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AXIOMATIZATIONS AND QUANTIFIER COMPLEXITY

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ABSTRACT

Given any theory T we give axiomatizations of $\exists_n T$ and $\forall_n T$, the $\exists_n$ and $\forall_n$ consequences of T. We show that if T is ω-categorical and $\forall_n$ axiomatized, but not $\exists_n$ axiomatized, then $\exists_n T$ is not finitely axiomatized (although T may be).

§0 INTRODUCTION

Given a theory T, let $\forall_n T$ and $\exists_n T$ be the theories axiomatized by the $\forall_n$ and $\exists_n$ sentences provable in T. If T is given by some axiomatization can we find reasonably simple axiomatizations of $\forall_n T$ and $\exists_n T$? The answer is in fact 'yes', and we provide ways of doing this in §1 below. Our motivation for doing this comes from the observation that T, $\exists_n T$ and $\forall_n T$ are all equiconsistent and so we have reduced the problem of constructing models of T to constructing models of less complex theories. Many properties of the theory T are transferred down to $\exists_n T$ and/or $\forall_n T$, one notable exception to this rule being that even if T is finitely axiomatized, $\exists_n T$ need not be. A full investigation of this phenomenon in the special case where T is one of the finite fragments $I\Sigma_n$ of Peano Arithmetic, or some related theory, was carried out in [KPD] and [K2] (see [K3] for a survey). In this case the non-finite axiomatizability of $I\Sigma_n^- = \Sigma_{n+2}(I\Sigma_n) = \exists_{n+2}(I\Sigma_n)$ takes a very strong form: finite fragments of $I\Sigma_n^-$ actually have a strictly weaker proof-theoretic strength (as measured in the usual way.)

I have suggested elsewhere that these simple model theoretic considerations may give new direction to research on some long-standing open problems. One example is the case where T is some theory of sets with a universal set (such as NF or Church's) which is finitely axiomatizable and the consistency problem notoriously open. (Here the axiomatizations of $\exists_n T$ suggest new hierarchies of theories and function classes that appear worthy of study.) The application that concerns us here however is that of the theories of bounded arithmetic (see e.g. [B] and [W]) and their relationships with complexity theory and the MRDP theorem. (I shall give explicit examples from this field of study in the text below.)

Consideration of §1 below and the examples cited in the last paragraph lead to the following general questions: Given an $\forall_{n+1}$ axiomatized theory T, what can one say about axiomatizations of $\exists_k T$ and $\forall_\ell T$ for various $k,\ell \leq n+1$? Are they finite? Are the obvious axiomatizations 'best possible' in terms of quantifier complexity? Even for very well behaved theories T these questions seem difficult and interesting, and well worth study. Proof theorists have studied these questions—§1 essentially consists of a model theoretic study of the crucial facts about Gentzen-style deduction systems relevant to this—however it seems that model theorists have an extra trick up their sleeves that is also relevant, namely the various notions of saturation. It was George Wilmers who suggested that the problems in weak systems of arithmetic may one day be settled by consideration of saturation properties of the nonstandard models of these theories—see in particular his paper [W]. Taking this suggestion seriously, in §2 we solve the analogous problem (that of the non-finite axiomatizability of $\exists_{n+1} T$ for an $\forall_{n+1}$ theory T) in the case where types over T and saturation properties are most easily understood, namely the case when T is ω-categorical. These results are hardly the last word on the subject: my object is to generate discussion on a new set of problems in model theory.

There are many people to thank: Dugald Macpherson, Angus Macintyre, Wilfrid Hodges and Alec Wilkie for interesting discussions and encouragement; Jesus College for financial support and the organizers of the Easter Model theory conference for providing the venue to air these ideas. As will be clear from the text below I have been greatly influenced by Wilfrid Hodges' book [H], and in particular all notation not explained below is taken from this book.

§1 THE AXIOMATIZATIONS

We start with a theory T axiomatized by,

$$\{ \forall \bar{y}\ \phi_i(\bar{y}) \mid i \in \mathbb{N} \}$$

(where $\phi_i \in \exists_n$ for each i and some fixed $n \geq 1$) in a countable language L (so T is $\forall_{n+1}$ axiomatized).

