Truth in generic cuts

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Abstract

In an earlier paper (*MLQ* 54, 128–144) the first author initiated the study of generic cuts of a model of Peano Arithmetic relative to a notion of an indicator in the model. This paper extends that work. We generalise the idea of indicator to a related *neighbourhood system*; this allows the theory to be extended to one that includes the case of elementary cuts. Most results transfer to this more general context, and in particular we obtain the idea of a generic cut relative to a neighbourhood system, which is studied in more detail. The main new result on generic cuts presented here is a description of truth in the structure (M, I), where I is a generic cut of a model M of Peano Arithmetic. The special case of elementary generic cuts provides a partial answer to a question of Kossak (*Notre Dame J. Formal Logic* 36, 519–530).

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1 Introduction

The first author has introduced the idea of a generic cut of a model M of Peano Arithmetic [2]. His paper, which we refer to as GCMA for convenience, considers the set of cuts or initial segments of a model of arithmetic as a topological space. An indicator serves to select a subspace of this space and give an idea of distance. A generic cut (relative to the indicator chosen) is an element of this subspace which is a member of each comeagre subset that is invariant under automorphisms of the original model M. It was shown in GCMA that generic cuts exist in all countable arithmetically saturated models of PA, and some of their properties were studied.

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The first aim of this paper is to generalise this to a setting that admits the case of elementary cuts as a special case. In Section 2, we give the basic definitions, namely that of a *neighbourhood system*, and that of a *species*. A neighbourhood system is an abstraction of the topological information obtained from an indicator, together with some conditions on definability in the model. A species is essentially the set of cuts that can be captured by a neighbourhood system. The main relaxation in the definitions here is in using classes or class functions in the usual sense of these words in models of arithmetic, instead of sets and functions which are definable outright.

In Section 3 we set up the topology in which we will work in. The major step there is proving any closed species in a countable model is (essentially) homeomorphic to the Cantor set. This enables us to apply the Baire Category Theorem to and play Banach–Mazur games on our space to obtain information about *enforceable* subsets. We go on to define the central notion of this paper, that of a generic cut. Although we are not in a position to prove existence theorems at this stage, we do prove a theorem showing the existence of generic cuts under rather general hypotheses (Theorem 3.8) that will be particularly useful in motivating later work.

Section 4 gives examples of enforceable properties and serves to provide a list of properties enjoyed by generic cuts when they do exist. Most of this section is rather similar to results in GCMA and serve to illustrate that this work lifts easily to the more general situation we are now in.

Section 5 gives the existence theorems for generic cuts in countable arithmetically saturated models of arithmetic. Once again, the proof models that in GCMA, but a more elegant approach turns out to be possible by looking at multi-variable versions of homogeneity notions in GCMA. Also, we have taken the time to extend this argument to showing the necessity of arithmetic saturation, and to analyse the proof into its finitistic core, with a view to extracting information about the true statements in the structure (M, I) where I is generic.

Section 6 studies how generic cuts behave under the action of the automorphism group of the model. The back-and-forth system that we took from GCMA is what most our results there are based on. A few new conjugacy and non-conjugacy properties are proved, including a characterisation of when two generic cuts are conjugate. We also give here a weak quantifier elimination result, the main theorem in this paper. It says that if I is a generic cut of a model M of PA, then the orbit of an element a of (M, I) under the action of $\mathrm{Aut}(M, I)$ is completely determined by classes that are relatively low in the formula hierarchy.

We conclude the paper and gather together various facts about *elementary* *generic cuts* in Section 7, and survey the relationships of them to the elementary cuts that appeared in the literature. In particular, we show that elementary generic cuts give new examples of *free cuts*, a notion introduced by Roman Kossak. This partially answers a question raised by him on the cardinality of orbits of free cuts, and possibly gives new ways to tackle his other problems too.

The notation used in this paper is standard, and follows that in GCMA, Kaye [1] and Kossak–Schmerl [9]. It is sometimes helpful to consider models of Peano Arithmetic as models of finite set theory via the usual Ackermann interpretation [3]. We assume some knowledge of semiregular, regular and strong cuts, the basic properties of which can be found in Kirby–Paris [5] and the book by Kossak and Schmerl already mentioned. Oxtoby [12] contains some useful background on Baire category.

Most of the results in this paper first appeared in the second author's qualifying MPhil dissertation at Birmingham University.

2 Neighbourhood systems and species of cuts

Throughout this paper, M is a nonstandard model of PA. We write $\mathscr{L}_{\mathrm{A}}$ for the usual first order language $\{+, \times, <, 0, 1\}$ for arithmetic, and $\langle\cdot,\cdot\rangle$ for the usual pairing function in $\mathscr{L}_{\mathrm{A}}$. Let $\mathrm{cl}(\bar{c})$ denote the definable closure of the tuple $\bar{c} \in M$, and $\overline{\mathrm{cl}}(\bar{c})$ the least initial segment of M containing $\mathrm{cl}(\bar{c})$.

We will sometimes consider adjoining to M a point at infinity, ∞. By definition we have $x < \infty$ and $\infty + x = \infty = \infty - x$ for every $x \in M$. If $B \in M \cup \{\infty\}$, then $M_{<B}$ and $M_{\leqslant B}$ denote respectively the coded sets

$$\{x \in M : x < B\} \text{ and } \{x \in M : x \leqslant B\}.$$

A *cut* of M is a nonempty initial segment closed under successors. We write $I \subseteq_{\mathrm{e}} M$ to mean 'I is a cut of M'. In distinction to GCMA, we do not require cuts to be $\mathscr{L}_{\mathrm{A}}$ structures here. For $a, b \in M \cup \{\infty\}$ we denote the set

$$\{x \in M : a \leqslant x \leqslant b\}$$

by $[a, b]$. Define

$$\mathcal{C} = \{I : I \subseteq_{\mathrm{e}} M\}$$

and

$$\mathcal{S} = \{[a, b] : a, b \in M \cup \{\infty\} \text{ with } a \leqslant b\}.$$

For $I \in \mathcal{C}$ and $[a, b] \in \mathcal{S}$, we write $I \in [a, b]$ to mean $a \in I < b$.

The automorphism group of M is denoted by $\mathrm{Aut}(M)$ and each automorphism extends in the obvious way to $M \cup \{\infty\}$. All actions by automorphisms are written as superscripts on the right. If $\bar{c} \in M$, then $\mathrm{Aut}(M, \bar{c})$ denotes the pointwise stabiliser of $\bar{c}$ in $\mathrm{Aut}(M)$. Similarly, if $I \in \mathcal{C}$, then $\mathrm{Aut}(M, I)$ denotes the setwise stabiliser of I in $\mathrm{Aut}(M)$.

Definition 2.1. Two cuts I, J are said to be *conjugate over* $c \in M$ if $I^g = J$ for some $g \in \mathrm{Aut}(M, c)$. They are *conjugate* if they are conjugate over 0. The *conjugacy class* of a cut I is the orbit of I under the action of $\mathrm{Aut}(M)$.

Extending ideas of Paris and Kirby, indicators were defined in GCMA. We first set the scene by abstracting the topological information given by an indicator.

Definition 2.2. A set $\mathcal{B} \subseteq \mathcal{S}$ is a *neighbourhood system* if and only if

(0) $\mathcal{B}$ is nonempty;
(1) $\mathcal{B}$ is invariant under the action of $\mathrm{Aut}(M)$;
(2) $\forall [a, b] \in \mathcal{B}\ (b > a + 1)$
(3) $\forall [a, b] \in \mathcal{B}\ \forall c \in M\ ([a, c] \in \mathcal{B}$ or $[c, b] \in \mathcal{B})$
(4) $\forall [a, b] \in \mathcal{B}\ \forall [u, v] \in \mathcal{S}\ ([a, b] \subseteq [u, v] \Rightarrow [u, v] \in \mathcal{B})$; and
(5) for every $B \in M$, there exists a recursive Σ_1 type $p(x, y)$ over M, possibly with finitely many parameters from M, such that

$$\forall a, b < B\ \left([a, b] \in \mathcal{B} \Leftrightarrow M \vDash \bigwedge\!\!\!\bigwedge p(a, b)\right).$$

An arbitrary element $[a, b]$ of $\mathcal{S}$ is called a *semi-interval*. If $\mathcal{B}$ is a neighbourhood system and $[a, b] \in \mathcal{B}$, then we say that $[a, b]$ is a $\mathcal{B}$*-interval*, or *interval* if $\mathcal{B}$ is clear from the context. We write $a \ll b$ or $a \ll_{\mathcal{B}} b$ to mean $[a, b]$ is a $\mathcal{B}$-interval. It is also helpful to have a notation for semi-intervals that identifies them as intervals: $[\![a, b]\!]$, or $[\![a, b]\!]_{\mathcal{B}}$ if $\mathcal{B}$ needs to be specified, will always denote a $\mathcal{B}$-interval whereas $[a, b]$ might or might not be an interval.

Proposition 2.3. *Let $\mathcal{B}$ be a neighbourhood system. Then for all intervals $[\![a, b]\!]$ with $b \neq \infty$ there is $c \in M$ such that*

$$a < c < b \text{ and } [a, c], [c, b] \in \mathcal{B}.$$

Proof. Let $[a, b] \in \mathcal{B}$, $b \neq \infty$, and $B \in M$ be greater than a, b. Suppose the formulas in the type $p(x, y)$ in clause (5) of the definition of a neighbourhood system be $\varphi_n(x, y)$, in increasing strength, so $\varphi_{n+1}(x, y)$ implies $\varphi_n(x, y)$ for all n and all $x, y < B$. By the Σ_1 recursive saturation of M it suffices to show that for each n there is $c \in [a, b]$ such that $\varphi_n(a, c)$ and $\varphi_n(c, b)$. Since $[a, b] \in \mathcal{B}$ let c be the least element of $[a, b]$ such that $\varphi_n(a, c)$. Then $c > a$ provided n is sufficiently large and $\neg\varphi_n(a, c - 1)$ hence $[a, c - 1] \notin \mathcal{B}$. It follows from (2) and (3) that $[a,c] \notin \mathcal{B}$, so from (3) again that $[c,b] \in \mathcal{B}$ and hence $\varphi_n(c,b)$, as required. □

Neighbourhood systems generalise the idea of an indicator in the sense that we may say that a neighbourhood system $\mathcal{B}$ *indicates* the property $\mathcal{Z}$ of cuts if and only if for each a, b there is a cut I with a property $\mathcal{Z}$ between a and b just in case that $[a,b] \in \mathcal{B}$.

Given a neighbourhood system $\mathcal{B}$ and $[a,b] \in \mathcal{B}$ there is always some $I \in \mathcal{C}$ with $a \in I < b$. In particular the next definition provides suitable I. For this definition, recall that, for a nonempty set $A \subseteq M$, $\inf A$ is the greatest initial part of M that is disjoint with A and $\sup A$ is the least initial part of M containing A.

Definition 2.4. Given a neighbourhood system $\mathcal{B}$ and $a, b \in M$, let

- $M_{\mathcal{B}}(a) = \inf\{c \in M : [a,c] \in \mathcal{B}\}$, and
- $M_{\mathcal{B}}[b] = \sup\{d \in M : [d,b] \in \mathcal{B}\}$.

The notation $M_{\mathcal{B}}(a)$ and $M_{\mathcal{B}}[b]$ hides the fact that these may not be defined for all a, b. We say that $M_{\mathcal{B}}(a)$ *exists* if

$$\exists y \in M \ [a,y] \in \mathcal{B}.$$

Similarly, $M_{\mathcal{B}}[b]$ *exists* if

$$\exists x \in M \ [x,b] \in \mathcal{B}.$$

In both cases, it is simple to check from the axioms that these are in $\mathcal{C}$ whenever they exist, and moreover, given $[\![a,b]\!]$, both $M_{\mathcal{B}}(a)$ and $M_{\mathcal{B}}[b]$ exist and are between a and b. $M_{\mathcal{B}}(a)$ and $M_{\mathcal{B}}[b]$ are respectively the smallest $I \in \mathcal{C}$ containing a and largest $I \in \mathcal{C}$ not containing b that are 'indicated' by $\mathcal{B}$. That $M_{\mathcal{B}}(a)$ and $M_{\mathcal{B}}[b]$ are distinct follows from Proposition 2.3 which says there is some $c \in M$ with $M_{\mathcal{B}}(a) < c < M_{\mathcal{B}}[b]$.

Definition 2.5. A class $\mathcal{Z} \subseteq \mathcal{C}$ is a *species of cuts* (*species* for short) if and only if

(0) $\mathcal{Z}$ is nonempty;
(1) $\mathcal{Z}$ is invariant under the action of $\mathrm{Aut}(M)$; and
(2) for every $B \in M$, there exists a recursive Σ_1 type $p(x,y)$ over M, possibly with finitely many parameters from M, such that

$$\forall a, b < B \ \left(\exists I \in \mathcal{Z} \ (a \in I < b) \Leftrightarrow M \vDash \bigwedge\!\!\!\bigwedge p(a,b) \right).$$

If I is an element of $\mathcal{Z} \subseteq \mathcal{C}$, then we say that I is a $\mathcal{Z}$*-cut*.

Each species of cuts $\mathcal{Z}$ comes equipped with a natural linear order, namely the subset relation, $\subseteq$.

Neighbourhood systems and species of cuts naturally arise from indicators $Y\colon M \times M \to M$ in the sense of GCMA. More generally, this Y might be a class in the sense of M, i.e. segments of Y are parameter-definable in M. Still more generally, our notion of indicator may not in fact be a function at all but is formed from a family of M-finite functions $Y_B\colon M_{<B} \times M_{<B} \to M$ for various $B \in M$ such that for $B_1 < B_2$ the functions Y_{B_1} and Y_{B_2} agree for all $x, y < B_1$ in the sense that $Y_{B_1}(x, y) > \mathbb{N}$ if and only if $Y_{B_2}(x, y) > \mathbb{N}$.

Definition 2.6. Let $B \in M$ and $Y\colon M_{<B} \times M_{<B} \to M$ be definable.

