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PARAMETER FREE INDUCTION IN ARITHMETIC

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We present a survey of results on fragments of first order Peano arithmetic axiomatized by schemes of parameter free induction and collection. Our two main themes are,
(a) conservation results relating these theories to the 'classical' fragments axiomatized with parameters, and
(b) that parameter free fragments are not finitely axiomatizable.

It turns out that there are many settings in which some sort of parameter-free or reduced-parameter scheme can be formulated with properties (a) and (b). Our examples are all taken from arithmetic, but I see no real reason why the ideas could not be applied to axiomatizations of set theory. We close by indicating how results and techniques developed here may turn out to be useful in connection with some open problems in arithmetic.

Attribution: Most of the work referred to here is presented in [3] or [2] and (except where stated otherwise) should be attributed to Kaye, Paris and Dimitracopoulos.

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1. Preliminaries and basic results

We work in the usual language $\mathcal{L} = \{0,1,+,\cdot,<\}$ and define the formula classes Δ_0, Σ_n, Π_n, $\exists_n$, $\forall_n$, E_n and U_n as usual. (Here E_n and U_n are the subclasses of $\exists_n$ and $\forall_n$ respectively in which every quantifier is bounded.) PA^- denotes a set of axioms whose models are exactly the class of nonnegative parts of discretely ordered rings. For a class Γ of formulae, $I\Gamma^-$ denotes PA^- together with,

$$(\theta(0) \wedge \forall x(\theta(x)\rightarrow\theta(x+1))) \rightarrow \forall x\theta(x)$$

for all $\theta\varepsilon\Gamma$ with only the free variable shown. $L\Gamma^-$ is PA^- together with,

$$\exists x\theta(x) \rightarrow \exists x(\theta(x) \wedge \forall y<x\neg\theta(y))$$

for all $\theta\varepsilon\Gamma$ with only one free variable, and $B\Gamma^-$ is $I\Delta_0^-$ together with,

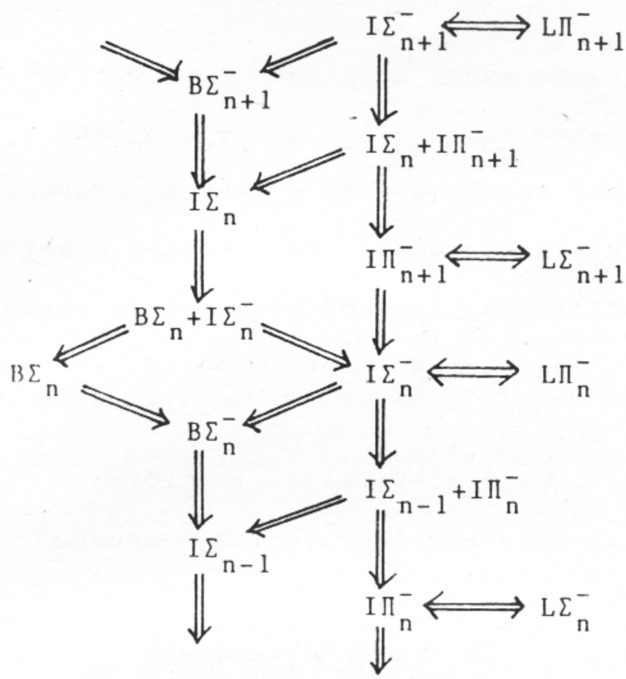
$$\forall x \exists y\theta(x,y) \rightarrow \forall t \exists z \forall x\leq t \exists y\leq z\theta(x,y)$$

for all $\theta\varepsilon\Gamma$. The parameter versions of these theories (denoted $I\Gamma$, $L\Gamma$ and $B\Gamma$) have been studied by many people, in particular by Paris and Kirby [6] who showed:

$$\begin{array}{l} I\Sigma_{n+1} \\ \Downarrow \\ B\Sigma_{n+1} \Leftrightarrow B\Pi_n \\ \Downarrow \\ I\Sigma_n \Leftrightarrow I\Pi_n \Leftrightarrow L\Sigma_n \Leftrightarrow L\Pi_n \end{array}$$

with the converses to the two vertical arrows being false. The main relationships between the various parameter free theories and their parameter analogues defined here are described in the following theorem:

<u>THEOREM 1</u>: ([3],[2]) For all $n \geq 1$,



also $L\Delta_0^- \Leftrightarrow I\Delta_0^- \Leftrightarrow I\Delta_0 \Leftrightarrow L\Delta_0$ and the absence of a directed path from one theory T to another, S, in the above diagram indicates $T \nvdash S$.