THEOREM 1.1: $\exists_{n+1}T$ is axiomatized by

$$\exists \bar{y}\ \eta(\bar{y}) \rightarrow \exists \bar{y}\ (\ \eta(\bar{y}) \wedge \phi_i(\bar{y})\ ) \qquad (\cdot)$$

over all $\eta \in \exists_n$ and $i \in \mathbb{N}$.

PROOF: Denote the theory $(\cdot)$ by T*, and suppose $M \models T^*$ is any model of T*. We shall show that there is a model

$$K \models T + \forall_{n+1}(M)$$

which clearly suffices.

The proof is by a model theoretic forcing argument.

(See [H] for other constructions of this kind.) <u>Conditions</u> are $\forall_{n-1}$ formulas $\lambda_i(x_0,..,x_{j_i})$ satisfying

a) $M \models \exists \bar{x}\lambda_i(\bar{x})$ for all i

b) $\vdash \forall \bar{x}\ (\lambda_{i+1}(\bar{x}) \to \lambda_i(\bar{x}))$ for all i

K will be the substructure $\{a_0,a_1,...\}$ of a model $M' \models Th(M)+ \bigwedge_i \lambda_i(a_0,..,a_{j_i})$. To make $K \prec_{\exists_n} M'$ the required Henkin condition on the λ_i's is:

c) If $\theta(\bar{u},\bar{v}) \in \forall_{n-1}$, and $\bar{y}$ is a tuple from $x_0,x_1,..$ then

<u>either</u> $\vdash \forall \bar{x}\ (\lambda_k(\bar{x}) \to \theta(\bar{y},\bar{z}))$ for some $k \in \mathbb{N}$ and some $\bar{z}$ from $x_0,..,x_{j_k}$

<u>or</u> $Th(M) \vdash \forall \bar{x}\ (\lambda_k(\bar{x}) \to \forall \bar{z} \neg\theta(\bar{y},\bar{z}))$ for some $k \in \mathbb{N}$.

This is satisfied in the usual way.

Finally to make $K \models T$ we need:

d) For each tuple $\bar{y}$ from $x_0,x_1,...$ and each $i \in \mathbb{N}$ there is $k \in \mathbb{N}$ s.t. $\vdash \forall \bar{x}\ (\lambda_k(\bar{x}) \to \phi_i(\bar{y}))$.

To check (d) can be satisfied, suppose $\lambda_k(\bar{x})$ has been constructed and we are considering ϕ_i and $\bar{y} \subseteq \{x_0,x_1,..\}$. By (a) $M \models \exists \bar{x}\lambda_k(\bar{x})$. Write $x_0,..,x_{j_k}$ as $\bar{y},\bar{z}$ where no variable in $\bar{z}$ occurs in $\bar{y}$. So: $M \models \exists \bar{y}[\exists \bar{z}\lambda_k(\bar{y},\bar{z})]$. The formula inside [..] is $\exists_n$ so applying ($\bullet$) we have

$$M \models \exists \bar{y},\bar{z}\ (\lambda_k(\bar{y},\bar{z}) \wedge \phi_i(\bar{y})).$$

So if $\phi_i(\bar{y})$ is $\exists \bar{w}\theta_i(\bar{y},\bar{w})$ with $\theta_i \in \forall_{n-1}$ then we just let $\lambda_{k+1}(\bar{x})$ be the formula $\lambda_k(\bar{y},\bar{z})\wedge\theta_i(\bar{y},\bar{w})$ for some new tuple of variables $\bar{w}$ from $x_0, x_1, \ldots$

<u>COROLLARY 1.2</u>: If T is an $\forall_{n+1}$ axiomatized theory where $n \geq 1$, then $\exists_{n+1}T$ is axiomatized by a set of $\exists_n \vee \forall_n$ sentences. If T is complete (or more generally, if T decides all $\exists_n$ and $\forall_n$ sentences) then $\exists_{n+1}T$ can be taken to be $\exists_n \cup \forall_n$.

<u>PROOF</u>: Immediate from 1.1.

