominus J$ and $(I \ominus J) \oplus J = I \oplus K = i + K$ for some $i \in I$, $(I - J) + J = I - K = i - K$ for some $i \in I$.
- If $K = \partial I = \partial J$ then $I - J = I \ominus J$ except when $I - J = x - K$ and $I \ominus J = x + K$ for some (equivalently all) $x \in (I \ominus J) \setminus (I - J)$. Also $(I \ominus J) \oplus J = I \oplus K$, which equals I unless I is a downwards K-limit; $(I \ominus J) \oplus J = I - K$, which equals I unless I is an upwards K-limit; and $(I \ominus J) + J$, $(I - J) \oplus J$ are both equal to one of these two.
- If $\partial I > \partial J$ then $I - J = I \ominus J$ and $(I \ominus J) \oplus J = (I \ominus J) + J = I$.

Proof. If $K = \partial J > \partial I$ Proposition 3.8 applies. Let $I \oplus K = I + K = i + K$ and $I \ominus K = I - K = i - K$ with $i \in I$, $k \in K$ and $i' = i + k > I$. Then $I - J = \sup\{i_1 - j_1' : i_1 \in I, j_1 > J\} = \sup\{i - j_1' : j_1 > J\}$ since each $i_1 \in I$ is bounded above by $i + k$ for some $k \in K = \partial J$. so $I - J = i - J$. Similarly $I \ominus J = i' - J$. But as $i' = i + k$, $k \in \partial J$, these are equal.

If $K = \partial I = \partial J$, and suppose either $I \ominus J$ is not an upwards K-limit or $I - J$ is not an downwards K-limit. Then there are $I \ominus J > x > x - K > y > I - J$ for some x, y, and as K is closed under addition there is $k' > K$ with $2k' < x - y$. Now take $i < I < i' = i + k'$ and $j < J < j'j + k'$ so $i' - j > I \ominus J > x$ and $i - j' < I - J < y$. But $(i' - j) - (i - j') = 2k' < x - y$ which is a contradiction. A previous calculation (Proposition 3.8) shows that $(I \ominus J) \oplus J = I \oplus K$ and Proposition 2.23 identifies when this equals I; similarly $(I \ominus J) \oplus J = I - K$, by Proposition 3.8, and Proposition 3.9 identifying when this is equal to I. Finally and $(I \ominus J) + J$, $(I - J) \oplus J$ are between these two with the same derivative, so each must be equal to one of these two.

If $K = \partial I < \partial J$ there is $j \in J$ and $k \in K$ with $j_k = j' > J$. It is easy to check from this that $I - J = \sup\{i - j' : i \in I\}$, $I \ominus J = \inf\{i' - j : i' > I\}$, and that these are equal because $j' = k + j$ and I is closed under addition by k. Thus $(I \ominus J) \oplus J = I \oplus \partial J = I$ and $(I - J) - J = I - \partial J = I$. $\square$

Example 3.13. Let K be a proper cut closed under addition and $a > b \in M$. Then $(a+K)-(b+K) = \sup\{(a+k)-(b+k') : k < K < k'\} = \sup\{(a-b)-k' : K < k'\} = (a-b) - K$. Other calculations can be carried out in the same way, we list them here.

$$
\begin{aligned}
&(a + K) - (b + K) = (a - b) - K \\
&(a + K) \ominus (b + K) = (a - b) + K \\
&(a + K) - (b - K) = (a + K) \ominus (b - K) = (a - b) + K \\
&(a - K) - (b + K) = (a - K) \ominus (b + K) = (a - b) - K \\
&(a - K) - (b - K) = (a - b) - K \\
&(a - K) \ominus (b - K) = (a - b) + K
\end{aligned}
$$

However, as in Example 2.18, these are not the only ways to get cuts I, J with given derivative K and $I \ominus J > I - J$.

Proposition 3.14. Given $I \in \overline{Z}$ and $I < x < y \in M$, the following are equivalent: $x + I = y + I$; $x - I = y - I$; and $y - x \in \partial I$.