In theorem 1, $I\Pi_1^- \nvdash B\Sigma_1^-$ and $I\Sigma_{n-1}+I\Pi_n^- \nvdash B\Sigma_n^-$ are due to Kaye [2]. All other parts are due to Kaye, Paris and Dimitracopoulos [3].

Theorem 1 indicates the study of parameter free induction should supply us with worthwhile imformation on the structure of the $I\Sigma_n/I\Pi_n$ hierarchy. That the parameter free schemes are strictly weaker than their parameter counterparts comes as no surprise since their axiomatizations, in terms of quantifier complexity, are less complex. The next result shows that their natural axiomatizations are 'best possible'.

<u>THEOREM 2</u>: ([3],[2]) For all $n \geq 1$,

i) $I\Sigma_n^-$ is $\Sigma_{n+1} \vee \Pi_{n+1}$ axiomatizable, but not Σ_{n+1} nor Π_{n+1}.

ii) $I\Pi_n^-$ is Σ_{n+1} axiomatizable, but not Π_{n+1}.

iii) $B\Sigma_n^-$ is $\Sigma_{n+1} \vee \Pi_{n+1}$ axiomatizable, but not Σ_{n+1} nor Π_{n+1}.

That $B\Sigma_1^-$ is not Π_2 is taken from [2] where I was also able to improve on (i) showing $I\Sigma_n^-$ is not $\Sigma_{n+1} \vee \Pi_{n+1}$. The rest of theorem 2 is from [3].

2. <u>Conservation</u> <u>results</u>.

Our first conservation result is,

<u>THEOREM 3</u>: ([3]) For all n $I\Sigma_n$ is a Σ_{n+2} conservative extension of $I\Sigma_n^-$. (We write this as $I\Sigma_n^- \equiv_{\Sigma_{n+2}} I\Sigma_n$.)

Notice that since $I\Sigma_n^-$ is $\Sigma_{n+1} \vee \Pi_{n+1}$ this has the surprising consequence that the consequences of $I\Sigma_n$ that are boolean combinations of Σ_{n+1} sentences form an axiomatization of the Σ_{n+2} consequences of $I\Sigma_n$. For $n \geq 1$ theorem 3 is best possible, that is $I\Sigma_n$ isn't a Π_{n+2} conservative extension of $I\Sigma_n^-$ (for example, since $I\Sigma_n$ itself is Π_{n+2} axiomatized.)

We now have several proofs of theorem 3 availible. Jeff Paris and Costas Dimitracopoulos first noticed that $I\Sigma_1 \equiv_{\Pi_2} I\Sigma_1^-$ (since they both Σ_1-define exactly the primitive recursive functions) and I realized that an analogous result for $I\exists_1$ and $I\exists_1^-$ was exactly what I needed to show that $I\exists_1^-$ proves the Matijasevic-Robinson-Davis-Putnam (MRDP) theorem. I was then able to show that for $\Gamma = E_n$, $\exists_n$ or Σ_n, $I\Gamma^-$ proves the 'reduced-parameter' scheme,

$$\forall \vec{a},x(\theta(x,\vec{a})\rightarrow\theta(x+1,\vec{a})) \rightarrow \forall \vec{a},x(\theta(0,\vec{a})\rightarrow\theta(x,\vec{a})) \qquad (*)$$

for all $\theta\varepsilon\Gamma$ with only the free variables shown. A simple Henkin-type argument then gives the result,

<u>THEOREM 4</u>: ([2]) For all $n\geq 1$, if Γ is E_n, $\exists_n$, or Σ_n then

$$I\Gamma \equiv_{\exists\forall\Gamma} I\Gamma^-$$

Jeff Paris then noticed that for $\Gamma = \Sigma_n$ the equivalence $I\Sigma_n^- \Leftrightarrow L\Pi_n^-$ gives a considerably shorter proof of theorem 3, by-passing (*). (This is the proof appearing in [3]). Finally we then realized that if $M\models I\Sigma_n^-$ then $K^{n+1}(M)$ (= the Σ_{n+1} definable elements of M) satisfies $M \succ_{\Sigma_{n+1}} K^{n+1}(M)\models I\Sigma_n$. (For details see [2]).