1.2 gives interesting information in the following example: IE_1 is the theory of arithmetic consisting of a weak base theory together with the scheme of induction for E_1 formulas, i.e. formulas of the form $\exists \bar{x}<t(\bar{a},y)\ \theta(\bar{a},y,\bar{x})$ with θ q.f. in $L_A = \{0,1,+,\cdot,<\}$ and t a term in this language. (Thus t is a polynomial with coefficients from $\mathbb{N}$.) See [W] for more details on IE_1. It is unknown if IE_1 proves the MRDP theorem on the diophantine representation of r.e. predicates, or if IE_1 has an $\forall_2$ axiomatization (the natural axiomatization is $\forall_3$) however $IE_1 \vdash MRDP \Rightarrow IE_1$ is $\forall_2$ (see [K1]). IU_1^- is the same base theory together with parameter-free induction on U_1 formulas, i.e. formulas of the form $\forall \bar{x}<t(y)\ \theta(\bar{x},y)$ with θ q.f.; IU_1^- is a very weak theory with recursive finitely generated models, and its natural axiomatization is $\exists_2$. Does IU_1^- have an $\forall_2$ axiomatization? We suspect not: forexample $I\forall_1^-$ (which is the same as IU_1^- except the induction formulas need not be bounded) has no $\forall_2$ axiomatization, although it too is $\exists_2$, [K4]. However (also from [K4]) we have $IU_1^- \vdash \forall_1 \vee \exists_1(IE_1)$, so if

$IE_1 \vdash MRDP$ it is $\forall_2$ and hence by 1.2 we would have the suprising consequences that IU_1^- is equivalent to $\exists_2(IE_1)$ and is $\forall_1 \vee \exists_1$ axiomatized.

Notice that our axiomatization of $\exists_{n+1}T$ in 1.1 is certainly not finite. Our next goal is to apply 1.1 to obtain information about $\forall_n T$. This will enable us to give a 'proof-theoretic' condition implying $\exists_{n+1}T$ is not finitely axiomatized.

As before, fix $n \geq 1$ and T axiomatized by $\{\forall \bar{y}\phi_i(\bar{y}) \mid i \in \mathbb{N}\}$ with each $\phi_i \in \exists_n$ in L. For any L-theory S define

$$D_T(S) = \left\{ \exists\bar{y}\eta(\bar{y}) \to \sigma \;\middle|\; \begin{array}{l} \eta \in \exists_n,\ \sigma \in \forall_n \text{ and} \\ S \vdash \exists\bar{y}(\eta(\bar{y}) \wedge \phi_i(\bar{y})) \to \sigma \quad \text{some } i \end{array} \right\}$$

so that $D_T(S)$ is $\forall_n$, and if S is also $\forall_n$ then $D_T(S) \vdash S$. Also define: $D_T^0(S)=S$; $D_T^{i+1}(S)=D_T(D_T^i(S))$; $D_T^\infty(S)=\bigcup_{i\in\mathbb{N}} D_T^i(S)$ and $D_T^i=D_T^i(\emptyset)$ for $i=0,1,2..,\infty$.

We prove the following simple property of D_T:

<u>PROPOSITION 1.3</u>: If σ,τ are sentences and $\sigma \in \exists_n$, $D_T^i \vdash \sigma$, $D_T^i(\sigma) \vdash \tau$ then $D_T^i \vdash \tau$.

<u>PROOF</u>: We show by induction on i that if $\sigma \in \exists_n$ then:

$$D_T^i + \sigma \vdash D_T^i(\sigma), \text{ for all } i \in \mathbb{N}.$$

$i=0$ is trivial. If $\exists\bar{y}\eta(\bar{y}) \to \xi$ is an axiom of $D_T^{i+1}(\sigma)$ with $\eta \in \exists_n$, $\xi \in \forall_n$ and $D_T^i(\sigma) \vdash \exists\bar{y}(\eta(\bar{y}) \wedge \phi_i(\bar{y})) \to \xi$ then by the induction hypothesis,

$$D_T^i + \sigma \vdash \exists\bar{y}\eta(\bar{y}) \wedge \phi_i(\bar{y}) \to \xi$$

so,

$$D_T^i \vdash \exists \bar{y}(\sigma \wedge \eta(\bar{y}) \wedge \phi_i(\bar{y})) \to \xi$$

hence

$$D_T^{i+1} \vdash \exists \bar{y}(\sigma \wedge \eta(\bar{y})) \to \xi$$

i.e. $$D_T^{i+1} + \sigma \vdash \exists \bar{y}\eta(\bar{y}) \to \xi$$

as required.