Proof. We show $x+I = y+I$ iff $y-x \in \partial I$. The other is similar. If $x+I = y+I$ then for all $i \in I$ there is $i_1 \in I$ with $y + i = x + i_1$ so $i + (y - x) \in I$ and hence $y - x \in \partial I$ as i was arbitrary. Conversely if $y - x \in \partial I$ then for all $i \in I$ there is $i_1 = i+(y-x) \in I$ such that $x+i_1 = x+(y-x)+i = y+i$, so $x+I \leqslant y+I$. $\square$

Using what we have learnt about addition and subtraction we shall now solve the following: given $I, J \in \overline{Z}$ we wish to find all possible $X \in \overline{Z}$ with $X + J = I$. In the case $\partial J > \partial I$ there will be no solutions X. So we will assume $K = \partial I \geqslant \partial J$.

Under these assumptions we have $(I - J) + J = I - \partial J$, by Proposition 3.8. If $\partial J < \partial I$ or if I is not an upwards K-limit then $I - \partial J = I$ and we have a solution $X = I - J$ to $X + J = I$.

So assume now that $I = i + K$ and $\partial J = \partial I = K$. Then $I - K = i - K$. We try $X = I \ominus J$. Here, $(I \ominus J) + J \leqslant (I \ominus J) \oplus J = I$ so $(I \ominus J) + J = I$ unless additionally $J = j' - K$ for some $j' \geqslant J$, by Proposition 2.25.

This proves one direction of the following.

Proposition 3.15. Suppose $I, J \in \overline{Z}$. Then there is a solution ($X = I - J$ or $I \ominus J$) to $X + J = I$ if and only if $\partial J \leqslant \partial I$ and the following holds: it is not the case that $\partial I = \partial J$, I is an upwards ∂I-limit and J is a downwards ∂J-limit.

Proof. One direction has been shown. For the other, suppose $X + J = I$, $K = \partial I = \partial J$, $I = i + K$ and $j = j' - K$. Then for some $k' > K$ and $x \leqslant X$ we must have $x + j' - k' = i$. But then by overspill there is $k'' > K$ with $k' - k'' > K$ so $x + j' - (k' - k'') = i + k'' > i + K$ contradicting $X + J = I$. $\square$

Proposition 2.30 says something about the uniqueness about such X. If $X_1 + J = I$ and $X_2 + J = I$ with $X_1 \leqslant X_2$, then for $\partial J < \partial I$ we have $X_1 = X_2$. This is because Proposition 2.30 implies $X_1 + \partial J \geqslant X_2$. But $\partial J < \partial X_1 = \partial I$ so $X_1 + \partial J = X_1 \geqslant X_2$ hence $X_1 = X_2$. On the other hand, if $\partial I = \partial J$, $X_1 \leqslant X_2$

and $X_1 + J = I$, $X_2 + J = I$, Proposition 3.14 says that X_1, X_2 must be 'close' in the sense that for all $X_1 \leqslant x_1 < x_2 \leqslant X_2$ we have $x_2 - x_1 \leqslant \partial J = \partial I$, but leaves open the possibility that they may not be identical. Proposition 2.30 gives the exact result.

Proposition 3.16. Suppose $I, J \in \overline{Z}$ and $\partial J < \partial I$. Then:

(a) if $\partial I = \partial J$ the solution to $X + J = I$ given by Proposition 3.15 when it exists is unique;

(b) if $\partial I > \partial J$ and $X + J = I$ then any X' such that $\partial X' = \partial I$ and $X' \oplus \partial J \geqslant X$, $X \oplus \partial J \geqslant X'$ is also a solution.

Proof. For (b) just use Proposition 2.30, noting that as $\partial J < \partial I = \partial X = \partial X'$ we have $X + \partial J = X \oplus \partial J$ and similarly for X'. □

4 Rationals and the logarithm

Additional structure on $\overline{Z}$, in particular for multiplication and division, becomes complicated by the fact that our elements of Z are integers, and so we often need to take integer parts. The theory goes more smoothly when we pass to the field of fractions $Q = Q(M)$ of $Z(M)$ and initial segments $I \subseteq_{\mathrm{e}} Q$. The main difference is that the order on Q is dense, rather than discrete, and this changes the formal details of some definitions though the spirit of the definitions remains the same. In particular for each $r \in Q$ there is an initial segment with no maximum element corresponding to r, i.e. the initial segment $\{p \in Q : p < r\}$.

Definition 4.1. We let $\overline{Q}$ denote the set of all initial segments of Q without maximum element and identify each $r \in Q$ with $\{p \in Q : p < r\} \in \overline{Q}$. We continue to use I, J, K as ranging over initial segments, the context indicating what these are initial segments of.