I used theorem 4 to derive the following result:

<u>THEOREM 5</u>: ([2])

i) $I\exists_1^-\vdash$ MRDP and hence $I\exists_n^-\Leftrightarrow I\Sigma_n^-$ for all $n\geq 1$, and $I\forall_n^-\Leftrightarrow I\Pi_n^-$ for all $n\geq 2$.

ii) If $IE_1\vdash$ MRDP then $IE_1^-\Leftrightarrow IE_1\Leftrightarrow I\Delta_0$.

The situation for $I\Pi_n^-$ is rather different, however with a little more work we can describe the relationship between $I\Pi_n^-$ and $I\Pi_n$ (= $I\Sigma_n$) quite precisely. The scheme $I\Pi_n^{-(k)}$ is "parameter free Π_n induction up to the ordinal ω^k" and is axiomatized by PA^- together with,

$$\theta(0,0,\ldots,0)\wedge \bigwedge_{i=1}^{k} \forall x_1,\ldots,x_i(\forall y_{i+1},\ldots,y_k\theta(\vec{x},\vec{y}) \rightarrow$$

$$\theta(x_1,\ldots,x_{i-1},x_i+1,0,\ldots 0)) \rightarrow \forall\vec{x}\theta(\vec{x})$$

for all $\theta\varepsilon\Pi_n$. $I\Pi_n^{-(\infty)}$ is $\bigcup_{k\in\mathbb{N}} I\Pi_n^{-(k)}$. There are corresponding least number principles, $L\Sigma_n^{-(k)} \Leftrightarrow I\Pi_n^{-(k)}$. Notice that $I\Pi_n^{-(\infty)}$ is Σ_{n+1} axiomatized, just as $I\Pi_n^-$ was.

<u>THEOREM 6</u>: ([3]) For all $n\geq 1$,

$$I\Pi_n^{-(\infty)} \equiv_{\Sigma_{n+1}} I\Pi_n .$$

To explain in more detail the role of $I\Pi_n^{-(k)}$ for each k, we shall restrict ourselves here to the case n=1. (A generalization of theorem 7 below to $n\geq 1$ appears in [2].) It is easy to define the following functions in $I\Sigma_1$ (or $I\Sigma_1^-$):

$$F_0(x) = x^2$$

$$F_{k+1}(x) = F_k^{(x)}(x)$$

These functions have Σ_1 graph and $I\Delta_0 + \{ \forall x \exists y\ F_k(x)=y \mid k \in \mathbb{N} \}$ is an axiomatization of the Π_2 consequences of $I\Sigma_1$ (because this theory defines exactly the primitive recursive functions). $F_1(x)$ is essentially the exponential function, ie. $I\Delta_0$+"$\forall x F_1(x)$ exists" is equivalent to $I\Delta_0$ + exp, the theory considered in [1].

<u>THEOREM 7</u>: ([3]) For all $k\geq 1$,

i) $I\Pi_1^{-(k)}$ proves all Σ_2 consequences of $I\Delta_0 + \forall x \exists y F_k(x)=y$

ii) $I\Delta_0 + \forall x \exists y F_k(x)=y$ proves all Π_2 consequences of $I\Pi_1^{-(k)}$.

<u>COROLLARY 8</u>: The $I\Pi_1^{-(k)}$ hierarchy is proper.

<u>PROOF</u>: $I\Delta_0 + F_{k+1} \vdash con(I\Delta_0 + F_k)$ but $I\Delta_0 + F_k \nvdash con(I\Delta_0+F_k)$ where con(T) denotes some natural Π_1 sentence expressing "T is consistent" - see [8].