In fact D_T^∞ (which can be considered as the closure of the empty L theory under the inference rules

$$\frac{\exists \bar{y}\ (\ \phi_i(\bar{y}) \wedge \eta(\bar{y})\ ) \to \sigma}{\exists \bar{y}\ \eta(\bar{y}) \to \sigma}$$

for $i \in \mathbb{N}$, $\sigma \in \forall_n$, $\eta \in \exists_n$) is $\forall_n T$. The following theorem shows a little more to be true:

<u>THEOREM 1.4</u>: Suppose $\sigma_j = \exists \bar{y}\eta_j(\bar{y}) \to \exists \bar{y}\ (\ \eta_j(\bar{y}) \wedge \phi_{i_j}(\bar{y})\ )$ for $j=1..k$ are k axioms of $\exists_{n+1}T$ and

$$\sigma_1 + \sigma_2 + \cdots + \sigma_k \vdash \tau \ \in \forall_n$$

then $D_T^k \vdash \tau$.

<u>PROOF</u>: By induction on k, k=0 being trivial. Suppose $D_T^k + \neg\tau$ is consistent, say $M \models D_T^k + \neg\tau$. We must show that $\sigma_1 + \cdots + \sigma_k + \neg\tau$ is consistent.

If $D_T^{k-1} \vdash \exists \bar{y}(\eta_j(\bar{y}) \wedge \phi_{i_j}(\bar{y})) \to \tau$ for each $j=1..k$, then

$$D_T^k \vdash \exists \bar{y}\ \eta_j(\bar{y}) \to \tau$$

for each j, and as $M \models D_T^k + \neg\tau$ it follows that $M \models \forall \bar{y} \neg\eta_j(\bar{y})$ each j, hence $M \models \sigma_1 + \sigma_2 + \cdots + \sigma_k + \neg\tau$ as required.

Otherwise for some $j \leq k$ we have:

$$D_T^{k-1} + \exists \bar{y}(\eta_j(\bar{y}) \wedge \phi_{i_j}(\bar{y})) \wedge \neg\tau \qquad \text{is consistent.}$$

Write τ' for $\neg(\exists \bar{y}(\eta_j(\bar{y}) \wedge \phi_{i_j}(\bar{y})) \wedge \neg\tau)$, so $\tau' \in \forall_n$. By the induction hypothesis

$$\sigma_1 + \cdots + \sigma_{j-1} + \sigma_{j+1} + \cdots + \sigma_k + \neg\tau' \qquad \text{is consistent}$$

i.e. $\neg\tau + \sigma_1 + \cdots + \sigma_{j-1} + \sigma_{j+1} + \cdots + \sigma_k + \exists \bar{y}(\eta_j(\bar{y}) \wedge \phi_{i_j}(\bar{y}))$ is consistent. But a model of this must clearly satisfy σ_j also, hence $\neg\tau + \sigma_1 + \cdots + \sigma_k$ is consistent, as required.

<u>COROLLARY 1.5</u>: If D_T^i forms a hierarchy (i.e. $D_T^i \nvdash D_T^\infty$ for all $i \in \mathbb{N}$) then $\exists_{n+1}T$ isn't finitely axiomatized.

<u>PROOF</u>: If $\sigma_1 + \cdots + \sigma_k \dashv\vdash \exists_{n+1}T$, then $\forall_n T = D_T^\infty = D_T^i$ for some i by 1.4.

<u>EXAMPLES</u>: (i) The subtheories $I\Sigma_n$ of PA are defined to be some base theory PA^- plus induction (with parameters) on Σ_n formulas. By prenex operations and 1.1, $\Sigma_{n+2}(I\Sigma_n)$ is axiomatized by

$$\forall \bar{a},x\ (\ \theta(\bar{a},x) \rightarrow \theta(\bar{a},x+1)\ ) \ \rightarrow$$
$$\forall \bar{a},x\ (\ \theta(\bar{a},0) \rightarrow \theta(\bar{a},x)\ )$$

over all $\theta \in \Sigma_n$. $\Pi_{n+1}(I\Sigma_n) = D_{I\Sigma_n}^\infty$ is the base theory PA^- together with closure under the inference rule

$$\frac{\forall \bar{a},x\ (\ \theta(\bar{a},x) \to \theta(\bar{a},x+1)\ )}{\forall \bar{a},x\ (\ \theta(\bar{a},0) \to \theta(\bar{a},x)\ )} \qquad (\dagger)$$

and $D^i_{I\Sigma_n}$ is the same except the rule (†) may only be used to depth i.