Complementation of initial segments and end segments is defined in Q in a symmetric way by $I^c = \{q \in Q : \exists r \notin I \ (q > r)\}$ for initial segments I and $E^c = \{q \in Q : \exists r \notin E \ (q < r)\}$ for end segments E. The order relation $\leqslant$ on $\overline{Q}$ is set inclusion and for $X \subseteq \overline{Q}$ we define

$$\inf X = \bigcup \{x \in \overline{Q} : \forall y \in X \ x < y\}$$

and

$$\sup X = \bigcap \{x \in \overline{Q} : \exists y \in X \ x < y\}.$$

We can also consider the 'localisation' of Q to a single denominator n. The result is similar to Z.

Definition 4.2. For positive $n \in Z$ we let $Q_n = \{z/n : z \in Z\} \subseteq Q$ and $\overline{Q}_n$ the set of initial segments of Q_n, possibly with maximum element. Thus $Z = Q_1$ and $\overline{Z} = \overline{Q}_1$. Because elements of $\overline{Q}_n$ may have maximum elements, complementation is defined in a way similar to that in Z.

To each $I \in \overline{Q}_n$ we associate $I_Q = \sup I = \inf(I^c)$ in $\overline{Q}$, the inf and sup taking place in $\overline{Q}$ and each $r \in Q$ being identified with its set of predecessors, as mentioned above. Similarly, if E is an end segment of Q_n we define $E_Q = (\sup E^c)^c = (\inf E)^c$.

The set $\overline{Q}$ has its own additive structure, with relations and operations $+, \oplus, -, \ominus, \partial, <$ and $\leqslant$ on $\overline{Q}$ all defined in an formally identical way to these operations of $\overline{Z}$ using the particular operations of inf, sup and complementation in $\overline{Q}$. Similarly, $+, \oplus, -, \ominus, \partial, <, \leqslant$ are all defined on $\overline{Q}_n$ for each positive $n \in Z$. The additive structure of $\overline{Q}_n$ is obviously isomorphic to that of $\overline{Z}$ via the map $z \mapsto z/n$. The relationship between each $\overline{Q}_n$ and $\overline{Q}$ is given by the next proposition.

Proposition 4.3. For each positive $n \in Z$, The map $I \mapsto I_Q$ from $\overline{Q}_n$ to $\overline{Q}$ is a embedding preserving all additive structure from $+, \oplus, -, \ominus, \partial, <, \leqslant$.

Proof. This is just axiom checking, and easy. The map is easily seen to be 1–1. Many statements are obvious, such as: $I \leqslant J$ implies $I_Q \leqslant J_Q$ and $I + J = K$ implies $I_Q + J_Q = K_Q$. That $(-I)_Q = -(I_Q)$ follows from the definition of $-I$ as $\{-n : n \in I^c\}$ and the fact that ${I^c}_Q = {I_Q}^c$. Hence $I \oplus J = K$ implies $-I + -J = -K$ which implies $-I_Q + -J_Q = -K_Q$ and hence $I_Q \oplus J_Q = K_Q$. The definition of ∂ does not cause difficulties either. ☐

Because of this, when passing from $\overline{Q}_n$ to $\overline{Q}$, we shall identify each $I \in \overline{Q}_n$ with I_Q. Thus our additive theory up to this point can be applied to each $\overline{Q}_n$ and to $\overline{Q}$ as well.

In the other direction, each $I \in \overline{Q}$ determines two (possibly different) initial segments $\overline{I}_n = \inf_{Q_n} I^c$ and $\underline{I}_n = \sup_{Q_n} I$ in each $\overline{Q}_n$, and this of course will occasionally require some care.

It will be helpful to have a definition of logarithms on $\overline{Q}$. Any base could be taken, but for simplicity we will use base e.

Definition 4.4. We denote by $\overline{Q}^+$ the set of $I \in \overline{Q}$ with $I > 0$. Our base model $M \vDash \mathrm{PA}$ can describe bounds on logarithms up to rational numbers in Q, and we define the logarithm operation $\log \colon \overline{Q}^+ \to \overline{Q}$ by

$$\log I = \inf\{q \in Q : \forall p \leqslant I \ (p > 0 \to q \geqslant \log_{\mathrm{e}} p)\}.$$

It is easy to check that this logarithm can also be written as

$$\log I = \sup\{q \in Q : \exists p \leqslant I \ (p > 0 \wedge q \leqslant \log_{\mathrm{e}} p)\}.$$

(Recall: by the definition of inf and sup in $\overline{Q}$ these cannot contain maximum elements.)