- If $\mathcal{B}$ is a neighbourhood system, then Y *indicates* $\mathcal{B}$ *below* B if and only if

$$\forall a, b < B \ ([a, b] \in \mathcal{B} \Leftrightarrow Y(a, b) > \mathbb{N}).$$

- If $\mathcal{Z}$ is a species of cuts, then Y *indicates* $\mathcal{Z}$ *below* B if and only if

$$\forall a, b < B \ (\exists I \in \mathcal{Z}\, a \in I < b \Leftrightarrow Y(a, b) > \mathbb{N}).$$

Definition 2.7. Let $B \in M$. A function $Y\colon M_{<B} \times M_{<B} \to M$ is *monotone* if and only if

$$\forall a, b, u, v \leqslant B \ \Big(a \leqslant u \wedge v \leqslant b \Rightarrow Y(a, b) \geqslant Y(u, v)\Big).$$

Proposition 2.8. *(a) Relative to the other axioms for a neighbourhood system, axiom (5) is equivalent to any of the statements that for all $B \in M$ the neighbourhood system below B is indicated by: a definable function; a monotone definable function; or a recursive type of bounded complexity.*

(b) Relative to the other axioms for a species of cuts, axiom (2) is equivalent to any of the statements that for all $B \in M$ the species is indicated below B by: a definable function; a monotone definable function; or a recursive type of bounded complexity.

Proof. (Sketch.)

Fix $\mathcal{B} \subseteq \mathcal{S}$. Let $p(x, y)$ be a recursive Σ_m type over M such that

$$\forall a, b < B \ \Big([a, b] \in \mathcal{B} \Leftrightarrow M \models \bigwedge\!\!\!\bigwedge p(a, b)\Big).$$

Let $\bar{d} \in M$ be the parameters that appear in $p(x, y)$, and write $p(x, y)$ as $p(x, y, \bar{d})$. Then $p(x, y, \bar{d})$ is coded in M by c, say, so that

$$\{(c)_n : n \in \mathbb{N}\} = \{\ulcorner\phi(x, y, \bar{z})\urcorner : \phi(x, y, \bar{d}) \in p(x, y, \bar{d})\}.$$

Define a function $Y\colon M_{<B} \times M_{<B} \to M$ by

$$Y(x,y) = (\mu n)\Big(\neg \operatorname{Sat}_{\Sigma_m}((c)_n, [x,y,\bar{d}])\Big)$$

for all $x, y < B$. This is a definable function that indicates $\mathcal{B}$ below B. To obtain a monotone indicator function replace Y with

$$Y'(x,y) = \max\{Y(a,b) : a, b \in [x,y]\}.$$

Since Y and Y' are definable and defined on $M_{<B} \times M_{<B}$ they are M-finite so can be coded as a sequence of values, by some $y \in M$ say. Then the type

$$p(u,v) = \{Y(u,v) > n : n \in \mathbb{N}\},$$

is a recursive Σ_1 type indicating $\mathcal{B}$ using the parameter y.

The argument for species is similar. □

Every neighbourhood system $\mathcal{B}$ gives rise to a 'largest' species of cuts that it indicates. Similarly, every species of cuts $\mathcal{Z}$ has a natural neighbourhood system that describes it. How to go from a neighbourhood system to a species of cuts and back again is defined next.

Definition 2.9. Given a neighbourhood system $\mathcal{B}$, define $\mathcal{Z}(\mathcal{B})$, *the species of cuts associated with* $\mathcal{B}$, by

$$\mathcal{Z}(\mathcal{B}) = \{I \in \mathcal{C} : \forall [a,b] \in \mathcal{S}\ (I \in [a,b] \Rightarrow [a,b] \in \mathcal{B})\}.$$

Definition 2.10. Given a species of cuts $\mathcal{Z}$, define $\mathcal{B}(\mathcal{Z})$, *the neighbourhood system associated with* $\mathcal{Z}$, by

$$\mathcal{B}(\mathcal{Z}) = \{[a,b] \in \mathcal{S} : \exists I \in \mathcal{Z}\ I \in [a,b]\}.$$

Proposition 2.11. *(a) If $\mathcal{B}$ is a neighbourhood system then $\mathcal{Z}(\mathcal{B})$ is a species of cuts, and if $\mathcal{Z}$ is a species of cuts then $\mathcal{B}(\mathcal{Z})$ is a neighbourhood system.*
(b) For any neighbourhood system $\mathcal{B}$, $\mathcal{B}(\mathcal{Z}(\mathcal{B})) = \mathcal{B}$.
(c) For any species of cuts $\mathcal{Z}$, $\mathcal{Z}(\mathcal{B}(\mathcal{Z})) \supseteq \mathcal{Z}$.

Proof. Straightforward applications of the axioms. □

It is time to give some examples.

Example 2.12. The set $\mathcal{B}^{\mathcal{C}} = \{[a,b] \in \mathcal{S} : \forall n \in \mathbb{N}\ a + n < b\}$ is easily seen to be a neighbourhood system. The corresponding species of cuts is $\mathcal{C}$, the set of all cuts of M.

Example 2.13. Let Y be an indicator in the sense of GCMA such that

$$M \vDash \exists x \exists y\, Y(x,y) \geqslant n$$

for every $n \in \mathbb{N}$ to avoid triviality. We call indicators in this old sense *GCMA indicators* in this paper. Set

$$\mathcal{B}^Y = \{[a,b] \in \mathcal{S} : Y(a,b) > \mathbb{N}\} \cup \{[a,\infty] \in \mathcal{S} : \exists b \in M\ Y(a,b) > \mathbb{N}\}.$$

Then $\mathcal{B}^Y$ is a neighbourhood system. The corresponding species of cuts is $\mathcal{Z}^Y = \mathcal{Z}(\mathcal{B}^Y)$ which is the largest set of cuts indicated by Y. For example, if Y is the Paris–Harrington indicator for $I \in \mathcal{C}$ satisfying PA, then $\mathcal{Z}^Y$ is the set of cuts satisfying the Π_2 consequences of PA, and is the topological closure of the set of $I \in \mathcal{C}$ satisfying PA.

Example 2.14. Let M be a short recursively saturated model of PA, i.e. such that each recursive type $p(x)$ with finitely many parameters from M and containing a formula of the form $x < a$ is realised in M. Fix a recursive sequence $(t_n(x))_{n\in\mathbb{N}}$ of $\mathscr{L}_\mathrm{A}$ Skolem functions with the following properties:

- $\forall n \in \mathbb{N}\ \forall x \in M\ (t_n(x) < t_{n+1}(x))$;
- $\forall n \in \mathbb{N}\ \forall x \in M\ (x < t_n(x) \leqslant t_n(x+1))$; and
- for each $\mathscr{L}_\mathrm{A}$ Skolem function $s(x)$ there is an $n \in \mathbb{N}$ such that for all $x \in M$, we have $s(x) < t_n(x)$.

Then the set

$$\mathcal{B}^{\mathrm{elem}} = \{[a,b] \in \mathcal{S} : \forall n \in \mathbb{N}\ (t_n(a) < b)\}$$

is a neighbourhood system. (This requires short recursive saturation to encode the values of all the function $t_n(x)$ for all $n \in \mathbb{N}$ and all $x < B$, in order to encode the set of intervals below each $B \in M$.) Intervals in $\mathcal{B}^{\mathrm{elem}}$ will sometimes be called *elementary intervals* and the corresponding species of cuts, $\mathcal{Z}^{\mathrm{elem}} = \mathcal{Z}(\mathcal{B}^{\mathrm{elem}})$ is the species of all elementary cuts of M. By a diagonalisation argument, it can be seen there is no definable function $Y\colon M^2 \to M$ such that

$$Y(a,b) > \mathbb{N} \text{ iff } [a,b] \in \mathcal{B}^{\mathrm{elem}}$$

for all $a, b \in M$. Therefore, our definition of a neighbourhood system is strictly more general than its counterpart in GCMA.

For $\mathcal{B} = \mathcal{B}^{\mathrm{elem}}$ the cuts $M_{\mathcal{B}}(a)$ and $M_{\mathcal{B}}[b]$ are familiar cuts, usually denoted $M(a)$ and $M[b]$. These are the smallest elementary cut containing a and the largest elementary cut not containing b, respectively.

In certain circumstances, this neighbourhood system can be regarded as the 'finest' such system, as the following proposition shows.

Proposition 2.15. *Suppose M is a recursively saturated model of* PA *and $\mathcal{B}$ is a neighbourhood system of M such that for each $a \in M$ there is $b \in M$ with $[a,b] \in \mathcal{B}$. Then $\mathcal{B} \supseteq \mathcal{B}^{\mathrm{elem}}$.*

Proof. Each $[a,\infty]$ is in $\mathcal{B}$ since there is some $c \in M$ with $[a,\infty] \supseteq [a,b] \in \mathcal{B}$.

Now let $a, b \in M$ with $[a, b] \in \mathcal{B}^{\text{elem}}$ and $c \in M$ with $[a, c] \in \mathcal{B}$. Then $b > M(a)$ and by saturation there is an automorphism g of M fixing a such that $c^g < b$. It follows from the axioms that $[a, b] \in \mathcal{B}$. □

It can easily be checked that some facts about indicators transfer to this more general setting. The following lemma is formulated in terms of the standard cut because the region around $\mathbb{N}$ is the place where we are mostly interested in. It is also true of other cuts, as we leave the reader to verify.

Lemma 2.16. *Let $\mathcal{B}$ be a neighbourhood system, $B \in M$ and $Y \in M$ be an indicator for $\mathcal{B}$ below B. If $[\![a, b]\!] \subseteq M_{<B}$ is a $\mathcal{B}$-interval, then*

$$\{n > \mathbb{N} : M \models \exists [u, v] \subseteq [\![a, b]\!]\ (Y(u, v) = n)\} \subseteq_{\text{dcf}} M \setminus \mathbb{N}.$$

Proof. Let $\mathcal{B}$ be a neighbourhood system, $B \in M$ and $Y \in M$ be an indicator for $\mathcal{B}$ below B. Take a $\mathcal{B}$-interval $[\![a, b]\!] \subseteq M_{<B}$ and define X to be the set

$$\{n \in \mathbb{N} : \exists [u, v] \subseteq [\![a, b]\!]\ (Y(u, v) = n)\}.$$

Note that X is nonempty.

Suppose that $X \not\subseteq_{\text{cf}} \mathbb{N}$. Then X has an upper bound in $\mathbb{N}$, say D. Now for every $x \in [\![a, b]\!]$,

$$\begin{aligned} [x, b] \in \mathcal{B} &\text{ iff } Y(x, b) > \mathbb{N} && \text{since } Y \text{ is an indicator for } \mathcal{B} \text{ below } B, \\ &\text{ iff } Y(x, b) > D && \text{by our choice of } D \text{ and axiom (4) for intervals.} \end{aligned}$$

Therefore, since the set $\{x \in [\![a, b]\!] : [x, b] \in \mathcal{B}\}$ contains a and is bounded above by b, it has a maximum element, say $x^* \in M$. So $[x^*, b] \in \mathcal{B}$ but $[x^*+1, b] \notin \mathcal{B}$. This contradicts (2) and (3) in the definition of a neighbourhood system. □

Question 2.17. Let $\mathcal{Z}$ be a species. Does there always exist a function $Y : M^2 \to M$ such that

- $\forall x, y \in M\ (\exists I \in \mathcal{Z}\ (x \in I < y) \Leftrightarrow Y(x, y) > \mathbb{N})$, and
- for every $B \in M$, the set $\{\langle x, y, Y(x, y)\rangle : x, y \leqslant B\}$ is coded in M?

3 The topology on $\mathcal{Z}$ and enforceable properties

The set $\mathcal{C}$ of all cuts of M has a natural topology, given by taking as basic open sets all intervals

$$U_{[a,b]} = \text{int}\{I \in \mathcal{C} : I \in [a, b]\}$$

for $[a,b] \in \mathcal{B}(\mathcal{C})$, where $\operatorname{int} \mathcal{I}$ is the 'interior' of $\mathcal{I} \subseteq \mathcal{Z}$, $\mathcal{I} \setminus \{\bigcap \mathcal{I}, \bigcup \mathcal{I}\}$, i.e. with end points removed if either of these exist in $\mathcal{I}$.

Each species of cuts $\mathcal{Z}$ can therefore be considered as a topological space, where the topology on $\mathcal{Z}$ is the subspace topology inherited from $\mathcal{C}$. Kotlarski seems to be the first person who explicitly studied a family of cuts with its topology obtained from the order relation. (See for example the appendix in Smoryński [14].)

Proposition 3.1. *Given a species of cuts $\mathcal{Z}$, the closure of $\mathcal{Z}$ in $\mathcal{C}$ is $\overline{\mathcal{Z}} = \mathcal{Z}(\mathcal{B}(\mathcal{Z}))$.*

Proof. If $I \notin \overline{\mathcal{Z}}$ then there is $[a,b] \in \mathcal{B}(\mathcal{C})$ with $I \in [a,b]$ and no $J \in \mathcal{Z} \cap U_{[a,b]}$. But this means $[a,b] \notin \mathcal{B}(\mathcal{Z})$ and hence $I \notin \mathcal{Z}(\mathcal{B}(\mathcal{Z}))$. Conversely if $I \in \overline{\mathcal{Z}}$ then every $[a,b] \in \mathcal{B}(\mathcal{C})$ with $I \in [a,b]$ contains some $J \in \mathcal{Z}$. Therefore $I \in \mathcal{Z}(\mathcal{B}(\mathcal{Z}))$. □

Paris and Kirby call two families of cuts *symbiotic* if they have the same indicators. This generalises immediately to our context, explaining perhaps our use of the word 'species'.

Definition 3.2. Two species of cuts $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are *symbiotic* if every open set U containing a cut from one species contains a cut from the other, i.e. if their closures are equal: $\overline{\mathcal{Z}_1} = \overline{\mathcal{Z}_2}$.

Proposition 3.3. *Let M be countable and $\mathcal{Z}$ a closed species of cuts. Then $\mathcal{Z}$ is either order-isomorphic (and hence homeomorphic) to the Cantor set 2^ω with its usual ordering and topology or else is order-isomorphic to $2^\omega + 1$, the Cantor set with an additional isolated point greater than all the others.*

Proof. Let $\mathcal{B} = \mathcal{B}(\mathcal{Z})$ be the corresponding neighbourhood system, so $\mathcal{Z} = \mathcal{Z}(\mathcal{B})$ as $\mathcal{Z}$ is closed.

Fix an enumeration $(x_n)_{n\in\mathbb{N}}$ of M.