Now for $n\geq 1$, $D^i_{I\Sigma_n}$ does form a hierarchy. In the case of $I\Sigma_1$ this is because the Σ_1 definable functions in $D^i_{I\Sigma_1}$ are exactly the functions in Grzegorczyk's class $\mathcal{E}^{i+2}$. (This is an easy modification of a result of Mints-Takeuti which is already provable using the machinery developed here. One just has to inductively Skolemize $D^i_{I\Sigma_1}$ by adding functions that are in $\mathcal{E}^{i+2}$, using Herbrand's theorem to show that if $\forall \bar{a},x\ (\ \theta(\bar{a},x)\to\theta(\bar{a},x+1)\ )$ is provable in $D^i_{I\Sigma_1}$ then the skolemization for $D^i_{I\Sigma_1}$ can be extended to one for $D^i_{I\Sigma_1} + \forall \bar{a},x(\theta(\bar{a},0)\to\theta(\bar{a},x))$ in $\mathcal{E}^{i+3}$. That $D^i_{I\Sigma_1}$ forms a hierarchy now follows since the $\mathcal{E}^i$'s form a hierarchy. A similar argument applies for $D^i_{I\Sigma_n}$ for $n>1$.) In fact we get the following extra information: If a function f is defined with Σ_n graph using k axioms of $\Sigma_{n+2}(I\Sigma_n)$, then f is in the k^{th}level of the analogous Grzegorczyk hierarchy, a result first proved by Adamowicz, Bigoraskja and Kaye by a different technique, [K2].

(ii) The theories $B\Sigma_n$ of Σ_n-collection (see [PK]) do not behave in the same way: $B\Sigma_n$ is Π_{n+2} axiomatized, and $\Sigma_{n+2}(B\Sigma_n)$ is the theory $B\Sigma_n^-$ considered in [KPD], which is not finitely axiomatizable, whereas $\Pi_{n+1}(B\Sigma_n)$ is just the theory $I\Sigma_{n-1}$, which (at least for n>1) is known to be finitely axiomatized.

Theorem 1.4 (together with Herbrand's theorem) can be considered as the content of the cut elimination theorem as used in Buss's characterization of the polynomial time computable function [B], and similar results (see for example [CT]). Indeed a proof of Buss's theorem can be deduced from 1.4 by inductively Skolemizing the appropriate D_T^∞ using Herbrand's theorem in exactly the same way as sketched above for $I\Sigma_1$.

§2 FINITE AXIOMATIZATIONS

In §§0,1 we raised the question: 'Given an $\forall_n$ theory T (where $n \geq 2$) when is $\exists_n T$ finitely axiomatizable?' and our feeling was that the answer should be: 'not very often'. An example: let T be a consistent $\forall_2$ theory in a pure relational language L with no finite models. Any $\exists_2$ sentence σ provable from T has a finite model, but of course $T \vdash \exists x_1,\ldots,x_k \bigwedge_{i<j\leq k} x_i \neq x_j$ so $\exists_2 T$ has no finite models and hence isn't finitely axiomatized. An example in arithmetic where even partial information on this question would be welcome is the $\forall_3$ theory IE_1 of [W] mentioned above. As we remarked before, if $IE_1 \vdash$ MRDP then IE_1 is $\forall E_1$ axiomatized and in fact this axiomatization can be taken to be finite. The corresponding theory $\exists\forall E_1(IE_1)$ has been identified in [K1] as IE_1^-, the same basic set of axioms together with parameter-free E_1 induction. Now the analogous theories $I\Sigma_n^-$ $(n \geq 1)$ and many other related theories turn out not to be finitely axiomatized—see [K2] or [KPD] for details. A similar result for IE_1^- would have the required consequence that $IE_1 \nvdash$ MRDP.