Proposition 4.5. The map $\log \colon \overline{Q}^+ \to \overline{Q}$ is 1–1 and onto.

We denote the inverse $\log^{-1}$ by exp.

Proposition 4.6. Given $I > 1$ is in $\overline{Q}$,

(a) I is closed under addition iff $\log I$ is closed under successor; and

(b) I is closed under multiplication iff $\log I$ is closed under addition.

The log and exp functions allow us to extend our additive theory to other operations, including multiplication. Multiplication and division is covered briefly in the next section. We conclude this section with a discussion of derivative.

Definition 4.7. The derivative $\partial\colon \overline{Q} \to \overline{Q}$ is defined in the same way as on Z, as $\partial I = \{x \in Q : \forall y \in I\ x + y \in I\}$. We also denote ∂ by ∂^1 and define

$$\partial^{n+1} I = \exp(\partial^n (\log I))$$

for all $I \in \overline{Q}$ for which $\partial^n (\log I)$ is defined.

Here, $\partial^1 (\log I)$ is defined for $I > 0$, $\partial^2 (\log I)$ for $I > 1$, and so on, so $\partial^2 I$ is defined for all $I > 0$. Also, since $\partial I \geqslant 0$ for all I, we have $\partial^2 I > 1$ for all $I > 0$.

$\partial^2 I$ is the multiplicative analogue of ∂I.

Proposition 4.8. For $I > 0$ in $\overline{Q}$, $\partial^2 I = \{x \in Q : \forall y \in I\ x \cdot y \in I\}$.

In other words, $\partial^2 I$ is the limit of i'/i where $i' \geqslant I \geqslant i$.

Corollary 4.9. For $I > 0$ in $\overline{Q}$, $\partial(\log I) = \log(\partial^2 I)$.

Proof. This is almost directly from the definition of ∂^2: $\partial^2 I = \exp(\partial(\log I))$ hence $\partial(\log I) = \log(\partial^2 I)$. □

Note that if $I > 1$ is closed under multiplication then $\partial^2 I = I$ so $\partial(\log I) = \log I$. Other cases for calculating $\partial \log I$ will be given in the next section.

Example 4.10. Given $q \in Q$ and $K \in \overline{Q}$ with $\infty > K > 0$ and working in $\overline{Q}$ we can define

$$\begin{aligned}
q + K &= \sup\{q + k : k < K\} = \inf\{q + k' : k' > K\} \\
q - K &= \sup\{q - k' : k' > K\} = \inf\{q - k : k < K\} \\
qK &= \sup\{qk : k < K\} = \inf\{qk' : k' > K\} \\
q/K &= \sup\{q/k' : k' > K\} = \inf\{q/k : k < K\}.
\end{aligned}$$

The first two of these just use additive structure of $\overline{Q}$ and $\partial(q+K) = \partial(q-K) = K$, $\log(q + K) = \sup\{\log(q + k) : k < K\}$, etc., and $\partial^2(q + K) = \sup\{(q + k')/(q + k) : k < K < k'\}$, with little extra information available on these in general. On the other hand, one can calculate $\log(qK) = \log(q) + \log(K)$, $\log(q/K) = \log(q) - \log(K)$, and $\partial^2(qK) = \partial^2 \log(q/K) = \partial^2 K$.

Example 4.11. For $0 < K < 1$ in $\overline{Q}$ one has $\partial^2 K = \inf\{k'/k : k' > K > k\}$ so $1 \leqslant \partial^2 K \leqslant 1/K$. In general $1 \leqslant \partial^2 K \leqslant \max(K, 1/K)$ for $K > 0$.

5 Multiplicative structure and beyond

Unsurprisingly, the structure of cuts in Z, Q_n or Q with addition and multiplication combined is rather complicated, and most calculations seem to require looking at a large number of cases. Where possible, problems should be solved in the muliplicative domain by using logarithms to reduce to additive problems, but there are limitations to this approach. This section concludes the paper with an incomplete overview of the multiplicative theory.