Define the sequence $([\![a_\sigma, b_\sigma]\!])_{\sigma\in 2^{<\omega}}$ of $\mathcal{B}$-intervals recursively as follows.

(a) By axioms (0) and (4) for a neighbourhood system, $0 \ll \infty$. Let $a_\varnothing = 0$ and if possible choose $b_\varnothing \in M$ such that $[b_\varnothing, \infty] \notin \mathcal{B}$. If there is no such $b_\varnothing \in M$ define $b_\varnothing = \infty$. Either way, $[a_\varnothing, b_\varnothing] \in \mathcal{B}$ by axiom (3).
(b) Let $n \in \mathbb{N}$ and $\sigma \in 2^n$ such that $[\![a_\sigma, b_\sigma]\!]$ is defined. If possible choose $c_\sigma \in M$ such that $a_\sigma \ll c_\sigma \ll b_\sigma$. Note that by Proposition 2.3 there is

always such a c_σ except possibly in the case $b_\sigma = \infty$. Define

$$\llbracket a_{\sigma 0}, b_{\sigma 0} \rrbracket = \begin{cases} \llbracket a_\sigma, x_n \rrbracket, & \text{if } a_\sigma \ll x_n \ll b_\sigma; \\ \llbracket x_n, c_\sigma \rrbracket, & \text{if } a_\sigma \leqslant x_n \text{ and } [a_\sigma, x_n] \notin \mathcal{B}; \\ \llbracket a_\sigma, c_\sigma \rrbracket, & \text{otherwise;} \end{cases}$$

and

$$\llbracket a_{\sigma 1}, b_{\sigma 1} \rrbracket = \begin{cases} \llbracket x_n, b_\sigma \rrbracket, & \text{if } a_\sigma \ll x_n \ll b_\sigma; \\ \llbracket c_\sigma, x_n \rrbracket, & \text{if } x_n \leqslant b_\sigma \text{ and } [x_n, b_\sigma] \notin \mathcal{B}; \\ \llbracket c_\sigma, b_\sigma \rrbracket, & \text{otherwise.} \end{cases}$$

(c) Let $n \in \mathbb{N}$ and $\sigma \in 2^n$ be such that $\llbracket a_\sigma, b_\sigma \rrbracket$ is defined where $b_\sigma = \infty$ and there is no such c_σ as in the last part. We define $\llbracket a_{\sigma 0}, b_{\sigma 0} \rrbracket = \llbracket a_{\sigma 1}, b_{\sigma 1} \rrbracket = \llbracket a_\sigma, \infty \rrbracket$. Note that in this case $[a_\sigma, c] \notin \mathcal{B}$ for all c with $\infty > c > a_\sigma$ since if $[a_\sigma, c]$ is an interval, either $[c, \infty]$ is also an interval, contradicting the fact that no such c_σ as in the last part could be found, or else $[c, \infty]$ is not an interval, contradicting the fact that there was no $b_\varnothing \in M$ such that $[b_\varnothing, \infty]$ is not an interval.

The case when $\mathcal{Z}$ turns out to be is order-isomorphic to $2^\omega + 1$ is when $b_\varnothing = \infty$ and for some σ, part (b) of the construction cannot be carried out because there is no suitable c_σ to take. If this happens we call such $a_\sigma \in M$ such that $a_\sigma \ll \infty$ but $[a_\sigma, c] \notin \mathcal{B}$ for all $\infty > c > a_\sigma$ *exceptional*. If there is some such exceptional a_σ then $M \in \mathcal{Z}$ because it is the only cut in all $\llbracket c, \infty \rrbracket$ for $c \geqslant a_\sigma$ and it is obviously an isolated greatest element in $\mathcal{Z}$.

The remainder of the proof is a straightforward application of the axioms and the enumeration of M to show that every cut $I \in \mathcal{Z}$ (except possibly M if there is an exceptional a_σ) is the limit of a sequence $(a_{\varepsilon \restriction n})_{n \in \omega}$ for some $\varepsilon \colon \omega \to 2$, and conversely any such limit is a cut in $\mathcal{Z}$. We omit the details. □

Example 3.4. The various cases implicit in the proof just given do all occur.

(a) Let $M \vDash \mathrm{PA}$ and let Y be the Paris–Harrington indicator for initial segments satisfying PA. The corresponding neighbourhood system is $\mathcal{B} = \mathcal{B}^Y$, and $\mathcal{Z} = \mathcal{Z}(\mathcal{B})$ is the set of initial segments satisfying the Π_2 consequences of PA. Then $\mathcal{Z} \cong 2^\omega$ and there are proper cuts in $\mathcal{Z}$ arbitrarily high in M and also proper nonstandard cuts in $\mathcal{Z}$ arbitrarily low in M, as well as both end points, M and $\mathbb{N}$ in $\mathcal{Z}$.
(b) Let $M \vDash \mathrm{PA} + \neg\mathrm{Con(PA)}$ and let Y be an indicator for initial segments satisfying $\mathrm{PA} + \mathrm{Con(PA)}$, and $\mathcal{B} = \mathcal{B}^Y$. Then once again $\mathcal{Z} = \mathcal{Z}(\mathcal{B}) \cong 2^\omega$ but this time there is some $B \in M$ above all $I \in \mathcal{Z}$.
(c) Let $M \vDash \mathrm{PA}$ be short, that is $M = M(a)$ for some $a \in M$ or, in other words, M has no proper elementary initial segments containing a, and suppose M is short recursively saturated. Then there is a neighbourhood system $\mathcal{B}$ for the (closed) species $\mathcal{Z} = \{I \in \mathcal{C} : I \prec_{\mathrm{e}} M\}$ by Example 2.14.

The full model M itself is clearly in $\mathcal{Z}$, but $\mathcal{Z}$ does not have arbitrarily large proper cuts of M since if $a \in I \prec_{\mathrm{e}} M$ then $I = M$. So in this case $\mathcal{Z} = \mathcal{Z}(\mathcal{B}) \cong 2^{\omega} + 1$.

Proposition 3.3 makes a whole range of topological tools available to us. For example, we now know that $\mathcal{Z}$, as a topological space, is perfect, compact, totally disconnected, of cardinality $2^{\aleph_0}$, and homeomorphic to a complete metric space. In addition, the Baire Category Theorem applies. Recall a set is *comeagre* if it contains a countable intersection of open dense sets.

Baire Category Theorem. A comeagre subset in a complete metric space is dense in this space.

In particular, comeagre sets in a complete metric space X are nonempty. In fact, by extending the proof of Baire's theorem using a tree argument one can show that if the complete metric space X is separable and has no isolated points then every comeagre set has size the continuum. The intersection of countably many comeagre sets is comeagre, and in a space X, the set $X \setminus \{x\}$ is comeagre for any non-isolated point $x \in X$. Hence the complement of any countable set of non-isolated points is comeagre.

Dense subsets of a complete species are exactly those that are *indicated* in the sense of Kirby–Paris [5]. This is one point of interest in comeagre sets of cuts. Comeagre sets have many nice properties, including a useful game-theoretic characterisation.

Definition 3.5. Let $\mathcal{B}$ be a neighbourhood system and $\mathcal{Z} = \mathcal{Z}(\mathcal{B})$ the corresponding closed species. The *Banach–Mazur game on* $\mathcal{B}$ is the following game.

- There are two players, called $\forall$ and $\exists$.
- Starting with $\forall$, the two players alternatingly choose a $\mathcal{B}$-interval that is a subinterval of the previously chosen one.
- The game terminates in ω many steps.

A play of this game gives rise to a sequence $(\llbracket a_n, b_n \rrbracket)_{n \in \mathbb{N}}$. The initial segment of M, $\sup\{a_n : n \in \mathbb{N}\}$, is called the *outcome* of the play. The player $\exists$ can always play in such a way to ensure that this is a cut lying in $\mathcal{Z}$.

A property P of cuts is *enforceable* if and only if $\exists$ has a way to ensure the outcome of a play has property P. Similarly, a subset $\mathcal{P}$ of $\mathcal{Z}$ is *enforceable* if and only if the property of being an element of $\mathcal{P}$ is enforceable.

By 'dovetailing' several strategies together, it is easy to see that $\exists$ can play to enforce countably many properties P_i simultaneously, provided she can enforce each one individually. This observation is part of the proof of Banach's characterisation of comeagre sets.

Theorem 3.6 (Banach)**.** *A subset $\mathcal{P} \subseteq \mathcal{Z}$ is enforceable if and only if it is comeagre in $\mathcal{Z}$.*

From the point of view of Baire category, an enforceable property P of cuts in $\mathcal{Z}$ is satisfied by a large set of cuts $I \in \mathcal{Z}$. So a 'general' (i.e. not carefully chosen or exceptional) example of a cut I in $\mathcal{Z}$ would be expected to have many such enforceable properties. It cannot satisfy all of them (unless I is actually isolated in $\mathcal{Z}$) as $\mathcal{Z} \setminus \{I\}$ is comeagre. A *generic cut* I of $\mathcal{Z}$ is one that satisfies as many enforceable properties as is reasonably possible. Say that $\mathcal{P} \subseteq \mathcal{Z}$ is *invariant under automorphisms of* M if $\{I^g : I \in \mathcal{P}\} = \mathcal{P}$ for each $g \in \mathrm{Aut}(M)$.

Definition 3.7. Let $\mathcal{Z}$ be a closed species of cuts of M and $J \in \mathcal{Z}$. We say that J is *generic in* $\mathcal{Z}$ or *$\mathcal{Z}$-generic* if J is an element of each comeagre $\mathcal{P} \subseteq \mathcal{Z}$ invariant under automorphisms of M.

For a simple example when generic cuts might exist, suppose $M \models \mathrm{PA}$ is countable and $\mathcal{Z}$ is a closed species of cuts of M. Suppose there is some cut $J \in \mathcal{Z}$ such that the set

$$\{I \in \mathcal{Z} : I \text{ is conjugate to } J\}$$

is comeagre. Then the cut J is generic. To see this, let $\mathcal{P}$ be an invariant enforceable property and play the Banach–Mazur game to enforce $\mathcal{P}$ and the property of being conjugate to J simultaneously. The resulting cut has both these properties hence J has $\mathcal{P}$. The next result gives a more useful generalisation of this observation.

Theorem 3.8. *Let M be countable and $\mathcal{Z}$ a closed species of cuts which does not contain M as an isolated point. Suppose further that there is a set of cuts $\mathcal{G} \subseteq \mathcal{Z}$ such that*

(i) $\mathcal{G}$ is a dense subset of $\mathcal{Z}$ that is invariant under automorphisms of M; and
(ii) for all $I \in \mathcal{G}$ and all $c \in M$, there is an interval $[\![a, b]\!] \in \mathcal{B}(\mathcal{Z})$ containing I in which all cuts in $\mathcal{G}$ are conjugate over c.

Then $\mathcal{G}$ is a comeagre set of cuts in $\mathcal{Z}$ and the cuts in $\mathcal{G}$ are precisely the $\mathcal{Z}$-generic cuts.

Proof. We start by showing that the property of being a cut in $\mathcal{G}$ is enforceable. This will show that $\mathcal{G}$ contains all generic cuts. We play the Banach–Mazur game. At stage n in the game we will have chosen $c_0, c_1, \ldots, c_{n-1} \in M$, a descending sequence of intervals $[\![a_0, b_0]\!], [\![a_1, b_1]\!], \ldots, [\![a_{n-1}, b_{n-1}]\!]$ in $\mathcal{B}(\mathcal{Z})$, $I_0, I_1, \ldots, I_{n-1} \in \mathcal{G}$ so that $I_i \in [\![a_i, b_i]\!]$ and all $\mathcal{G}$-cuts in $[\![a_i, b_i]\!]$ are conjugate over $c_0, c_1, \ldots, c_i$ for each i, and also $g_1, g_2, \ldots, g_{n-1} \in \mathrm{Aut}(M)$ so that

$I_{i-1}^{g_i} = I_i$ and $g_i \in \mathrm{Aut}(M, c_0, \ldots, c_{i-1})$ for each i. The intervals $[\![a_n, b_n]\!]$ will be our plays in the game.

Given our opponent's move $[\![u, v]\!]$ in the game, we first choose $I_n \in \mathcal{G}$ with $I_n \in [\![u, v]\!]$ using the density of $\mathcal{G}$. If $n > 0$ we will also need to choose an automorphism $g_n \in \mathrm{Aut}(M, c_0, \ldots, c_n)$ such that $I_{n-1}^{g_n} = I_n$. This can be done using $I_{n-1} \in [\![a_{n-1}, b_{n-1}]\!] \supseteq [\![u, v]\!]$ and the previous choice of $[\![a_{n-1}, b_{n-1}]\!]$.

We next select some $c_n \in M$. If n is even, $n = 2k$ say, we select c_n to be the kth element x_k in some fixed enumeration $M = \{x_k : k \in \mathbb{N}\}$, to ensure that at the end of the construction $M = \{c_n : n \in \mathbb{N}\}$. If n is odd, $n = 2k + 1$ say, we choose $c_n = c_{2k+1} = x_k^{g_0 g_1 \cdots g_n}$ instead.

We now choose $[\![a_n, b_n]\!] \subseteq [\![u, v]\!]$ containing I_n such that all $\mathcal{G}$-cuts $[\![a_n, b_n]\!]$ are conjugate over $c_0, c_1, \ldots, c_n$. We play the interval $[\![a_n, b_n]\!]$ in the game.

The play continues in this fashion and constructs a cut $J \in \mathcal{Z}$ which is the limit of the intervals $[\![a_n, b_n]\!]$. We must show that $J \in \mathcal{G}$ and it suffices to show that I_0 and J are conjugate.

Observe that, since g_k fixes c_i for $k \geq i$, for each $x \in M$ there is some k such that $x = x_k$ and hence $c_{2k+1} = x^{g_0 g_1 \cdots g_{2k} g_{2k+1}}$ is fixed by g_{2k+2}, g_{2k+3}, etc. Therefore for each $x \in M$ there is $k \in \mathbb{N}$ such that

$$x^{g_1 g_2 \cdots g_k} = x^{g_1 g_2 \cdots g_k g_{k+1}} = \cdots = x^{g_1 g_2 \cdots g_k \cdots g_l}$$

for $l \geq k$. We define $g \colon x \mapsto x^g$ so that x^g is the eventual value $x^{g_1 g_2 \cdots g_l}$. It is easy to see that g preserves $\mathscr{L}_{\mathrm{A}}$-structure and is injective. It is onto since each $y \in M$ is $x_k = c_{2k}$ for some k so g maps $c_{2k}^{g_{2k}^{-1} \cdots g_1^{-1} g_0^{-1}}$ onto y. Finally g maps I_0 to J since by construction $g_0 g_1 \cdots g_n$ maps I_0 to some initial segment in $[\![a_n, b_n]\!]$ and the limit of these intervals is J. This completes the proof that $\mathcal{G}$ is enforceable and every generic cut is in $\mathcal{G}$.