<u>COROLLARY 9</u>:

i) $I\Pi_1^- \vdash \Delta_0$-PHP, the Δ_0 pigeonhole principle,

ii) $I\Pi_1^- \vdash \forall x>1\ \exists y\leq x^2$ (prime(y)$\wedge$ y>x).

Corollary 9 holds since both the statements on the RHS are Π_1 and true in $I\Delta_0$ + exp. This corollary is suprising because one feels that, since $I\Pi_1^-$ can not define functions of greater than polynomial growth, $I\Delta_0$ and $I\Pi_1^-$ should be "very close", whereas it has proved notoriously difficult to settle whether or not the two statements in corollary 9 are provable in $I\Delta_0$.

It is also possible to derive a conservation result for $B\Sigma_n^-$. By considering a suitable pairing function it is easy to see that $B\Sigma_n^-$ proves the scheme,

$$\forall \vec{a},x \exists y\theta(\vec{a},x,y) \rightarrow \forall \vec{a},t \exists z \forall x{\le}t \exists y{\le}z\theta(\vec{a},x,y)$$

for $\theta\varepsilon\Sigma_n$ with only the free variables shown. A straightforward Henkin type argument then gives,

<u>THEOREM 10</u>: ([3]) For all $n\geq 1$,

$$B\Sigma_n^- \equiv_{\Sigma_{n+2}} B\Sigma_n$$

Once again this is best possible (since $B\Sigma_n^- \not\Rightarrow B\Sigma_n$ and $B\Sigma_n$ is Π_{n+2} axiomatizable) and since $B\Sigma_n^-$ is $\Sigma_{n+1}\vee \Pi_{n+1}$ we deduce that the Σ_{n+2} consequences of $B\Sigma_n$ are $\Sigma_{n+1}\vee\Pi_{n+1}$ axiomatized.

The Δ_n induction and least number principles have occasionally been studied. $I\Delta_n$ is PA^- together with,

$$\forall \vec{a}\ ((\forall x(\theta(x,\vec{a}) \leftrightarrow \psi(x,\vec{a})) \wedge \theta(0,\vec{a}) \wedge \forall x(\theta(x,\vec{a}) \rightarrow \theta(x+1,\vec{a})))$$
$$\rightarrow \forall x\theta(x,\vec{a}))$$

for all $\theta\varepsilon\Sigma_n$ and $\psi\varepsilon\Pi_n$. $L\Delta_n$, $I\Delta_n^-$, and $L\Delta_n^-$ are defined in the obvious way also. It is known that $L\Delta_n \Leftrightarrow B\Sigma_n$ (an unpublished result due to Gandy, see [2]) but it is still open whether $I\Delta_n \leftrightarrow L\Delta_n$.

Combining theorem 3 with the Paris-Friedman conservation result, $I\Sigma_n \equiv_{\Pi_{n+2}} B\Sigma_{n+1}$ (see [5]) we have,

$$I\Delta_n^- \equiv_{\Delta_{n+1}} I\Delta_n \quad \text{and} \quad L\Delta_n^- \equiv_{\Delta_{n+1}} L\Delta_n,$$

but although we know $L\Delta_n$ isn't a Σ_{n+2} conservative extension of $L\Delta_n^-$ (because $I\Pi_n^- \vdash L\Delta_n^-$ and $I\Pi_n^- \nvdash B\Sigma_n^- = \Sigma_{n+2}(L\Delta_n)$) the situation is still rather unsatisfactory as we don't know if these results are best possible.

Instead, define $UI\Delta_n$ (for "uniform $I\Delta_n$") to be PA^- with,

$$\forall \vec{a}, x(\theta(x,\vec{a}) \leftrightarrow \psi(x,\vec{a})) \rightarrow$$
$$\forall \vec{a}((\theta(0,\vec{a}) \wedge \forall x(\theta(x,\vec{a}) \rightarrow \theta(x+1,\vec{a}))) \rightarrow \forall x\theta(x,\vec{a}))$$

for all $\theta \in \Sigma_n$ and $\psi \in \Pi_n$, and define $UL\Delta_n$ similarly. Then,

<u>THEOREM 11</u>: ([2])

i) $UI\Delta_n \equiv_{\Sigma_{n+2}} I\Delta_n$, and

ii) $UL\Delta_n \equiv_{\Sigma_{n+2}} L\Delta_n$.