As for addition and subtraction, there are two obvious definitions of multiplication, depending on whether one looks from below or from above. And as for Dedekind's construction of the reals, the definitions are made slightly more complicated because we need to consider the cases of positive and negative cuts separately.

Definition 5.1. Let $I, J \in \overline{Z}$ where $I \geqslant 0$ and $J \geqslant 0$. Then we define

$$I \cdot J = \sup\{ij : 0 \leqslant i \in I, 0 \leqslant j \in J\}$$

and

$$I \odot J = \inf\{i'j' : i' \geqslant I, j' \geqslant J\}.$$

We extend this to other $X, Y \in \overline{Z}$ using the unary subtraction operation as follows. Assuming that $I \geqslant 0$ and $J \geqslant 0$, we let

$$I \cdot (-J) = (-I) \cdot J = -(I \odot J),$$
$$I \odot (-J) = (-I) \odot J = -(I \cdot J),$$
$$(-I) \cdot (-J) = (I \odot J)$$

and

$$(-I) \odot (-J) = (I \cdot J).$$

Our guiding principle here is that $X \cdot Y$ is always a supremum and $X \odot Y$ is always a (larger) infimum.

Proposition 5.2. For all $X, Y \in \overline{Z}$, $X \cdot Y \leqslant X \odot Y$.

In the sequel we will mainly consider structural properties of positive I, J. This is for simplicity: results on other cuts can be read off using the above definitions and use of the absolute value operation. For example, it is easy to check that $\partial(aI) = |a|\partial I$.

Example 4.10 defines $1/I$ for positive $I \in \overline{Q}$. We may also define $1/(-I) = -(1/I)$. This enables us to define two division operations on cuts I, J.

Definition 5.3. For $I, J \in \overline{Q}$, (or via the usual identifications in $\overline{Q}_n, \overline{Z}$) we define I/J and $I \oslash J$ by

$$I/J = I \cdot (1/J) \quad \text{and} \quad I \oslash J = I \odot (1/J).$$

Note that division yields cuts in $\overline{Q}$. These may be approximated in $\overline{Q}_n, \overline{Z}$ with possible ambiguity and/or loss of precision.

A number of straightforward basic results hold for multiplication and division by translation between the multiplicative and additive structure using the log and exp functions.

Proposition 5.4. Let $I > 0$ be in $\overline{Q}$. Then $\partial^2 I$ is closed under multiplication.

Proposition 5.5. Let $I, J > 0$ be in $\overline{Q}$. Then $I \cdot J = \exp(\log I + \log J)$ and $I \odot J = \exp(\log I \oplus \log J)$.

Proposition 5.6. Let $I, J > 0$ be in $\overline{Q}$. Then $I \cdot J = I \odot J$ unless $\partial^2 I = \partial^2 J$ and for $K = \partial^2 I$ and any $a \in I \odot J \setminus I \cdot J$ we have $I \cdot J = a/K$, $I \odot J = aK$.

As expected, examples where $\cdot$ and $\odot$ differ do occur: let $K \in \overline{Z}$ be a positive cut closed under multiplication. Then one can calculate $aK \cdot b/K = (ab)/K$ whereas $aK \odot b/K = (ab)K$.

Probably the most interesting results here concern combinations of the families of additive and multiplicative operations.

The next result is analogous to the familiar result $\mathrm{d}/\mathrm{d}x(\log y) = (1/y)(\mathrm{d}y/\mathrm{d}x)$. We will see that it is not in general particularly exact.

Proposition 5.7. Let $I \in \overline{Q}$ be such that $I > 0$. Then

$$\partial I \cdot \frac{1}{I} \leqslant \partial \log I \leqslant \partial I \odot \frac{1}{I}.$$

Proof. Let $k \in \partial I \cdot (1/I)$ and $i \leqslant I$. To show $\log(i) + k \leqslant \log I$ we may assume $k \leqslant \mathrm{d}i/i'$ where $\mathrm{d}i \in \partial I$ and $i' \geqslant I$. Note too that $2\mathrm{d}i = \mathrm{d}i + \mathrm{d}i \leqslant I$ so $\mathrm{d}i/i' < 1/2$. Then

$$\log(i) + k \leqslant \log(i) + \mathrm{d}i/i' \leqslant \log(i + i\,\mathrm{d}i/i')$$

which holds by the familiar alternating power series for $\log(1 + x)$ (formalised and proved in PA) showing $\log(1 + x) \leqslant x$ for $0 < x < 1/2$ and $\mathrm{d}i/i' < 1/2$. But then $i/i' \leqslant 1$ and so $i + i\,\mathrm{d}i/i' \leqslant i + \mathrm{d}i \leqslant I$, so $\log(i) + k \leqslant \log I$.