To show that every $I \in \mathcal{G}$ is generic, let $\mathcal{P}$ be an enforceable $\mathrm{Aut}(M)$-invariant property and $[\![a, b]\!]$ is chosen so that $I \in [\![a, b]\!]$ and every $J \in [\![a, b]\!]$ in $\mathcal{G}$ is conjugate to I. Then we play the Banach–Mazur game starting with $[\![a, b]\!]$ enforcing $\mathcal{P}$ and $\mathcal{G}$ simultaneously to construct some $K \in \mathcal{G} \cap \mathcal{P}$ with $K \in [\![a, b]\!]$. Then I is conjugate to K and hence has $\mathcal{P}$, as required. $\square$

Question 3.9. Suppose M is a countable model of PA, $\mathcal{Z}$ is a closed species of cuts of M, and the set $\mathcal{G}$ of $\mathcal{Z}$-generic cuts is comeagre in $\mathcal{Z}$. Does it follow that conditions (i) and (ii) in the statement of Theorem 3.8 hold?

4 Examples of enforceable properties of cuts

In this section we make the global assumption that *our model* $M \models \mathrm{PA}$ *is countable and nonstandard, and our species of cuts* $\mathcal{Z}$ *is closed in* $\mathcal{C}$ *and order-isomorphic to* 2^ω. We let $\mathcal{B} = \mathcal{B}(\mathcal{Z})$ be the corresponding neighbourhood system. In other words, we assume that we are not in the exceptional case when $M \in \mathcal{Z}$ is isolated. (To apply the results under these assumptions when $\mathcal{Z} \cong 2^\omega + 1$ and $M \in \mathcal{Z}$ is isolated we can replace $\mathcal{Z}$ with $\mathcal{Z}_0 = \mathcal{Z} \setminus \{M\}$, which is also closed.) The object of this section is to extend the results of enforceability of various properties of cuts in GCMA to the current setting.

Proposition 4.1. *It is enforceable that a* $\mathcal{Z}$*-cut is not an* ω*-limit.*

Proof. By assumption, no $I \in \mathcal{Z}$ is isolated so $\mathcal{Z} \setminus \{I\}$ is comeagre. The proposition follows from the countability of M as there are countably many cuts which are ω-limits. □

Proposition 4.2. *It is enforceable that*

$$I \neq M_{\mathcal{B}}(a) \text{ and } I \neq M_{\mathcal{B}}[a] \text{ whenever } a \in M$$

for a $\mathcal{Z}$*-cut* I.

Proof. There are countably many cuts of the form $M_{\mathcal{B}}(a)$ or $M_{\mathcal{B}}[a]$. □

In a similar way one can see that it is enforceable that a cut is not definable over finitely many parameters from M in any reasonable logic, such as infinitary logic or second order logic, since there are only countably many conjugates of these parameters.

Proposition 4.3. *It is enforceable that a* $\mathcal{Z}$*-cut* I *has the property that* $\mathbb{N}$ *is* Π_2 *definable with parameters in* (M, I) *for a* $\mathcal{Z}$*-cut* I*. In particular we may force* I *so that* $\mathbb{N}$ *is defined by a formula of the form* $\forall x \in I\ \exists y \in I\ \theta(x, y, z, \bar{a})$ *where* $\theta(x, y, z, \bar{a})$ *is a* Δ_0 *formula of the language* $\mathscr{L}_{\mathrm{A}}$ *and* $\bar{a} \in M$ *are parameters.*

Proof. We play a Banach–Mazur game on $\mathcal{B}$. Suppose $\forall$ plays $[\![a, b]\!]$ in his first move, and without loss of generality we may assume b is finite. Let $Y \in M$ be a monotone indicator for $\mathcal{B}$ below $b+1$. We show that $\exists$ can force the outcome of the play I to satisfy

$$\{n \in M : M \models \forall x \in I\ \exists y \in I\ Y(x, y) \geqslant n\} = \mathbb{N}.$$

Note that since $I \in \mathcal{Z}$, it is clear that $\{n \in M : M \models \forall x \in I\ \exists y \in I\ Y(x, y) \geqslant n\} \supseteq \mathbb{N}$ for each outcome I. Let $n \in M$ be nonstandard, and suppose that $\exists$ is given $[\![u, v]\!] \subseteq [\![a, b]\!]$ to play in. Using Lemma 2.16, let $[\![x_n, y_n]\!] \subseteq [\![u, v]\!]$

such that $Y(x_n, y_n) < n$. Using the countability of M, player $\exists$ can do this for every nonstandard $n \in M$ in any single play. Now, if I is an outcome of this play and $n \in M$ is nonstandard, then we have $x_n \in I < y_n$ such that

$$Y(x_n, y) \leqslant Y(x_n, y_n) < n$$

for each $y \in I$ by the monotonicity of Y. This proves the claim. □

Remark. In the terminology of Kirby [4, Definition 4.5], the above proof shows that one can enforce the *index* of a cut corresponding to an indicator to be $\mathbb{N}$.

Corollary 4.4. *It is enforceable that a $\mathcal{Z}$-cut I has the property that (M, I) is not Π_2 recursively saturated.*

Proof. If not, apply Π_2 recursive saturation to the set of formulas $\{z > n : n \in \mathbb{N}\} \cup \{\forall x \in I\ \exists y \in I\ \theta(x, y, z, \bar{a})\}$ where $\theta(x, y, z, \bar{a})$ is from the last proposition. □

Question 4.5. How much saturation can we enforce in the structure (M, I) for a $\mathcal{Z}$-cut I? In particular, can Σ_2 recursive saturation be enforced?

Enforceability results related to the Kirby–Paris notions of semiregularity and regularity are proved in GCMA. A slight modification of the Grzegorczyk hierarchy as used there gives us the following.

Definition 4.6. The neighbourhood system $\mathcal{B}$ is said to be *relatively indestructible* if and only if for every $[\![a, b]\!] \in \mathcal{B}$, there is an element $c \in M$ such that

$$a = (c)_0 \ll (c)_1 \ll \cdots \ll (c)_{a+1} = b.$$

Using the same ideas it is straightforward to modify the combinatorial arguments given as Theorem 4.13 and Theorem 4.15 in GCMA to obtain the following results showing that semiregularity is the best one can hope for in the sense of the 'classical' Paris–Kirby hierarchy of combinatorial properties.

Proposition 4.7. *Semiregularity is enforceable if and only if $\mathcal{B}$ is relatively indestructible.*

Proposition 4.8. *The property of being not regular is enforceable.*

5 Pregenerics and the existence of generic cuts

Throughout this section, we work with a recursive enumeration $(\theta_i(x, y, z))_{i \in \mathbb{N}}$ of $\mathscr{L}_\mathrm{A}$ formulas in the free variables x, y, z. We fix a neighbourhood system

$\mathcal{B}$, and its corresponding closed species $\mathcal{Z} = \mathcal{Z}(\mathcal{B})$ and continue the global assumption of the last section that $\mathcal{Z}$ has no isolated point.

Our objective is to prove results showing the existence of generic cuts. Our motivation is Theorem 3.8 and the problem we address is to identify those intervals which are sufficiently homogeneous for many cuts in them to be conjugate. The existence of generic cuts relative to an indicator Y was shown in GCMA by a related 'self-similarity' property of intervals, that of being 'constant', together with a 'smallness' notion. We give the first of these definitions here.

Definition 5.1. Let $c \in M$. An interval $[\![a, b]\!] \in \mathcal{B}$ is *constant over c (with respect to $\mathcal{B}$)* if and only if

$$\forall x \in [\![a, b]\!]\ \forall [\![u, v]\!] \subseteq [\![a, b]\!]\ \exists x' \in [\![u, v]\!]\ \ \mathrm{tp}(x, c) = \mathrm{tp}(x', c).$$

We shall present a two-variable version of this self-similarity idea, which seems to give a more elegant approach. Intervals having this stronger self-similarity property will be called *pregeneric*, and it is clear that a pregeneric interval is constant in the sense of GCMA. (This notion of 'pregeneric' also implies 'smallness'.)

It will turn out that, by an argument similar to one in GCMA, pregeneric intervals exist in abundance in countable arithmetically saturated models of PA. We shall study this argument much more closely. This investigation will reveal that although arithmetic saturation is essential for the full argument, a large part of the proof goes through without any countability or saturation assumption. For applications to understanding truth in expanded structures of the form (M, I) we will be particularly interested in how the arguments can be adapted to notions of self-similarity with respect to finite sets of formulas. This increases the number of technical details but in other respects the main ideas are straightforward and similar to those in the earlier paper.

Definition 5.2. Let $x, y, x', y', c \in M$ and $n \in \mathbb{N}$. We write $(x, y, c) \equiv_n (x', y', c)$ to mean

$$\bigwedge_{i \leqslant n} (\theta_i(x, y, c) \leftrightarrow \theta_i(x', y', c)),$$

and write $(x, y, c) \equiv (x', y', c)$ to mean

$$\bigwedge_{i \in \mathbb{N}} (\theta_i(x, y, c) \leftrightarrow \theta_i(x', y', c)).$$

If M is recursively saturated, $(x, y, c) \equiv (x', y', c)$ is equivalent to the assertion that there is $g \in \mathrm{Aut}(M)$ such that $x^g = x'$, $y^g = y'$ and $c^g = c$.

Definition 5.3. Let $n, k, c \in M$, $[a, b] \in \mathcal{S}$ finite, and Y an indicator for $\mathcal{B}$ below $b+1$. We say that $[a, b]$ is $(n, k)_Y$*-pregeneric over* c if and only if $Y(a, b) \geqslant k$ and for all $x, y \in [a, b]$

$$\forall [u, v] \subseteq [a, b] \left(Y(u, v) \geqslant k \rightarrow \exists x', y' \in [u, v] \left((x, y, c) \equiv_n (x', y', c) \right) \right).$$

We shall omit the subscript Y if the indicator in consideration is clear from context.

To prove the existence of (n, k)-pregeneric intervals, we use the tree argument given in GCMA. The only difference here is that the tree is now finite.

Definition 5.4. Let $[a, b] \in \mathcal{S}$ be finite and Y be a monotone indicator for $\mathcal{B}$ below $b+1$. Fix $c \in M$. For $i \in \mathbb{N}$, define $e_i \colon M_{\leqslant b} \times M_{\leqslant b} \rightarrow M$ by setting $e_i(r, s)$ to be

$$\max \left\{ l \in M : \exists [r', s'] \subseteq [r, s] \left(Y(r', s') = l \wedge \forall x, y \in [r', s'] \, \neg\theta_i(x, y, c) \right) \right\}$$

for each $r, s \leqslant b$. The *tree of possibilities from* $[a, b]$ *over* c *with respect to* Y is a sequence $([r_\sigma, s_\sigma])_{\sigma \in 2^{<\omega}}$ of semi-intervals defined recursively as follows.

- Set $[r_\varnothing, s_\varnothing] = [a, b]$.
- Let $m \in \mathbb{N}$ and $\sigma \in 2^m$ such that $[r_\sigma, s_\sigma]$ is defined. Set $[r_{\sigma 0}, s_{\sigma 0}] = [r_\sigma, s_\sigma]$ and let $[r_{\sigma 1}, s_{\sigma 1}] \subseteq [r_\sigma, s_\sigma]$ be the unique semi-interval such that $r_{\sigma 1}$ is the least r in $[r_\sigma, s_\sigma]$ such that

  $$\exists s \in [r_\sigma, s_\sigma] \left(Y(r, s) \geqslant e_m(r_\sigma, s_\sigma) \wedge \forall x, y \in [r, s] \, \neg\theta_m(x, y, c) \right),$$

  and $s_{\sigma 1}$ is the greatest s in $[r_\sigma, s_\sigma]$ such that

  $$\forall x, y \in [r_{\sigma 1}, s] \, \neg\theta_m(x, y, c).$$

Remark. Note that the function e_i defined above is dependent on and uniquely determined by the choice of $c \in M$ and the indicator Y. Note also that both e_i and the tree of possibilities are uniformly definable in (M, Sat) for all partial inductive satisfaction class Sat for M. This is also true for $(n, k)_Y$-pregenericity over an element c of M.

The idea is that given a large enough finite semi-interval $[a, b]$ and a formula $\theta(x, y)$, exactly one of two things has to happen: either there is a large subinterval of $[a, b]$ in which no pair of elements satisfy $\theta(x, y)$, or there is not. In the first case, the witnessing subinterval is homogeneous for $\theta(x, y)$, simply because no pair of elements in there satisfies this formula. In the second case, the whole semi-interval is already homogeneous for $\theta(x, y)$, because by assumption, every large enough subinterval contains a pair of elements satisfying $\theta(x, y)$. In either case, we get a sufficiently large subinterval that is homogeneous for $\theta(x, y)$.

We can repeat this argument with all $\mathscr{L}_{\mathrm{A}}$ formulas. It is sometimes quite hard to find out which case we are in, but we definitely know what possibilities we can have. This gives rise to the tree of possibilities defined above. We do not need to know which way down the tree we have to go. We only need to know there is a way that works.

Lemma 5.5. *Let $[a, b]$ be a finite semi-interval, Y be a monotone indicator for $\mathcal{B}$ below $b+1$, and $c, k \in M$ such that $Y(a, b) \geqslant k$. If $([r_\sigma, s_\sigma])_{\sigma \in 2^{<\omega}}$ is the tree of possibilities from $[a, b]$ over c with respect to Y, then*

$$\forall m \in \mathbb{N}\ \exists! \sigma \in 2^m \Big(Y(r_\sigma, s_\sigma) \geqslant k \wedge \forall i < m\ (\sigma(i+1) = 0 \leftrightarrow e_i(r_{\sigma\restriction_i}, s_{\sigma\restriction_i}) < k)\Big).$$

Proof. This can be proved by an easy induction on m. □

It is then down to checking how many formulas we need to guarantee a certain amount of pregenericity.