Here we aim to prove the corresponding result for an $\forall_{n+1}$ ω-categorical theory T, where $n \geq 1$. We start with some definitions and standard properties of ω-categorical theories (for this background material see e.g. [H].)

Fix an $\forall_{n+1}$ axiomatized ω-categorical T with no finite models. Then T is $\exists_n$-complete, i.e. any $\sigma \in \exists_n$ is either provable ($T \vdash \sigma$) or disprovable ($T \vdash \neg\sigma$). An $\exists_n$-type over T is a (not necessarily maximal) set $\Phi(\bar{x})$ of $\exists_n$ formulas consistent with T in a finite number of variables $\bar{x}$. There are finitely many such maximal $\exists_n$-types over T for each tuple $\bar{x}$, and each maximal $\exists_n$-type $\Phi(\bar{x})$ is isolated by some $\phi(\bar{x}) \in \Phi$, i.e.

$$T \vdash \forall \bar{x} \; (\; \phi(\bar{x}) \rightarrow \psi(\bar{x}) \;)$$

for all $\psi \in \Phi$. The crucial property is of course that of $\exists_n$-saturation: If $M \models T$ is countable, $\bar{a} \in M$ and $\Phi(\bar{a},\bar{x}) \subseteq \exists_n$ s.t. $M \models \exists \bar{x} \bigwedge\limits_{\phi \in S} \phi(\bar{a},\bar{x})$ for each finite $S \subseteq \Phi$ then $M \models \exists \bar{x} \bigwedge\limits_{\phi \in \Phi} \phi(\bar{a},\bar{x})$. (In fact an ω-categorical theory T is saturated in the above sense without the restriction $\Phi \subseteq \exists_n$, however we shall only need saturation for $\exists_n$-types.)

Now for the theorem:

<u>THEOREM 2.1</u>: Suppose $n \geq 1$ and T is an $\forall_{n+1}$ axiomatized ω-categorical theory with no finite models. Then

(a) If $n=1$ then $\exists_2 T$ is not finitely axiomatized over $\forall_1 T$.

(b) If $n>1$ and $\exists_{n+1} T \nvdash T$ (i.e. T isn't $\exists_{n+1}$ axiomatized) then $\exists_{n+1} T$ is not finitely axiomatized over $\forall_n T$.

<u>REMARK</u>: The extra condition in (b) is necessary: for example T = 'DLO with end points' is a finitely axiomatized $\exists_3 \wedge \forall_3$ ω-categorical theory.

<u>PROOF OF 2.1</u>: We first prove (a). Suppose T is $\forall_2$ and ω-categorical, and suppose $\exists_2 T \dashv\vdash \exists \bar{x} \phi(\bar{x}) + \forall_1 T$ where ϕ is quantifier free. Let $M \models T$ with $\bar{a} \in M \models \phi(\bar{a})$, and let K be the closure of $\bar{a}$ under the function symbols of L. Then $\bar{a} \in K \subseteq M$

and so $K \models \phi(\bar{a}) + \forall_1 M$, but by the Ryll-Nardzewski theorem K is finite, for otherwise the types

$\Phi_k(x)$ = "x is obtained from $\bar{a}$ by k+1 applications of L-functions but not by any k applications of L-functions"

would yield infinitely many types over $\bar{a}$ in the variable x. It follows that K doesn't even satisfy the $\exists_1$ consequence "there are at least k elements" of T for large k.

To prove (b), suppose $\exists_{n+1} T \dashv\vdash \forall_n T + \exists \bar{x}\phi(\bar{x})$ with $\phi \varepsilon \forall_{n-1}$ and $n \geqq 2$, and consider forcing with $\forall_{n-2}$ conditions with $\forall_n T + \phi(\bar{a})$ in the language of T expanded by adding constant symbols $\bar{a}$. With this notion of forcing it is clear that the compiled structure satisfies $\forall_n T + \phi(\bar{a})$, hence by our supposition satisfies $\exists_{n+1} T$. Denote "it is enforcable that the compiled structure satisfies S" (for this notion of forcing) by $\Vdash S$. 2.1 then follows from the following two lemmas:

<u>LEMMA 2.2</u>: If $\exists_{n+1} T \nvdash T$ then $\nVdash T$.