For the other inequality, suppose $k' > \partial I \oslash I$. Then there are $\mathrm{d}i' > \partial I$ and $i \in I$ with $i + \mathrm{d}i' > I$ and $k > \mathrm{d}i'/i$. So

$$\log(i) + k \geqslant \log(i) + \mathrm{d}i'/i \geqslant \log(i + \mathrm{d}i') > \log I$$

as required. □

Corollary 5.8. For $I > 0$ we have

$$\partial I \cdot \frac{1}{I} \leqslant \log \partial^2 I \leqslant \partial I \odot \frac{1}{I}.$$

Example 5.9. Let $N \subseteq_{\mathrm{e}} Z$ be a positive cut containing $\mathbb{N}$ and closed under multiplication. Let $a > N$ in Z. We look at the cuts $I = aN$ and $J = a/N$.

For $I = aN$ we can calculate $\partial^2 I = \inf\{n'/n : n' > N > n\} = N$, since N is closed under multiplication. Also $\partial I = aN$, $\partial I/I = \sup\{an/an' : n' > N > n\} = 1/\mathbb{N}$, and $\partial(I) \oslash I = \inf\{an'/an : n' > N > n\} = N$, again by closure under multiplication. Also $\log(aN) = \log a + \log N$ so $\partial \log(aN) = \log N$ since $\log N$ is closed under addition.

Similarly for $J = a/N$, $\partial^2 J = \inf\{(a/n)/(a/n') : n' > N > n\} = N$, $\partial(J) = a/N$, $1/(a/N) = N/a$, $\partial(J)/J = \sup\{(a/n') \cdot (n/a) : n' > N > n\} = 1/N$ and $\partial(J) \oslash J = \inf\{(a/n) \cdot (n'/a) : n' > N > n\} = N$. Also $\log(a/N) = \inf\{\log(a/n) : n \in N\} = \inf\{\log(a) - \log(n) : 0 < n \in N\} = \log(a) - \log N$, so $\partial \log(a/N) = \log N$.

The derivatives of products $I \cdot J$, and of powers of I are interesting.

Proposition 5.10. Suppose $I, J > 0$. Then either $\partial(I \cdot J) = \partial(I \odot J)$ or $\partial(I \cdot J) = a/K$ and $\partial(I \cdot J) = aK$ for $K = \partial^2 I = \partial^2 J$ and $I \cdot J < a < I \odot J$.

Proof. There is nothing to prove if $I \cdot J = I \odot J$. Assuming they are different, $I \cdot J = a/K$ and $I \odot J = aK$, with $a > 0$ and $K = \partial^2 I = \partial^2 J$. Then a direct calculation (using the closure of K under multiplication) shows that $\partial(a/K) = a/K$ and $\partial(aK) = aK$. □

The following is analogous to the product rule for differentiation.

Proposition 5.11. For $I, J > 0$ in $\overline{Q}$,

$$\max\{I \cdot \partial J,\ \partial I \cdot J\} \leqslant \partial(I \cdot J) \leqslant \partial(I \odot J) \leqslant \max\{I \odot \partial J,\ \partial I \odot J\}.$$

Proof. $\partial(I \cdot J) \leqslant \partial(I \odot J)$ holds by the previous result.

For the first inequality, assume that $I \cdot \partial J \leqslant \partial I \cdot J$ and suppose $i \in I$, $j \in J$, $\mathrm{d}i \in \partial I$ and $j_1 \in J$. We must show $ij + j_1 \mathrm{d}i \in I \cdot J$. Without loss we may assume $j = j_1$ for if $j < j_1$ simply replace j by j_1 and prove a stronger result. Then $ij + j\mathrm{d}i = (i + \mathrm{d}i)j \in I \cdot J$ as $\mathrm{d}i \in \partial I$.