Definition 5.6. Let $\beta\colon \mathbb{N} \to \mathbb{N}$ be the function defined by: for all $n \in \mathbb{N}$, the number $\beta(n)$ is the least $m \in \mathbb{N}$ such that

> if $\phi(x, y, z)$ is a Boolean combination of formulas in $\{\theta_i(x, y, z) : i \leqslant n\}$, then there is a formula $\phi'(x, y, z) \in \{\theta_i(x, y, z) : i \leqslant m\}$ that is logically equivalent to $\phi(x, y, z)$.

Theorem 5.7. *Let n be a natural number, $[a, b]$ be a finite semi-interval, $k, c \in M$, and Y a monotone indicator for $\mathcal{B}$ below $b+1$ such that $Y(a, b) \geqslant k$. Then $[a, b]$ contains a semi-interval that is $(n, k)_Y$-pregeneric over c. Moreover, if* Sat *is a partial inductive satisfaction class for M, then one such semi-interval is definable in (M, Sat) uniformly in the parameters a, b, c, Y, n, k.*

Proof. Let $[a, b]$ be a finite semi-interval, $k, c \in M$, and Y be a monotone indicator for $\mathcal{B}$ below $b + 1$ such that $Y(a, b) \geqslant k$. Let $([r_\sigma, s_\sigma])_{\sigma \in 2^{<\omega}}$ be the tree of possibilities from $[a, b]$ over c with respect to Y. Using Lemma 5.5, define the function $\pi\colon \mathbb{N} \to 2^{<\omega}$ by setting $\pi(m)$ to be the unique $\sigma \in 2^m$ such that

$$Y(r_\sigma, s_\sigma) \geqslant k \wedge \forall i < m\ (\sigma(i + 1) = 0 \leftrightarrow e_i(r_{\sigma\restriction_i}, s_{\sigma\restriction_i}) < k)$$

for each $m \in \mathbb{N}$. It can then be checked that $[r_{\pi(\beta(n))}, s_{\pi(\beta(n))}] \subseteq [a, b]$ is $(n, k)_Y$-pregeneric over c for every $n \in \mathbb{N}$.

The 'moreover' part can be proved by a careful check of all the steps, and is left to the reader. □

By noting that almost everything in the above argument is coded in M, one can prove the same statement with fully pregeneric intervals in a similar way.

Definition 5.8. Let $c \in M$. An interval $[\![a, b]\!] \in \mathcal{B}$ is *pregeneric over c with* *respect to $\mathcal{B}$* if and only if

$$\forall x, y \in [\![a, b]\!]\ \forall [\![u, v]\!] \subseteq [\![a, b]\!]\ \exists x', y' \in [\![u, v]\!]\ (x, y, c) \equiv (x', y', c).$$

We say that a $\mathcal{B}$-interval is *pregeneric with respect to $\mathcal{B}$* if and only if it is pregeneric over 0.

Theorem 5.9. *Suppose M is arithmetically saturated. Let $c \in M$. Then every $\mathcal{B}$-interval contains a subinterval pregeneric over c.*

Proof. Suppose M is arithmetically saturated. Let $c \in M$ and $[\![a, b]\!] \in \mathcal{B}$. Without loss of generality, assume $b \neq \infty$. Fix a monotone indicator Y for $\mathcal{B}$ below $b+1$, and let $([r_\sigma, s_\sigma])_{\sigma \in 2^{<\omega}}$ be the tree of possibilities from $[\![a, b]\!]$ over c with respect to Y. By recursive saturation, this tree of possibilities and thus $(Y(r_\sigma, s_\sigma))_{\sigma \in 2^{<\omega}}$ are coded in M. Using the strength of $\mathbb{N}$ in M, let $d > \mathbb{N}$ such that

$$\forall \sigma \in 2^{<\omega}\ (Y(r_\sigma, s_\sigma) > d \Leftrightarrow Y(r_\sigma, s_\sigma) > \mathbb{N}).$$

In particular, $Y(a, b) > d$ since $[\![a, b]\!] \in \mathcal{B}$. By Lemma 5.5, we have

$$\forall m \in \mathbb{N}\ \exists! \sigma \in 2^m\ \Big(Y(r_\sigma, s_\sigma) > d \wedge \forall i < m\ (\sigma(i+1) = 0 \leftrightarrow e_i(r_{\sigma\restriction_i}, s_{\sigma\restriction_i}) \leqslant d)\Big).$$

Using recursive saturation of M, let $n > \mathbb{N}$ and $\sigma \in 2^n$ such that

$$\begin{aligned} Y(r_\sigma, s_\sigma) > d \wedge \forall i < n\ \Big(\sigma(i+1) = 0 \leftrightarrow e_i(r_{\sigma\restriction_i}, s_{\sigma\restriction_i}) \leqslant d\Big) \\ \wedge\ \forall i < n\ \Big([r_{\sigma\restriction_i}, s_{\sigma\restriction_i}] \supseteq [r_{\sigma\restriction_{i+1}}, s_{\sigma\restriction_{i+1}}]\Big). \end{aligned}$$

It can then be checked that $[\![r_\sigma, s_\sigma]\!] \subseteq [\![a, b]\!]$ is pregeneric over c. □

One can try to strengthen the definition of pregeneric intervals to one involving tuples of length more than two. However this does not give us anything much stronger, at least when the model is recursively saturated.

Proposition 5.10. *Suppose M is recursively saturated, and let $c \in M$. Then an interval $[\![a, b]\!] \in \mathcal{B}$ is pregeneric over c if and only if*

$$\forall \bar{x} \in [\![a, b]\!]\ \forall [\![u, v]\!] \subseteq [\![a, b]\!]\ \exists \bar{x}' \in [\![u, v]\!]\ (\bar{x}, c) \equiv (\bar{x}', c).$$

Proof. One direction is obvious. For the other, note that if $g \in \mathrm{Aut}(M, c)$ maps $\min\{\bar{x}\}$ and $\max\{\bar{x}\}$ into $[\![u, v]\!]$, then it must also map all other elements in $\bar{x}$ into $[\![u, v]\!]$. □

Remark. The above argument also shows that modulo recursive saturation, pregenericity of a $\mathcal{B}$-interval $[\![a, b]\!]$ over an element c in M is equivalent to

$$\forall [\![u, v]\!] \subseteq [\![a, b]\!]\ \exists a', b' \in [\![u, v]\!]\ (a, b, c) \equiv (a', b', c).$$

Another way to strengthen the notion of pregenericity is to require an interval to be pregeneric over all elements in a cut I. In some very particular cases, this works.

Proposition 5.11. *Suppose M is recursively saturated and $\mathcal{B} = \mathcal{B}^{\text{elem}}$. Let $c, c' \in M$ and $[\![a, b]\!]$ be an elementary interval such that $c, c' \ll a$ and* $\text{tp}(c) = \text{tp}(c')$. *Then*

$$\forall [\![u, v]\!] \subseteq [\![a, b]\!]\ \exists a', b' \in [\![u, v]\!]\ (a, b, c) \equiv (a', b', c').$$

In particular, if $c = c'$, then $[\![a, b]\!]$ is pregeneric over c.

Proof. Suppose M is recursively saturated and $\mathcal{B} = \mathcal{B}^{\text{elem}}$. Let $[\![a, b]\!]$ be a finite elementary interval, $c, c' \ll a$ and $[\![u, v]\!] \subseteq [\![a, b]\!]$.

First, we find $a' > u$ with $(a, c) \equiv (a', c')$ and $a' \ll v$. Consider the recursive type

$$\begin{aligned} p(x) = \{&\phi(x, c') \leftrightarrow \phi(a, c) : \phi(x, y) \in \mathscr{L}_{\text{A}}\} \\ &\cup \{t_n(x) < v : n \in \mathbb{N}\} \cup \{u < x\}. \end{aligned}$$

Take $n \in \mathbb{N}$ and $\phi(x, y) \in \mathscr{L}_{\text{A}}$ such that $M \models \phi(a, c)$. Pick an elementary cut I in $[\![u, v]\!]$. Since $c \ll a$, we see that $M \models \mathsf{Q}x\ \phi(x, c)$ where Q denotes 'there are cofinally many'. Our hypothesis on c and c' then implies that $M \models \mathsf{Q}x\ \phi(x, c')$. By elementarity of I in M, we have $M \models \mathsf{Q}x \in I\ \phi(x, c')$. In particular, $M \models \exists x > u\ (t_n(x) < v \wedge \phi(x, c'))$. So $p(x)$ is finitely satisfied in M. Using recursive saturation, let $a' \in M$ realise $p(x)$, so that

$$(a, c) \equiv (a', c') \text{ and } u < a' \ll v. \tag{$*$}$$

Next, consider the recursive type

$$q(y) = \{\theta(a, b, c) \leftrightarrow \theta(a', y, c') : \theta(x, y, z) \in \mathscr{L}_{\text{A}}\} \cup \{y < v\}.$$

Let $\theta(x, y, z) \in \mathscr{L}_{\text{A}}$ such that $M \models \theta(a, b, c)$. We need to show $M \models \exists y < v\ \theta(a', y, c')$. Now, we know that $M \models \exists y\, \theta(a, y, c)$ and so $M \models \exists y\, \theta(a', y, c')$ by $(*)$. Thus

$$(\mu y)(\theta(a', y, c')) \in \text{cl}(a', c') \subseteq M(\langle a', c' \rangle) < v,$$

proving that $q(y)$ is finitely satisfied in M. Using recursive saturation again, let b' realise $q(y)$ in M. Then

$$(a, b, c) \equiv (a', b', c') \text{ and } u < a' < b' < v,$$

as required. $\square$

However, in most other cases, this does not work.

Proposition 5.12. *For every $B > \mathbb{N}$, there exists cofinally many $Y \in M$ such that, for every $\mathcal{B}$-interval $[\![a,b]\!] \subseteq M_{<B}$ and every $d > \mathbb{N}$, there exists a nonstandard $c < d$ with $[\![a,b]\!]$ not pregeneric over $\langle c, Y\rangle$.*

Proof. Let $B > \mathbb{N}$. Using Proposition 2.8, let $Y \in M$ be a monotone indicator for $\mathcal{B}$ below B. Note that by requiring Y to be an indicator below a sufficiently large number, one can make the code Y arbitrarily large.

Let $[\![a,b]\!] \subseteq M_{<B}$ be a $\mathcal{B}$-interval and $d > \mathbb{N}$. Without loss of generality, suppose $Y(a,b) > d$. Using Lemma 2.16, pick $[\![u,v]\!] \subseteq [\![a,b]\!]$ such that $\mathbb{N} < Y(u,v) < d$. Let $c = Y(u,v)$. Then for all $[a',b'] \subseteq [\![u,v]\!]$, we have

$$Y(a',b') \leqslant Y(u,v) = c$$

by monotonicity of Y, and

$$Y(a,b) > d > Y(u,v) = c.$$

Hence $(a,b,\langle c,Y\rangle) \not\equiv (a',b',\langle c,Y\rangle)$ for every $[a',b'] \subseteq [\![u,v]\!]$. Therefore, $[\![a,b]\!]$ is not pregeneric over $\langle c,Y\rangle$. □

These show that pregenericity is stable and optimal. More evidence of this comes from its relationship with arithmetic saturation.

Proposition 5.13. *If for every $f \in M$, there are $B \in M$ and an indicator Y for $\mathcal{B}$ below B such that a pregeneric interval over $\langle f, Y\rangle$ exists in $M_{<B}$, then $\mathbb{N}$ is strong in M.*

Proof. Suppose the hypothesis in the proposition holds. Let $f\colon \mathbb{N} \to M$ be a coded function in M. Abusing notation, we let f be a code for this function in M. Using the hypothesis, let $B \in M$ and Y be an indicator for $\mathcal{B}$ below B, and pick a $\mathcal{B}$-interval $[\![a,b]\!] \subseteq M_{<B}$ that is pregeneric over $\langle f,Y\rangle$. Note that by the proof of Proposition 2.8, we may assume Y to be monotone.

We claim that $f(n) > \mathbb{N}$ if and only if $f(n) > Y(a,b)$ for all $n \in \mathbb{N}$. Note that since $[\![a,b]\!] \in \mathcal{B}$, the 'if part' is obvious. So let $n \in \mathbb{N}$ such that $f(n) > \mathbb{N}$. Using Lemma 2.16, let $[\![u,v]\!] \subseteq [\![a,b]\!]$ such that $\mathbb{N} < Y(u,v) < f(n)$. Recalling that $[\![a,b]\!]$ is pregeneric over $\langle f,Y\rangle$, let $a',b' \in [\![u,v]\!]$ such that

$$(a,b,\langle f,Y\rangle) \equiv (a',b',\langle f,Y\rangle). \qquad (\dagger)$$

By monotonicity of Y, we have $Y(a',b') \leqslant Y(u,v) < f(n)$. Thus by (†), we get $Y(a,b) < f(n)$ as required. □

While pregeneric intervals are interesting in their own right, the original reason for their introduction is to construct generic cuts. In doing this we shall prove the following characterisation of generic cuts in countable arithmetically saturated models.

Theorem 5.14. *Let M be countable and arithmetically saturated. A cut I is generic if and only if it is contained in a pregeneric $\mathcal{B}$-interval over c for every $c \in M$.*

The proof of this will emerge in the course of following discussion. Let us say that a cut $I \in \mathcal{C}$ is *strongly generic for $\mathcal{B}$* if and only if it is contained in a pregeneric $\mathcal{B}$-interval over c for every $c \in M$. It is easy to check that all such I are in $\mathcal{Z}(\mathcal{B})$. For if I is strongly generic and $a, b \in M$ are such that $a \in I < b$ then there is $[\![u, v]\!]$ pregeneric over $\langle a, b\rangle$ containing I. Then $a, b \notin [\![u, v]\!]$, so $[a, b] \supseteq [\![u, v]\!]$ is a $\mathcal{B}$-interval by axiom (4) for a neighbourhood system.

It is now straightforward to show that strongly generic cuts exist using the Banach's characterisation of comeagre sets.