It follows from theorem 11 (ii) that $UL\Delta_n \Leftrightarrow B\Sigma_n^-$. A direct proof of this appears in [2].

3. Finite axiomatizability

It is well known that $I\Sigma_n$ (for $n\geq 1$) and $B\Sigma_n$ (for $n\geq 2$) are finitely axiomatizable. In contrast to this it appears to be the case that none of the parameter free fragments we have considered is finitely axiomatizable. For example in [3] we show that,

THEOREM 12: For all $n\geq 1$, if T is a theory with $\Pi_{n+1}(\mathbb{N}) \vdash T \vdash B\Sigma_n^-$ then T is not finitely axiomatizable.

The proof goes by taking $\sigma\in\Pi_{n+1}(\mathbb{N})$ such that $\sigma\vdash T$ and $M\models PA+\sigma$ with nonstandard Σ_n definables. Then $K^n(M) \prec_{\Sigma_n} M$ as in [6], but one can show that $K^n(M)\models\sigma+\neg B\Sigma_n^-$ so we have $T\nvdash B\Sigma_n^-$ a contradiction.

To see that this means none of $I\Pi_{n+1}^{-(\infty)}$, $I\Pi_{n+1}^{-(k)}$, $I\Sigma_n^-$, $B\Sigma_n^-$ is finitely axiomatizable, notice that

$$\Pi_{n+1}(\mathbb{N}) \vdash I\Pi_{n+1}^{-(\infty)} \vdash I\Pi_{n+1}^{-(k)} \vdash I\Sigma_n^- \vdash B\Sigma_n^- .$$

(It will follow from a later result that neither $I\Pi_1^{-(\infty)}$ nor $I\Pi_1^{-(k)}$ is finitely axiomatizable.)

In [2] I investigated the proof theoretic strength of a general finite fragment $I\Sigma_{n-1}+I\phi_1+I\phi_2+\ldots I\phi_k$ of $I\Sigma_n^-$, each ϕ_i having only one free variable. Using the notion of a set being α-large for ordinals $\alpha<\varepsilon_0$ taken from Ketonen and Solovay's paper [4] we have,

THEOREM 13: ([2]) For all $n,k\leq 1$, if $\sigma\in\Pi_2$ is provable from some finite fragment $I\Sigma_{n-1}+I\phi_1+\ldots+I\phi_k$ of $I\Sigma_n^-$, then,

$$I\Delta_0 + \{ \forall x \exists y[x,y] \text{ is } \omega_{n-1}^{\omega^k \cdot m}\text{-large} \mid m\in\mathbb{N} \} \vdash \sigma$$

and moreover this is best possible, ie. there are instances

$\phi_1, \phi_2, \ldots, \phi_k$ of $I\Sigma_n^-$ such that

$$I\Sigma_{n-1} + I\phi_1 + .. + I\phi_k \vdash \forall x \exists y[x,y] \text{ is } \omega_{n-1}^{\omega^{k} \cdot m}\text{-large}$$

for each $m \in \mathbb{N}$.

(Zofia Adamowicz has informed me that Zygmunt Ratajczyk has obtained similar results independently.)

We thus have an intriguing situation which one feels ought to be useful - in some different setting perhaps. For $n \geq 1$, $I\Sigma_n$ and $I\Sigma_n^-$ have the same consistency strength (by theorem 3) yet $I\Sigma_n$ is finitely axiomatizable, whereas each finite fragment of $I\Sigma_n^-$ is strictly "easier" to prove consistent than the full $I\Sigma_n^-$ itself.