For the other inequality, suppose $I \odot \partial J \leqslant \partial I \odot J$ and choose $\mathrm{d}i' \geqslant \partial I$, $j' \geqslant J$. So $j'\mathrm{d}i'$ is a 'typical' element in $(\partial I \odot J)^c$ and it will suffice to find $i \in I$ and $j \in J$ with $ij + j'\mathrm{d}i' \geqslant I \odot J$. Choose $i_0 \leqslant I \leqslant i_0'$ with $i_0 + \mathrm{d}i' = i_0'$. Now from $I \odot \partial J \leqslant \partial I \odot J$ there are $i' \geqslant I$ and $\mathrm{d}j' \geqslant \partial J$ with $i'\mathrm{d}j' \leqslant j'\mathrm{d}i'$. Without loss we may assume $i' \leqslant i_0'$ and we let $i = i_0 - i_0' + i'$ so $i \leqslant I \leqslant i' = i + \mathrm{d}i'$. Now choose $j \in J$ with $j' = j + \mathrm{d}j' \geqslant J$. Hence

$$ij + j\mathrm{d}i' + i'\mathrm{d}j' = (i + \mathrm{d}i')(j + \mathrm{d}j') = i'j' \geqslant I \odot J.$$

Thus one or both of $j\mathrm{d}i'$, $i'\mathrm{d}j'$ is greater than or equal to $\partial(I \cdot J)$. But $j\mathrm{d}i' \leqslant j'\mathrm{d}i'$ and $i'\mathrm{d}j' \leqslant j'\mathrm{d}i'$ so $j'\mathrm{d}i' \geqslant \partial(I \cdot J)$, as required. □

Note that $I \cdot \partial J$ and $\partial I \cdot J$ are closed under addition since ∂I, ∂J are, so

$$\max\{I \cdot \partial J,\ \partial I \cdot J\} = I \cdot \partial J + \partial I \cdot J = I \cdot \partial J \oplus \partial I \cdot J$$

Similarly for $\max\{I \odot \partial J,\ \partial I \odot J\}$.

Example 5.12. If $N > \mathbb{N}$ is closed under addition and $a > N$, setting $I = aN$ and $J = a/N$, then I, J and hence $I \cdot J$, $I \odot J$ are closed under addition and we have $\partial(I \cdot J) = I \cdot J = a^2/N$, and $\partial(I \odot J) = I \odot J = a^2 N$.

Observe that, in calculating powers of cuts by standard natural number powers, $I^n = I \cdot I \cdot \dots \cdot I$ or $I^n = I \odot I \odot \dots \odot I$ it doesn't matter whether one takes $\cdot$ or $\odot$ since two lower bounds $i, j \leqslant I$ can be replaced by $\max(i, j)$, and similarly $i', j' \geqslant I$ can be replaced by $\min(i', j')$. In other words, for $I > 0$ and standard $n \in \mathbb{N}$, $I^n = \sup\{i^n : 0 < i \leqslant I\} = \inf\{i'^n : i' \geqslant I\}$. Hence we deduce from the previous proposition that,

Corollary 5.13. For $I > 0$ in $\overline{Q}$ and $n \in \mathbb{N}$, $I^{n-1} \cdot \partial I \leqslant \partial(I^n) \leqslant I^{n-1} \odot \partial I$.

At present I am unable to give an example of a cut $I > 1$ such that $I^{n-1} \cdot \partial I$ and $I^{n-1} \odot \partial I$ are different.

For the derivative of $1/I$, observe first the following.

Proposition 5.14. For $I > 0$ closed under $+$ we have $\partial(1/I) = 1/I$.

Proof. For $i \leqslant I \leqslant i'$,

$$\frac{1}{i'} + \frac{1}{i'} = \frac{1}{i'/2}$$

so $\partial(1/I) \geqslant 1/I$. But $\partial(1/I) \leqslant 1/I$ holds by definition, hence the result. □

We wonder if, more generally, it is true that $\partial(1/I) = \partial I/I^2$. The following gets close to proving this.