Theorem 5.15. *If M is countable and arithmetically saturated, then strong genericity is an enforceable property of $\mathcal{Z}$-cuts.*

Proof. Let M be countable and arithmetically saturated. We play a Banach–Mazur game on $\mathcal{B}$. If $c \in M$, then $\exists$ can make the outcome of a play be contained in a pregeneric interval over c using Theorem 5.9 in a single step. Since M is countable and $\exists$ has ω many steps to play, she can actually ensure that the outcome is contained in a pregeneric interval over c for every $c \in M$. In other words, strong genericity is enforceable. □

Corollary 5.16. *If M is countable and arithmetically saturated then strongly generic cuts I for $\mathcal{B}$ exist. Furthermore every generic cut I for $\mathcal{B}$ is strongly generic.*

In fact, a direct consequence of Proposition 5.13 and the definition of strong genericity is that the strength of $\mathbb{N}$ in the hypothesis of the above theorem is necessary.

Corollary 5.17. *If M contains a strongly generic cut for $\mathcal{B}$ then $\mathbb{N}$ is strong in M.*

The other implication, that a strongly generic cut is generic will follow from looking at conjugacy properties of strongly generic cuts.

Theorem 5.18. *Let M be countable and arithmetically saturated. Let $c \in M$ and $[\![a, b]\!] \in \mathcal{B}$ be a pregeneric interval over c. Then any two strongly generic cuts contained in $[\![a, b]\!]$ are conjugate over c.*

Proof. We use a back-and-forth argument.

Let $c \in M$ and $[\![a, b]\!] \in \mathcal{B}$ be a pregeneric interval over c. Pick two strongly generic cuts I and I' in $[\![a, b]\!]$. At any stage of the back-and-forth, we have

- an interval $[\![u, v]\!]$ containing I,
- an interval $[\![u', v']\!]$ containing I', and
- tuples $\bar{r}, \bar{r}' \in M$

such that

- $[\![u, v]\!]$ is pregeneric over $\langle c, \bar{r} \rangle$,
- $[\![u', v']\!]$ is pregeneric over $\langle c, \bar{r}' \rangle$, and
- $(u, v, c, \bar{r}) \equiv (u', v', c, \bar{r}')$.

We show how to add an arbitrary ${}^*r$ to $\bar{r}$. In the process, we find ${}^*u, {}^*v$ to replace u, v and choose corresponding ${}^*u', {}^*v', {}^*r'$ while keeping $\bar{r}'$ fixed. This constitutes the 'forth' step. The 'back' step is similar.

Using the definition of 'strongly generic', choose an interval $[\![{}^*u, {}^*v]\!]$ that contains I and is pregeneric over $\langle u, v, c, \bar{r}, {}^*r \rangle$. Pick an automorphism $g \in \mathrm{Aut}(M, c)$ such that $\langle u, v, \bar{r} \rangle^g = \langle u', v', \bar{r}' \rangle$, which is possible since $(u, v, c, \bar{r}) \equiv (u', v', c, \bar{r}')$ and M is recursively saturated. It follows that $[\![{}^*u^g, {}^*v^g]\!] \subseteq [\![u', v']\!]$. Using pregenericity of $[\![u', v']\!]$ and recursive saturation, let $h \in \mathrm{Aut}(M, c, \bar{r}')$ such that $[\![u', v']\!]^h \subseteq [\![{}^*u^g, {}^*v^g]\!]$. The back-and-forth then continues by setting

$$[\![{}^*u', {}^*v']\!] = [\![{}^*u^{gh^{-1}}, {}^*v^{gh^{-1}}]\!] \text{ and } {}^*r' = {}^*rgh^{-1}.$$

The required isomorphism is given by $\bar{r} \mapsto \bar{r}'$ at the end. □

Corollary 5.19. *If M be countable and arithmetically saturated, then every strongly generic cut is generic.*

Proof. Use Theorem 3.8. □

6 Conjugacy properties and truth

We continue working with a fixed neighbourhood system $\mathcal{B}$ and its species of cuts $\mathcal{Z} = \mathcal{Z}(\mathcal{B})$ which will be assumed not to have any isolated point. Additionally, in this section we assume that our model M is countable and arithmetically saturated.

Results in the last section show that, in this context, the set $\mathcal{G}$ of $\mathcal{Z}$-generic cuts is comeagre in $\mathcal{Z}$ and satisfies the hypotheses of Theorem 3.8. The neighbourhood of a generic cut is fuzzy or blurred in some sense, and this agrees with our idea that pregeneric intervals should be homogeneous. In fact, Theorem 3.8 says that this blurry nature actually characterises genericity. It is natural to ask exactly how large the blurry zone around a generic cut is. The following shows that one can improve Theorem 5.18 slightly.

Corollary 6.1. *If* $[\![a,b]\!]$ *is an interval satisfying*

$$\exists x \in [\![a,b]\!]\ \forall [\![u,v]\!] \subseteq [\![a,b]\!]\ \exists x' \in [\![u,v]\!]\ (x,c) \equiv (x',c),$$

then all generic cuts in $[\![a,b]\!]$ *are conjugate over* c.

Proof. Let $[\![a,b]\!]$ be an interval and $x, c \in M$ such that

$$\forall [\![u,v]\!] \subseteq [\![a,b]\!]\ \exists x' \in [\![u,v]\!]\ (x,c) \equiv (x',c). \tag{‡}$$

Pick two generic cuts I_1 and I_2 from $[\![a,b]\!]$. Using Corollary 5.16, let $[\![u_1,v_1]\!]$ and $[\![u_2,v_2]\!]$ be pregeneric intervals over $\langle a,b,c\rangle$ that contain I_1 and I_2 respectively. Note that $[\![u_1,v_1]\!]$ and $[\![u_2,v_2]\!]$ have to be subintervals of $[\![a,b]\!]$.

Our plan is to map I_1 close enough to I_2 via x, so that Theorem 5.18 can be applied. Using the axioms for a neighbourhood system, let $[\![u_2',v_2']\!]$ be a pregeneric subinterval of $[\![u_2,v_2]\!]$ over c containing I_2 such that

$$u_2 \ll u_2' \ll v_2' \ll v_2. \tag{§}$$

Using (‡) and recursive saturation, let $g_1, g_2 \in \mathrm{Aut}(M,c)$ such that $x^{g_1} \in [\![u_1,v_1]\!]$ and $x^{g_2} \in [\![u_2',v_2']\!]$. It follows from (§) that $[\![u_1,v_1]\!]^{g_1^{-1}g_2} \cap [\![u_2,v_2]\!] \in \mathcal{B}$. By Theorem 5.18, both $I_1^{g_1^{-1}g_2}$ and I_2 are conjugate over c to the generic cuts in this intersection. Therefore, $(M,I_1,c) \cong (M,I_2,c)$. □

This turns out to be the best possible.

Proposition 6.2. *Let* $[\![a,b]\!]$ *be a* $\mathcal{B}$*-interval,* $\mathcal{D} \subseteq \mathcal{Z}$ *and* $c \in M$ *such that* $\mathcal{D}$ *is dense in* $[\![a,b]\!]$*. If all* $\mathcal{D}$*-cuts in* $[\![a,b]\!]$ *are conjugate over* c*, then*

$$\exists x \in [\![a,b]\!]\ \forall [\![u,v]\!] \subseteq [\![a,b]\!]\ \exists x' \in [\![u,v]\!]\ (x,c) \equiv (x',c).$$

Proof. Let $[\![a,b]\!] \in \mathcal{B}$ and $\mathcal{D} \subseteq \mathcal{Z}$ such that $\mathcal{D}$ is dense in $[\![a,b]\!]$. Fix $c \in M$, and suppose all $\mathcal{D}$-cuts in $[\![a,b]\!]$ are conjugate over c. Using Theorem 5.9, let $[\![r,s]\!] \subseteq [\![a,b]\!]$ be a pregeneric interval of c, and pick $x \in [\![r,s]\!]$. We show that this x works.

Let $[\![u,v]\!] \subseteq [\![a,b]\!]$ be arbitrary. We apply a similar trick as in the previous proof again. Using the axioms for a neighbourhood system, let $[\![u',v']\!]$ be a subinterval of $[\![u,v]\!]$ such that

$$u \ll u' \ll v' \ll v.$$

Using the density of $\mathcal{D}$ in $[\![a,b]\!]$, take $\mathcal{D}$-cuts $I \in [\![r,s]\!]$ and $J \in [\![u',v']\!]$. By assumption, I is conjugate to J over c. Let $h \in \mathrm{Aut}(M,c)$ such that $I^h = J$. Then $[\![r,s]\!]^h \cap [\![u,v]\!]$ is an interval whose preimage under h is a subinterval of $[\![r,s]\!]$. Let $[\![r',s']\!]$ be this preimage. Recall that $[\![r,s]\!]$ is a pregeneric interval over c. So there exists an automorphism $g \in \mathrm{Aut}(M,c)$ such that $x^g \in [\![r',s']\!]$ and hence $x^{gh} \in [\![u,v]\!]$, as required. □

We now start to prove some new results that have no counterparts in GCMA. The main theorem is a syntactic characterisation of conjugacy for generic cuts. As a corollary, we obtain a description of the orbits of M under the action of $\mathrm{Aut}(M,I)$ where I is a generic cut.

Definition 6.3. Let $\mathscr{L}_{\mathrm{A}}^I$ denote the language obtained from $\mathscr{L}_{\mathrm{A}}$ by adding an extra unary relation symbol, which will usually represent a cut of M. The language obtained from $\mathscr{L}_{\mathrm{A}}^I$ by adding all $\mathscr{L}_{\mathrm{A}}$ Skolem functions is denoted by $\mathscr{L}_{\mathrm{Sk}}^I$.

Definition 6.4. Let $I \in \mathcal{C}$ and $\bar{c}, \bar{c}' \in M$. We write $(\bar{c}, I) \equiv (\bar{c}', I)$ to mean that $\bar{c}$ and $\bar{c}'$ are of the same length, and

$$(M,I) \vDash \varphi(\bar{c}) \leftrightarrow \varphi(\bar{c}')$$

for all $\mathscr{L}_{\mathrm{A}}^I$ formulas $\varphi(\bar{x})$.

Our first objective is to count the number of conjugacy classes of generic cuts. It will turn out that in some cases there will be exactly $\aleph_0$ conjugacy classes, and in other cases just one. We have already proved results showing that under certain conditions two generic cuts are conjugate. To characterise conjugacy, we additionally need to know when two generic cuts are not conjugate. It is obvious that if two cuts are separated by a definable point, then they cannot be conjugate, and this observation gives us one set of examples.

Example 6.5. Let $\mathcal{D}$ be a dense set in $\mathcal{Z}$ that is invariant under the action of $\mathrm{Aut}(M)$, and suppose $\mathcal{B} = \mathcal{B}^Y$ for some GCMA indicator Y. If $M \nvDash \mathrm{Th}(\mathbb{N})$, then there are at least countably infinitely many conjugacy classes of $\mathcal{D}$-cuts that are contained in $\overline{\mathrm{cl}}(\varnothing)$, the smallest elementary cut of M.

Proof. Let $\mathcal{D}$, Y and $\mathcal{B} = \mathcal{B}^Y$ be as in the statement and $M \nvDash \mathrm{Th}(\mathbb{N})$. By the closure of $\mathcal{Z}$, $M_{\mathcal{B}}(0)$ exists and is in $\mathcal{Z}$. Note that

$$M_{\mathcal{B}}(0) = \sup\{(\mu y)(Y(0,y) \geqslant n) : n \in \mathbb{N}\} \subsetneq_{\mathrm{e}} \overline{\mathrm{cl}}(\varnothing).$$

Take $a \in \mathrm{cl}(\varnothing)$ such that $a > M_{\mathcal{B}}(0)$. Then $[\![0,a]\!] \in \mathcal{B}$ by the definition of $M_{\mathcal{B}}(0)$. Using an argument similar to that in the proof of Proposition 2.3 one can divide the $\mathcal{B}$-interval $[\![0,a]\!]$ indefinitely into smaller subintervals by definable points. Since $\mathcal{D}$ is dense in $\mathcal{Z}$, we get any finite number of mutually non-conjugate $\mathcal{D}$-cuts in $\overline{\mathrm{cl}}(\varnothing)$. □

When $M \vDash \mathrm{Th}(\mathbb{N})$, this trick does not work because there is no nonstandard definable point. Instead we may make use of a function H that grows like an ascending sequence of gaps. The cuts in consecutive gaps cannot be conjugate because the maximum w such that $H(w)$ is in the cut are all in different congruence classes modulo a sufficiently large natural number. The following technical lemma allows this to work.

Lemma 6.6. *Let Y be a GCMA indicator. If $M \vDash \forall x \exists y\, Y(x, y) \geqslant n$ for each $n \in \mathbb{N}$, then there is a strictly increasing function $H\colon M \to M$ definable in M without parameters such that*

$$H(k) \ll_{\mathcal{B}^Y} H(k+1)$$

for all large enough $k \in M$.

Proof. Let Y be a GCMA indicator. Suppose $M \vDash \forall x \exists y\, Y(x, y) \geqslant n$ for each $n \in \mathbb{N}$.

If $M \vDash \forall n \forall x \exists y\, Y(x, y) \geqslant n$, then let H be the function defined recursively by

$$H(0) = 0 \wedge \forall z \Big(H(z+1) = (\mu y)(Y(H(z), y) \geqslant z+1)\Big).$$

If $M \vDash \exists n \exists x \forall y\, Y(x, y) < n$, then define H by

$$H(0) = 0 \wedge \forall z \Big(H(z+1) = (\mu y)(Y(H(z), y) \geqslant n)\Big),$$

where $n = (\max m)(\forall x \exists y\, Y(x, y) \geqslant m)$. □

Proposition 6.7. *Let Y be a GCMA indicator such that $\mathcal{B} = \mathcal{B}^Y$, and $\mathcal{D}$ be a dense set of $\mathcal{Z}$-cuts that is closed under the action of $\mathrm{Aut}(M)$.*

(a) If $M \nvDash \forall x \exists y\, Y(x, y) \geqslant n$ for some $n \in \mathbb{N}$, then no $\mathcal{Z}$-cut can contain $\overline{\mathrm{cl}}(\varnothing)$.
(b) If $M \vDash \forall x \exists y\, Y(x, y) \geqslant n$ for all $n \in \mathbb{N}$, then there are at least countably infinitely many mutually non-conjugate $\mathcal{D}$-cuts containing $\overline{\mathrm{cl}}(\varnothing)$.