There are also interesting connections between the finite fragments $I\Sigma_{n-1}+I\phi_1+\ldots+I\phi_k$ of $I\Sigma_n^-$ and the $I\Pi_n^{-(k)}$'s:

<u>THEOREM 14</u>: ([2]) For all $n \geq 1$ and $k \geq 1$,

i) if $\sigma \in \Pi_{n+1}$ and $I\Sigma_{n-1} + I\Pi_n^{-(k)} \vdash \sigma$ then there are $\phi_1, \phi_2, \ldots \phi_k$ in Σ_n with one free variable each such that,

$$I\Sigma_{n-1}+I\phi_1+\ldots+I\phi_k \vdash \sigma$$

ii) if $\sigma \in \Sigma_{n+1}$, $\phi_1, \ldots \phi_k \in \Sigma_n$ with one free variable each and $I\Sigma_{n-1}+I\phi_1+\ldots+I\phi_k \vdash \sigma$ then,

$$B\Sigma_n^- + I\Pi_n^{-(k)} \vdash \sigma$$

and $$I\Pi_n^{-(k+1)} \vdash \sigma$$

if σ is Δ_{n+1} then $I\Sigma_{n-1}+I\Pi_n^{-(k)} \vdash \sigma$.

I don't know whether, in the conclusion to part (ii), the $B\Sigma_n^-$ can be dropped. Notice that theorem 14 (ii) together with theorem 3 provide a refinement on theorem 6.

For $I\Pi_n^-$ we have the following strong way of saying "$I\Pi_n^-$ isn't finitely axiomatizable":

<u>THEOREM 15</u>: ([3]) For all $n\geq 1$, $\Pi_n(\mathbb{N})$ is the only consistent Π_n theory (up to deductive equivalence) to imply $I\Pi_n^-$.

Combining this with the results that $\forall_n(\mathbb{N}) \Leftrightarrow \Pi_n(\mathbb{N})$ and $I\Pi_n^- \Leftrightarrow I\forall_n^-$ for $n\geq 2$, (by theorem 5 and the fact that $I\Delta_0$ + exp is $\forall\exists$ and proves the MRDP theorem) we deduce a positive answer to problem 5.4 in [7]:

<u>THEOREM 16</u>: For all $n\geq 1$ $\forall_n(\mathbb{N})$ is the only consistent $\forall_n$ theory (up to deductive equivalence) to imply $I\forall_n^-$.

(This result was proved by Wilkie in [7] for the case n=1)

We are also able to prove results analogous to theorem 15 for a (proper?)-hierarchy of theories within $I\Pi_1^-$:

Let $\forall E_n = \{ \forall\vec{x}\phi(\vec{x},\vec{y}) \mid \phi\varepsilon E_n\}$. Then,

<u>THEOREM 17</u>: ([2]) For all n and all $\sigma\varepsilon \exists\forall E_n$,

i) if $I\Delta_0$ + exp $\vdash\sigma$ then $I\forall E_n^- \vdash\sigma$

ii) $\forall E_n(\mathbb{N})$ is the only consistent $\forall E_n$ theory (up to deductive equivalence) to imply $I\forall E_n^-$.

Thus if $I\forall E_n^- \vdash I\forall E_{n+1}^-$ then $\forall E_n(\mathbb{N}) \Leftrightarrow \forall E_{n+1}(\mathbb{N})$. It would be nice to be able to relate this to the possible collapse of the $E_n^{\mathbb{N}}$ hierarchy of relations on $\mathbb{N}$, but I see no way at present of doing this.

A possibly more fruitful application of ideas here is the following: By a result of George Wilmers [9], if $M\models IE_1$ is nonstandard then its reduct to addition, $M\restriction +$ is recursively saturated. Now although IE_1 may turn out to be finitely axiomatizable, we suspect (by analogy with theorem 12) that IE_1^- is not. Suppose we could construct for each finite fragment $T = I\phi_1+\ldots+I\phi_k$ of IE_1^- a nonstandard model M_T of T with $M_T\restriction +$ not recursively saturated, then $T\nvdash IE_1$ for each such T. Now if $IE_1 \vdash$ MRDP then it is $\forall E_1$ axiomatized ([2]) and also finitely axiomatized (a result due to Paris and Dimitracopoulos, proved using truth definitions for E_n formulas). Thus there is a sentence $\sigma \in \forall E_1$ with $\sigma \Leftrightarrow IE_1$. By the conservation result $IE_1^- \equiv_{\exists\forall E_1} IE_1$ there is a finite fragment T of IE_1^- that proves σ. But then $T \vdash IE_1$, a contradiction. Hence we could conclude that $IE_1 \nvdash$ MRDP.

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