<u>LEMMA 2.3</u>: $\Vdash \exists_{n+1} T$ iff $\Vdash T$.

<u>PROOF OF 2.2</u>: Suppose that $\Vdash T$ and that $\sigma = \forall \bar{x} \exists \bar{y} \forall \bar{z} \psi(\bar{x},\bar{y},\bar{z})$ is $\forall_{n+1}$ with $\psi \ \varepsilon \ \exists_{n-2}$ and provable in T but not in $\exists_{n+1} T$. Denote the compiled structure K, and suppose $K \models \chi(\bar{a})$ for any $\chi \ \varepsilon \ \exists_{n+1}$. Then $K \models \exists \bar{x} \chi(\bar{x})$ and since $K \models T$ and T is complete we have $\exists_{n+1} T \vdash \exists \bar{x} \chi(\bar{x})$.

Now since $\Vdash T$ we have $\Vdash \sigma$. This clearly means we have:

$$(1)\quad \left\{\begin{array}{l}\text{For any condition } \alpha(\bar{x},\bar{a}) \text{ there is a condition} \\ \beta(\bar{x},\bar{y},\bar{a}) \supseteq \alpha(\bar{x},\bar{a}) \text{ s.t: whenever } \gamma(\bar{x},\bar{y},\bar{z},\bar{a}) \supseteq \beta \text{ is} \\ \text{a condition there is another condition} \\ \delta(\bar{x},\bar{y},\bar{z},\bar{a}) \supseteq \gamma \text{ s.t.} \\ \forall_n T+\phi(\bar{a}) \vdash \forall \bar{x},\bar{y},\bar{z}(\delta(\bar{x},\bar{y},\bar{z},\bar{a}) \rightarrow \psi(\bar{x},\bar{y},\bar{z}))\end{array}\right.$$

(Here a 'condition' is of course an $\exists_{n-1}$ formula consistent with $\forall_n T+\phi(\bar{a})$, and $\alpha \subseteq \beta$ denotes α is contained in β as a conjunct.) The proof of (1) is standard. We also have:

$$(2)\quad \left\{\begin{array}{l}\text{For any tuple } \bar{z} \text{ there is a finite collection} \\ \phi_j(\bar{z},\bar{a}) \quad (j \leq N \in \mathbb{N}) \text{ of conditions s.t. the compiled} \\ \text{structure satisfies } \forall \bar{z} \bigvee\!\!\!\bigvee_{j \leq N} \phi_j(\bar{z},\bar{a}) \text{ , and for} \\ \text{each } j \text{ if } \pi(\bar{z},\bar{a}) \supseteq \phi_j \text{ is a condition then} \\ \forall_n T+\phi(\bar{a}) \vdash \forall \bar{z} \; (\; \phi_j(\bar{z},\bar{a}) \rightarrow \pi(\bar{z},\bar{a}) \;).\end{array}\right.$$

To see (2), notice that in the compiled structure K each tuple $\bar{z} \in K$ realises a maximal $\exists_{n-1}$ type over $\bar{a}$. Let $\Phi(\bar{z},\bar{a})$ be a typical such type. We claim Φ is principal, i.e. isolated by some $\phi(\bar{z},\bar{a}) \in \Phi$. If not let $\Phi = \{\phi_0,\phi_1,\phi_2,..\}$ s.t. $\forall_n T+\phi(\bar{a})+\phi_0+\cdots+\phi_k \nvdash \phi_{k+1}$ for all k. But then each $\phi_0 \wedge .. \wedge \phi_k(\bar{z},\bar{a}) \wedge \neg\phi_{k+1}(\bar{z},\bar{a})$ extends to a maximal complete type over T, and these are all distinct, contradicing Ryll-Nardzewski's theorem. Thus in K there are at most countably many $\exists_{n-1}$ types over $\bar{a}$, and these are all principal, isolated by $\phi_0,\phi_1,\phi_2,...$ say. Thus

$$K \models \forall \bar{z} \bigvee\!\!\!\bigvee_{k \in \mathbb{N}} \phi_k(\bar{z},\bar{a}),$$

and so for some $N \in \mathbb{N}$ we have

$$K \models \forall \bar{z} \bigvee\!\!\!\bigvee_{k \leq N} \phi_k(\bar{z},\bar{a})$$

for otherwise $K \models \exists \bar{z} \bigwedge_{k \in N} \neg \phi_k(\bar{z},\bar{a})$ for each N, so $K \models \exists \bar{z} \bigwedge_{k \in \mathbb{N}} \neg\phi_k(\bar{z},\bar{a})$ by saturation, a contradiction. This gives (2) as required.