Proof. Let Y be a GCMA indicator such that $\mathcal{B} = \mathcal{B}^Y$, and $\mathcal{D}$ be a dense set of $\mathcal{Z}$-cuts that is closed under the action of $\mathrm{Aut}(M)$.

(a) Take $n \in \mathbb{N}$ such that $M \vDash \exists x \forall y\, Y(x, y) < n$. Let $x^* = (\mu x)(\forall y\, Y(x, y) < n)$. Then $x^* \in \mathrm{cl}(\varnothing)$ and no $\mathcal{B}$-interval is above x^* because $n \in \mathbb{N}$. So, there cannot be any $\mathcal{Z}$-cut above $\overline{\mathrm{cl}}(\varnothing)$.
(b) Suppose $M \vDash \forall x \exists y\, Y(x, y) \geqslant n$ for each $n \in \mathbb{N}$. Let H be a fast growing function whose existence is guaranteed by Lemma 6.6. Pick $x > \overline{\mathrm{cl}}(\varnothing)$ such that $\Big(\llbracket H(x+k), H(x+k+1)\rrbracket\Big)_{k \in \mathbb{N}}$ is a sequence of $\mathcal{B}$-intervals,

which is possible by recursive saturation. Using the density of $\mathcal{D}$ in $\mathcal{Z}$, take a $\mathcal{D}$-cut $I_k \in [\![H(x+k), H(x+k+1)]\!]$ for each $k \in \mathbb{N}$. Noting that

$$(\max w)(H(w) \in I_k) = x + k$$

for each $k \in \mathbb{N}$, it can easily be verified that the cuts in $(I_k)_{k\in\mathbb{N}}$ are mutually non-conjugate.

□

Corollary 6.8. *If $\mathcal{B} = \mathcal{B}^Y$ for some GCMA indicator Y, then there are exactly countably infinitely many conjugacy classes of generic cuts in M.*

Proof. Let Y be a GCMA indicator such that $\mathcal{B} = \mathcal{B}^Y$.

Recall that Theorem 5.18 says that if two generic cuts are in the same pregeneric interval, then they are conjugate. By the countability of M, this implies that there can be at most countably infinitely many conjugacy classes of generic cuts in M.

On the other hand, note that it is not possible to have $M \models \mathrm{Th}(\mathbb{N})$ and

$$M \models \exists x \forall y\, Y(x, y) < n \text{ for some } n \in \mathbb{N}$$

both true at the same time. Otherwise, the truth of $\exists x \exists y\, Y(x, y) \geqslant n$ in M for every $n \in \mathbb{N}$ then implies the existence of a nonstandard definable element. Therefore we are done by Example 6.5, Proposition 6.7, Theorem 5.15, and the Baire Category Theorem. □

Remark. Note that there is exactly one conjugacy class of generic cuts for $\mathcal{B}^{\mathrm{elem}}$ by Theorem 5.18 and Proposition 5.11.

All the above non-conjugacy claims are actually proved by cooking up a sentence that is true in one structure but not the other. One may ask whether we are able to find non-conjugate cuts that are elementary equivalent in the expanded language. The following suggests that this may not be possible.

Example 6.9. Suppose $\mathcal{B} = \mathcal{B}^{\mathrm{elem}}$, and let I be a generic cut for $\mathcal{B}^{\mathrm{elem}}$. If $a, b \in I$ such that $\mathrm{tp}(a) = \mathrm{tp}(b)$, then $(M, I, a) \cong (M, I, b)$.

Proof. Suppose $\mathcal{B} = \mathcal{B}^{\mathrm{elem}}$ and let $a, b \in I \prec_{\mathrm{e}} M$ such that I is generic and $\mathrm{tp}(a) = \mathrm{tp}(b)$. Using Corollary 5.16, let $[\![r, s]\!]$ be a pregeneric interval over $\langle a, b\rangle$ that contains I. Then we necessarily have $a, b \ll r$.

Using Proposition 5.11 and recursive saturation, let $g \in \mathrm{Aut}(M)$ such that

$$a = b^g \text{ and } [\![r, s]\!]^g \subseteq [\![r, s]\!].$$

Let $J = I^g$. Then
$$J = I^g \in [\![r,s]\!]^g \subseteq [\![r,s]\!]$$
so that both I and J are generic cuts in $[\![r,s]\!]$. However, $[\![r,s]\!]$ is pregeneric over a by Proposition 5.11. So by Theorem 5.18, there is an automorphism $h \in \mathrm{Aut}(M,a)$ such that $J^h = I$ and thus

$$(M,I,b) \cong (M,I^g,b^g) = (M,J,a) \cong (M,J^h,a^h) = (M,I,a),$$

as required. □

This essentially says that the formula '$x \in I$' of $\mathscr{L}_{\mathrm{A}}^I$ tells us a lot about an element x when I is generic for $\mathcal{B}^{\mathrm{elem}}$. On the other hand, the formula '$x \notin I$' is much weaker.

Proposition 6.10. *Suppose that all $\mathcal{Z}$-cuts are closed under addition and multiplication. If I is a generic cut, $c \in M$ and $B > I$, then there are $d, d' \in M$ such that $I < d, d' < B$ and $(d,c) \equiv (d',c)$, but $(d,c,I) \not\equiv (d',c,I)$.*

Proof. Under the hypotheses of the proposition, using Corollary 5.16, let $[\![a,b]\!] \in \mathcal{B}$ be a pregeneric interval over $\langle c, B\rangle$ containing I.

By Proposition 4.2 and Corollary 5.19, $I \neq M_{\mathcal{B}}[b]$, so $I < M_{\mathcal{B}}[b]$. Let $w \in M_{\mathcal{B}}[b] \setminus I$. By Proposition 2.3, $M_{\mathcal{B}}(w) \neq M_{\mathcal{B}}[b]$. Take $z \in M_{\mathcal{B}}[b] \setminus M_{\mathcal{B}}(w)$ and let $d = \langle w, z\rangle$. Note that $M_{\mathcal{B}}[b] \in \mathcal{Z}$ is closed under addition and multiplication, and thus $d \in M_{\mathcal{B}}[b]$. So now, we have

$$a \in I < w \ll z < \langle w, z\rangle = d \in M_{\mathcal{B}}[b] < b.$$

Using Theorem 5.15 and the Baire Category Theorem, pick a generic cut $J \in [\![w,z]\!] \subseteq [\![a,b]\!]$. Then I and J are conjugate over $\langle c, B\rangle$ by Theorem 5.18. Let $g \in \mathrm{Aut}(M, \langle c, B\rangle)$ such that $J^g = I$. Let $d' = d^g$ so that $(d,c,B) \equiv (d',c,B)$. In particular, as $d < B$, we have $d' < B$ as well. Note also that since $J < d$, we have
$$I = J^g < d^g = d'.$$

Let π_L be the Skolem function defined by

$$\forall p\, \Big(\pi_L(p) = (\mu x)\Big(\exists y (p = \langle x, y\rangle)\Big)\Big).$$

Then $\pi_L(d) = \pi_L(\langle w, z\rangle) = w > I$, but since $w \in J$, we have

$$\pi_L(d') = \pi_L(d^g) = (\pi_L(d))^g = w^g \in J^g = I.$$

Therefore, $(d,c,I) \not\equiv (d',c,I)$. □

Again, the above proof uses an $\mathscr{L}_{\mathrm{A}}^I$ formula that is true in one structure but not in the other to prove non-conjugacy. This seems to provide evidence supporting the conjecture that the $\mathscr{L}_A^I$ theory of (M, I) determines its conjugacy type when I is generic. We shall now show that this conjecture is in fact true. Surprisingly, the formulas used in the proof of Proposition 6.7 are already sufficient to describe the theory of (M, I). The next definition sets up the notation we shall need properly.

Definition 6.11. Let $\phi(\bar{x}, y)$ be an $\mathscr{L}_A$ formula, $I \in \mathcal{C}$ and $\bar{c} \in M$. We write $\nu^I_{\phi(\bar{x},y)}(\bar{c})\downarrow$ for

$$\exists y \in I \ \Big(\phi(\bar{c}, y) \wedge \forall y' \in I \ (y' > y \to \neg\phi(\bar{c}, y))\Big).$$

The expression $\nu^I_{\phi(\bar{x},y)}(\bar{c})\uparrow$ is the negation of $\nu^I_{\phi(\bar{x},y)}(\bar{c})\downarrow$. Define

$$\nu^I_{\phi(\bar{x},y)}(\bar{c}) = \begin{cases} (\max y \in I)(\phi(\bar{c}, y)), & \text{if } \nu^I_{\phi(\bar{x},y)}(\bar{c})\downarrow; \\ 0, & \text{otherwise.} \end{cases}$$

Note that the statements $\nu^I_{\phi(\bar{x},y)}(\bar{c})\uparrow$, $\nu^I_{\phi(\bar{x},y)}(\bar{c}) = d$, etc., are all first order statements of the $\mathscr{L}_A^I$ structure (M, I).

Lemma 6.12. *Let $I \in \mathcal{Z}$ be generic. If $c \in M$, and $[\![a, b]\!] \in \mathcal{B}$ is pregeneric over c and contains I, then $\nu^I_{\phi(x,y)}(c) < a$ for every $\mathscr{L}_A$ formula $\phi(x, y)$ such that $\nu^I_{\phi(x,y)}(c)\downarrow$.*

Proof. Let $I \in \mathcal{Z}$ be generic, $c \in M$, and $[\![a, b]\!] \in \mathcal{B}$ be pregeneric over c that contains I. Fix an $\mathscr{L}_A$ formula $\phi(x, y)$. Clearly $0 < a$. Suppose $M \models \nu^I_{\phi(x,y)}(c)\downarrow$. Let $A = \nu^I_{\phi(x,y)}(c) + 1 \in I$. Then $M_{\mathcal{B}}(A) < I$ by Proposition 4.2 and Corollary 5.19. Let $B \in M$ such that $M_{\mathcal{B}}(A) < B \in I$. If $A > a$, then $[\![A, B]\!] \subseteq [\![a, b]\!]$ and

$$M \models \nu^I_{\phi(x,y)}(c) \in [\![a, b]\!] \wedge \phi(c, \nu^I_{\phi(x,y)}(c))$$

while $M \models \forall y \in [\![A, B]\!] \ \neg\phi(c, y)$ by the maximality of $\nu^I_{\phi(x,y)}(c)$, which is not possible since $[\![a, b]\!]$ is pregeneric over c. Therefore, $\nu^I_{\phi(x,y)}(c) < A \leqslant a$. □

Theorem 6.13. *Let $c \in M$ and $I, J \in \mathcal{Z}$ be generic. Then $(M, I, c) \cong (M, J, c)$ if and only if*

$$(M, I) \models \nu^I_{\alpha(x,y)}(c)\downarrow \Leftrightarrow (M, J) \models \nu^J_{\alpha(x,y)}(c)\downarrow$$

for every $\mathscr{L}_A$ formula $\alpha(x, y)$.

Proof. One direction is obvious. For the other direction, let $c \in M$ and $I, J \in \mathcal{Z}$ be generic such that $M \models \nu^I_{\alpha(x,y)}(c)\downarrow \leftrightarrow \nu^J_{\alpha(x,y)}(c)\downarrow$ for every $\mathscr{L}_A$ formula $\alpha(x, y)$. Without loss of generality, assume $I < J$. Using Corollary 5.16, pick a pregeneric interval $[\![a, b]\!]$ over c containing I, and a pregeneric interval $[\![u, v]\!]$

over c containing J. By genericity and Proposition 4.2 we have $M_{\mathcal{B}}(a) < I$. Let $A \in M$ such that $M_{\mathcal{B}}(a) < A \in I < b$.

Consider the recursive type

$$p(y) = \{u \leqslant y \leqslant v\} \cup \{\alpha(c,y) \leftrightarrow \alpha(c,A) : \alpha(x,y) \in \mathscr{L}_{\mathrm{A}}\}.$$

We show that this is finitely satisfied in M. Let $\alpha(x,y) \in \mathscr{L}_{\mathrm{A}}$ such that $M \models \alpha(c,A)$. Now if $M \models \nu^I_{\alpha(x,y)}(c){\downarrow}$, then by Lemma 6.12 and the maximality of $\nu^I_{\alpha(x,y)}(c)$, we have

$$a \ll A \leqslant \nu^I_{\alpha(x,y)}(c) < a,$$

which is a contradiction. So $M \models \nu^I_{\alpha(x,y)}(c){\uparrow}$. By our hypothesis, we have $M \models \nu^J_{\alpha(x,y)}(c){\uparrow}$. Note that $A \in I < J$ and $M \models \alpha(c,A)$. So there are cofinally many $y \in J$ such that $M \models \alpha(c,y)$. In particular, there is a $y \in J$ such that $M \models y \geqslant u \wedge \alpha(c,y)$. Thus $M \models \exists y \in [\![u,v]\!]\ \ \alpha(c,y)$, as required.

Let B realise $p(y)$ in M. By construction, $\mathrm{tp}(A,c) = \mathrm{tp}(B,c)$. Using recursive saturation of M, let $g \in \mathrm{Aut}(M,c)$ such that $A^g = B \in [\![u,v]\!]$. Since $a \ll A \ll b$, the intersection $[\![a,b]\!]^g \cap [\![u,v]\!]$ is a $\mathcal{B}$-interval. Using Theorem 5.15 and the Baire Category Theorem, pick a generic cut J' in this interval. By Theorem 5.18, J is conjugate to J' over c, and $(J')^{g^{-1}}$ is conjugate to I over c. Therefore, I is conjugate to J over c. □

Apart from giving alternative proofs of Proposition 6.2 and Example 6.9 for generic cuts, this theorem also implies a weak quantifier elimination result.

Definition 6.14. Define $\mathscr{L}^I_\nu$ to be the language obtained from $\mathscr{L}^I_{\mathrm{Sk}}$ by adding a new predicate

$$\nu^I_{\alpha(\bar{x},y)}(\bar{x}){\downarrow}$$

for each $\mathscr{L}_{\mathrm{A}}$ formula $\alpha(\bar{x},y)$. $\mathscr{L}^I_{\mathrm{A}}$ structures are interpreted as $\mathscr{L}^I_\nu$ structures in the natural way.