Proposition 5.15. For $I > 0$ we have

$$\partial I \cdot \frac{1}{I^2} \leqslant \partial \frac{1}{I} \leqslant \partial I \odot \frac{1}{I^2}.$$

Proof. For $\mathrm{d}i \in \partial I$ and $i' \geqslant I$,

$$\mathrm{d}i \cdot \frac{1}{i'^2} + \frac{1}{i'} \leqslant \frac{1}{i' - \mathrm{d}i}$$

hence $\partial I \cdot (1/I^2) \leqslant \partial(1/I)$. For the other inequality, given $\mathrm{d}i' \geqslant \partial I$ and $i \leqslant I$, we want $i' \geqslant I$ with $1/i' + \mathrm{d}i'/i^2 \geqslant 1/I$. Choose (without loss of generality, increasing i if necessary) $i' = i + \mathrm{d}i'/2 \geqslant I$ so

$$\frac{1}{i'} + \frac{\mathrm{d}i'}{i^2} \geqslant \frac{1}{i' - (i'^2/i^2)\mathrm{d}i'/2} \geqslant \frac{1}{i' - \mathrm{d}i'/2} = \frac{1}{i} \geqslant \frac{1}{I},$$

which follows from the series expansion $1/(x-h) = 1/x + h/x^2 + h^2/x^3 + \cdots$, as required. □

We conclude with some results concerning the interesting cut $\exp(1/\mathbb{N}) = \inf\{\sqrt[n]{\mathrm{e}} : n \in \mathbb{N}\}$.

Proposition 5.16. Suppose $I \in \overline{Q}$ and $I > 0$. Suppose also that $\partial^2 I > \exp(1/\mathbb{N})$. Then I is closed under addition.

Proof. By assumption $\partial \log I > 1/\mathbb{N}$. Let $0 < i \leqslant I$ and $\mathrm{d}l \leqslant \partial \log I$ with $\mathrm{d}l \geqslant 1/n$ for some $n \in \mathbb{N}$, so $n\mathrm{d}l \geqslant 1$. Then $\log I \geqslant \log i + n\mathrm{d}l$, since $\log I$ is closed under repeated addition by $\mathrm{d}l$. But $\log i + n\mathrm{d}l \geqslant \log(i + in\mathrm{d}l)$ hence $i + in\mathrm{d}l \leqslant I$ and so $2i \leqslant I$. □

Corollary 5.17. If $I \in \overline{Q}$, $I > 0$ and $\partial^2 I > \exp(1/\mathbb{N})$ then $\partial I \cdot I^k = \partial I \odot I^k$ for all $k \in \mathbb{N}$, and in particular $\partial(I^{k+1}) = \partial I \cdot I^k$.

Proof. By the previous proposition I is closed under addition so $I = \partial I$ and the result is a triviality. □

Example 5.18. Let I be the cut $\exp(1/\mathbb{N})$ in $\overline{Q}$. We can check that $\partial^2 I = \exp \partial(1/\mathbb{N}) = \exp(1/\mathbb{N}) = I$. Also, given $n' \geqslant \mathbb{N}$ we have $2\mathrm{e}^{1/n'} > 2 > \mathrm{e}^{1/2}$ and hence I is not closed under $+$.

We can also show that $\partial \exp(1/\mathbb{N}) = 1/\mathbb{N}$. In one direction, suppose that $0 < x \leqslant 1/n'$ for some $n' \geqslant \mathbb{N}$ and $n'' > \mathbb{N}$. Then we must find an upper bound for $\exp(1/n'') + x$. But in PA one can prove that for sufficiently large $z > y > 0$, $\exp(1/(y/3)) \geqslant \exp(1/y) + 1/z$, since by expanding $\exp(1/y) + 1/z = 1 + 1/y + 1/2y^2 + \cdots + 1/z$ which is at most $1 + 3/y$. Thus $\exp(1/n'') + x \leqslant \exp(1/n'') + 1/n' \leqslant \exp(1/(n''/3)) \leqslant \exp(1/\mathbb{N}$ as required, showing $\partial \exp(1/\mathbb{N}) \geqslant 1/\mathbb{N}$. The other direction is easier, as given $x = 1/n$ ($n \in \mathbb{N}$) we have $2n/2 = 2 > \exp(1/\mathbb{N})$ so for $a \leqslant \exp(1/\mathbb{N})$ we have $a + 2nx > \exp(1/\mathbb{N}) \geqslant a$ hence for some $i < n$ we have $a + (i+1)x > \exp(1/\mathbb{N}) \geqslant a + ix$ showing $x \geqslant \partial \exp(1/\mathbb{N})$. Note also that $1/\mathbb{N} = 1/\mathbb{N}^2 \exp(1/\mathbb{N})$, which is the answer for $\partial \exp(1/\mathbb{N})$ expected from elementary calculus.

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