Corollary 6.15. *Let $I \in \mathcal{Z}$ be generic and $a,b \in M$. Then $(M,I,a) \cong (M,I,b)$ if and only if a and b satisfy the same quantifier free $\mathscr{L}^I_\nu$ formulas with respect to I. In particular, (M,I) is ω-homogeneous.*

The following example shows that the new predicates $\nu^I_{\alpha(\bar{x},y)}(\bar{x}){\downarrow}$ are necessary for the previous corollary. The idea is very similar to that in Proposition 6.7(b).

Example 6.16. Suppose $\mathcal{B} = \mathcal{B}^{\mathrm{elem}}$, and let $I \in \mathcal{Z}$ be generic. Then the formula

$$(\max j)((x)_j \in I) \text{ is even}$$

which is equivalent to

$$\exists w\Big((x)_{2w} = \nu^I_{\exists j(y=(x)_j)}(x)\Big)$$

is not equivalent in (M, I) to a quantifier-free $\mathscr{L}_{\mathrm{Sk}}^{I}$ formula. In fact, it is not even equivalent to an infinite conjunction of quantifier-free $\mathscr{L}_{\mathrm{Sk}}^{I}$ formulas.

Proof. Suppose $\mathcal{B} = \mathcal{B}^{\mathrm{elem}}$ and let $I \in \mathcal{Z}$ be generic. Using recursive saturation, let $c \in M$ code an ascending sequence of gaps of length ω, i.e., c codes a sequence of nonstandard length such that $(c)_i \ll (c)_{i+1}$ for each $i \in \mathbb{N}$. Let $l \in M$ be the length of this sequence. Without loss of generality, assume this sequence is strictly increasing on its domain. Pick an indicator Y for $\mathcal{B}$ below $\max_{i<l}(c)_i + 1$. Using the strength of $\mathbb{N}$ in M, let $\nu \in M$ be nonstandard such that

$$Y((c)_i, (c)_{i+1}) > \mathbb{N} \text{ iff } Y((c)_i, (c)_{i+1}) > \nu.$$

for every $i \in \mathbb{N}$. By overspill, let $m > \mathbb{N}$ such that

$$\forall i < m \;\; Y((c)_i, (c)_{i+1}) > \nu.$$

Using arithmetic saturation, let $i < m$ be nonstandard such that $i \notin \mathrm{cl}(c)$.

Pick generic cuts $I \in [\![(c)_{i-1}, (c)_i]\!]$ and $J \in [\![(c)_i, (c)_{i+1}]\!]$. Notice that Proposition 5.11 and Theorem 5.18 imply that I and J are conjugate. Let $g \in \mathrm{Aut}(M)$ such that $I = J^g$ and set $d = c^g$. Then by our choices of I and J,

$$(\max j)((c)_j \in I) \text{ and } (\max j)((c)_j \in J)$$

are of different parities. Hence

$$(c, I) \not\equiv (c, J) \cong (c^g, J^g) = (d, I).$$

On the other hand, if t is a Skolem function such that $t(c) \in [\![(c)_{i-1}, (c)_{i+1}]\!]$, then i is definable from $(\mu j)((c)_j \geqslant t(c)) \in \mathrm{cl}(c)$, which is contradictory to our choice of i. So for every Skolem function t, we either have $t(c) < (c)_{i-1}$, or $(c)_{i+1} < t(c)$. It follows that

$$t(c) \in I \text{ iff } t(c) < (c)_{i-1} \text{ iff } t(c) \in J \text{ iff } t(c^g) \in J^g \text{ iff } t(d) \in I$$

for every Skolem function t in $\mathscr{L}_{\mathrm{A}}$. Thus, c and d have the same quantifier-free $\mathscr{L}_{\mathrm{Sk}}^{I}$ type since $c^g = d$. Therefore, the formula

$$(\max j)((x)_j \in I) \text{ is even}$$

is not equivalent to an infinite conjunction of quantifier-free $\mathscr{L}_{\mathrm{Sk}}^{I}$ formulas. □

We are not yet able to prove a real quantifier elimination result, and whether such a result is possible is the main open question arising from this work.

Question 6.17. Let $M \models \mathrm{PA}$ be countable and arithmetically saturated, $\mathcal{Z}$ a closed species of cuts without isolated point and $I \in \mathcal{Z}$ a $\mathcal{Z}$-generic cut.

Is it the case that every $\mathscr{L}_{\mathrm{A}}^{I}$ formula $\theta(\bar{x})$ is equivalent in (M, I) to a single quantifier-free formula $\theta_{\mathrm{qf}}(\bar{x})$ in the language $\mathscr{L}_{\nu}^{I}$ with the same free variables?

The main obstruction to answering this question at present is the observation that (M, I) is not recursively saturated and may not be recursively saturated for types built from quantifier-free $\mathscr{L}_{\nu}^{I}$ formulas.

7 Elementary generic cuts

Elementary cuts are so important and often studied that we feel it useful to highlight them as a special case of the general theory above. Throughout this section we assume that our model M of PA is countable and arithmetically saturated.

In the case when $\mathcal{B} = \mathcal{B}^{\mathrm{elem}}$ and $\mathcal{Z} = \mathcal{Z}^{\mathrm{elem}} = \mathcal{Z}(\mathcal{B})$ of Example 2.14, we have shown that generic cuts for this species exist; we shall call these cuts *elementary generic cuts*.

One useful property of the neighbourhood system of elementary intervals is the following.

Proposition 7.1. *The notion of elementary intervals $\mathcal{B}^{\mathrm{elem}}$ is relatively indestructible.*

Proof. Let $[\![a, b]\!] \in \mathcal{B}$. Consider the recursive type

$$p(x) = \{\forall i < a\ (t_n((x)_i) < (x)_{i+1}) : n \in \mathbb{N}\} \cup \{(x)_0 = a \wedge (x)_a \leqslant b\}.$$

This is finitely satisfied in M since $[\![a, b]\!]$ contains an elementary cut. Any element realizing $p(x)$ in M witnesses the relative indestructibility of $[\![a, b]\!]$. □

Therefore, by Propositions 4.7 and 4.8, an elementary generic I is semiregular but not regular in M. It follows that M is never a conservative extension of an elementary generic cut I, since I would be strong and hence regular in any conservative extension.

Elementary generic cuts, like generic cuts for other species, are not definable over a finite set of parameters in any logic. This means for example elementary generic cuts cannot be of the form $M_{\mathcal{B}}(a)$ or $M_{\mathcal{B}}[b]$ (Proposition 4.2). Using Corollary 4.4 and the well-known idea of chronic resplendency (see for example the presentation in Kaye [1, Theorem 15.8]) it is also easy to see that there is no Σ_1^1 formula characterising genericity below any $B \in M$.

Proposition 4.2 also gives us some information about automorphisms fixing I

pointwise via a theorem by Kotlarski [11].

Theorem 7.2 (Kotlarski [11, Theorem 4.1])**.** *Let J be an elementary cut of a countable arithmetically saturated M. If $J \neq M[b]$ for any $b \in M$, then J is* closed in *M, i.e.,*

$$\forall b > J \ \exists g \in \mathrm{Aut}(M) \left(\forall x \in J \ x^g = x \text{ and } b^g \neq b\right).$$

Corollary 7.3. *All elementary generic cuts are closed.*

It also follows from Proposition 4.2 and and Lemmas 2 and 4 of Kotlarski [10] that an elementary generic cut I of M is recursively saturated as an $\mathscr{L}_{\mathrm{A}}$ structure. The standard systems of I and M are the same (since I is nonstandard) and so by general results, I and M are isomorphic. This proves the following.

Proposition 7.4. *If M is countable and arithmetically saturated then there is a countable arithmetically saturated elementary end-extension N of M such that M is elementary generic in N.*

Similarly, any countable and arithmetically saturated M is $K[b]$ for some countable arithmetically saturated elementary end-extension K of M and some $b \in K$. So we have the following.

Proposition 7.5. *If M is countable and arithmetically saturated then there is an elementary end-extension N of M such that M is not elementary generic in N.*

Although an elementary generic cut I is 'rich' considered as a model in its own right, the pair of models (M, I) (i.e. M with a I realising a new predicate symbol) is not recursively saturated (Corollary 4.4). The proof of that corollary gives an example of a recursive set of formulas that is finitely satisfied but not realised. It is instructive in the case of elementary generic cuts to give a more straightforward example.

The idea of *sequences of skies* or *gaps*, introduced by Smoryński and Stavi [15] and discussed further by Smoryński [13] and Kossak and Schmerl [9], gives us a particularly nice necessary condition on (M, J) being recursively saturated, where J is an elementary cut of M.

Fact 7.6 (Smoryński [13, Theorem 2.8])**.** *If J is an elementary cut such that (M, J) is recursively saturated as an $\mathscr{L}_{\mathrm{A}}^{I}$ structure, then J is the limit of an ascending sequence of gaps of length J.*

Proposition 7.7. *An elementary generic cut I of a countable arithmetically saturated M is not the limit of an ascending sequence of gaps of length I.*

Proof. Suppose $c \in M$ codes an ascending sequence of gaps of length I such that

$$\sup\{(c)_i : i \in I\} = I.$$

Using Corollary 5.16, pick a pregeneric interval $[\![a,b]\!] \in \mathcal{B}$ over c that contains I. Note that the sequence $((c)_i)_{i\in I}$ is cofinal in I. So let $i \in I$ such that $(c)_i > a$. By Theorems 5.15 and 5.18, I is conjugate to a generic cut in $[\![(c)_i,(c)_{i+1}]\!] \subseteq [\![a,b]\!]$ over c. This is impossible since no $\mathcal{Z}$-cut $J \in [\![(c)_i,(c)_{i+1}]\!]$ can satisfy

$$\{(c)_j \in J : j \in M \text{ is less than the length of } c\} \subseteq_{\mathrm{cf}} J,$$

as required. □

All our known examples of elements $c \in M$ for which $\mathbb{N}$ is definable in (M, I, c) are above I. So we ask the following.

Question 7.8. Suppose M is countable and arithmetically saturated and I is elementary generic for $\mathcal{B}$, what is the set

$$\{c \in M : \mathbb{N} \text{ is definable in } (M, I, c)\}?$$

In particular, is it a subset of $M \setminus I$?

We conjecture that the elements of M definable in (M, I, c) are precisely the elements in the Skolem closure of $\{\nu^I_{\alpha(x,y)}(c) : \alpha \in \mathscr{L}_\mathrm{A}\}$. In the case when c is absent, by using a theorem by Kossak and Bamber [8], one can verify that all elements definable without parameters in (M, I) are in $\mathrm{cl}(\varnothing)$.

Theorem 7.9 (Kossak and Bamber [8, Theorem 4.1])**.** *If $J \in \mathcal{C}$ is closed under exponentiation, then every element definable in (M, J) without parameters is in $\mathrm{cl}(c)$ for some $c \in J$.*

To return to the topic of conjugacy properties, recall that exceptionally all elementary intervals are pregeneric (over 0) by Proposition 5.11. A consequence of this result is Example 6.9, which says that

$$\forall a, b \in I\ \big(\mathrm{tp}(a) = \mathrm{tp}(b) \Rightarrow (M, I, a) \cong (M, I, b)\big)$$

for an elementary generic cut I. This relates generic cuts to the notion of *free cuts* defined by Kossak.

Definition 7.10 (Kossak [6,7])**.** An elementary cut I is *free* if and only if whenever $a, b \in I$ with $\mathrm{tp}(a) = \mathrm{tp}(b)$, we have $(a, I) \equiv (b, I)$.

Corollary 7.11. *All elementary generic cuts are free.*

This provides new examples of free cuts. By Theorem 5.18 and Proposition 5.11, all elementary generic cuts are conjugate, and hence by Theorem 5.15

the orbit of I under the action of $\mathrm{Aut}(M)$ has cardinality $2^{\aleph_0}$. This partially answers a question by Kossak [7, Problem 4.7]. Proposition 6.10 also says something about the degree of freeness of I. In Kossak's terminology [6], it says that I is the largest initial segment J of M such that I is J*-free* in M.

However, in view of the above discussion, this does not provide us with an example of a free cut I such that (M, I) is recursively saturated. One possible way to pursue this problem is to relax the axioms for a neighbourhood system so that Proposition 7.7 cannot be proved but enough freeness is retained. The statement of Proposition 2.3 seems to be a good candidate for a weakening of axiom (5). Another way is to use arguments similar to those in Section 6 of GCMA. A positive answer to the following question will also help.

Question 7.12. If M is arithmetically saturated, I is generic for some species $\mathscr{Z}$, and $\bar{a} \in M$, is the theory $\mathrm{Th}(M, I, \bar{a})$ coded in M?

In view of the interesting work that has been done on the automorphism group of a countable recursively saturated or arithmetically saturated models of PA, it would seem that the automorphism group $\mathrm{Aut}(M, I)$ is begging to be explored, where I is elementary generic or (more generally) generic for some other neighbourhood system. Theorem 5.18 and Corollary 6.15 provide useful ways to construct automorphisms in this group. The new back-and-forth system taken from GCMA, together with the well-known ones, suggest that the structure of such groups is quite rich.

We only state two questions relating to this group here, and leave it to the reader's imagination to come up with others. In the next two questions, let I be elementary generic in M, or more generally $\mathscr{Z}$-generic for some closed species $\mathscr{Z}$.

Question 7.13. Is $\mathrm{Aut}(M, I)$ a maximal subgroup of $\mathrm{Aut}(M)$?

Note that $\mathrm{Aut}(M, I)$ is naturally equipped with a topology, namely that generated by cosets of pointwise stabilisers of finite tuples from M. It is straightforward to see that $G_{(I)}$ is a closed normal subgroup of $G_{\{I\}}$.

Question 7.14. Other than $G_{(I)}$, what are the other closed normal subgroups of $G_{\{I\}}$? In particular, if $M \models \mathrm{Th}(\mathbb{N})$, is $G_{(I)}$ the only closed normal subgroup of $G_{\{I\}}$?

Another topic that is worth looking into is about $\mathscr{L}_{\mathrm{A}}^{I}$ elementary extensions of the structure (M, I), where I is elementary generic in M. By standard model theoretic techniques, we know that there is a countable elementary extension of (M, I) that is recursively saturated in the expanded language. So genericity is not preserved in all such extensions by Corollary 4.4. However, is there any proper elementary extension $(N, J) \succ (M, I)$ such that J is generic in N?

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