We use (1) and (2) together to write down a proof of $\forall \bar{x} \exists \bar{y} \forall \bar{z} \psi(\bar{x},\bar{y},\bar{z})$ in $\exists_{n+1}T$.

By (2) we have maximal conditions $\alpha_j(\bar{x},\bar{a}) \in \exists_{n-1}$ such that

(3) $$K \models \forall \bar{x} \bigvee_{j \leq N_0} \alpha_j(\bar{x},\bar{a}).$$

For each α_j let $\beta_j \supseteq \alpha_j$ be found satisfying the property in (1). Then $\exists \bar{y} \beta_j(\bar{x},\bar{y},\bar{a})$ is a condition extending α_j so by the maximality of α_j we have,

(4) $$\forall_n T + \phi(\bar{a}) \vdash \forall \bar{x} \ (\ \alpha_j(\bar{x},\bar{a}) \rightarrow \exists \bar{y} \beta_j(\bar{x},\bar{y},\bar{a}) \ )$$

and so in particular we have

(5) $$K \models \forall \bar{x} \bigvee_{j \leq N_0} \exists \bar{y} \beta_j(\bar{x},\bar{y},\bar{a}).$$

We once again apply (2) to obtain maximal conditions $\xi_\ell(\bar{x},\bar{y},\bar{z},\bar{a})$ for $\ell \leq N_1 \in \mathbb{N}$ such that

(6) $$K \models \forall \bar{x},\bar{y},\bar{z} \bigvee_{\ell \leq N_1} \xi_\ell(\bar{x},\bar{y},\bar{z},\bar{a})$$

and for each $\gamma_{j\ell}(\bar{x},\bar{y},\bar{z},\bar{a}) = \beta_j(\bar{x},\bar{y},\bar{a}) \wedge \xi_\ell(\bar{x},\bar{y},\bar{z},\bar{a})$ that is a condition obtain using (1) and our choice of β_j a condition $\delta_{j\ell}(\bar{x},\bar{y},\bar{z},\bar{a}) \supseteq \gamma_{j\ell}$ such that

(7) $$\begin{cases} \forall_n T + \phi(\bar{a}) \vdash \forall \bar{x},\bar{y},\bar{z} \ (\ \gamma_{j\ell} \rightarrow \delta_{j\ell} \ ) \quad \text{and} \\ \forall_n T + \phi(\bar{a}) \vdash \forall \bar{x},\bar{y},\bar{z} \ (\ \delta_{j\ell} \rightarrow \psi \ ) \end{cases}$$

by maximality of ξ_ℓ.

Putting (3), (4), (5), (6), (7) together we have

$$K \models \exists \bar{a} \left\{ \begin{array}{l} \forall \bar{x} \bigvee_{j \le N_0} \exists \bar{y} \beta_j(\bar{x},\bar{y},\bar{a}) \\ \forall \bar{x},\bar{y},\bar{z} \bigvee_{\ell \le N_1} \xi_\ell(\bar{x},\bar{y},\bar{z},\bar{a}) \\ \bigwedge_{\ell,j} \bar{x},\bar{y},\bar{z} \ (\ \beta_j \wedge \xi_\ell \to \delta_{j\ell} \ ) \\ \bigwedge_{\ell,j} \bar{x},\bar{y},\bar{z} \ (\ \delta_{j\ell} \to \psi \ ) \end{array} \right\}$$

and this sentence is $\exists_{n+1}$, so true in any model of $\exists_{n+1}T$, because T is complete. It now follows immediately from this sentence that $\exists_{n+1}T \vdash \forall \bar{x} \exists \bar{y} \forall \bar{z} \psi(\bar{x},\bar{y},\bar{z})$, concluding the proof of 2.2.

<u>PROOF OF 2.3</u>: One direction is trivial. For the other, suppose $\Vdash \exists_{n+1}T$. Then by 1.1 the compiled structure K satisfies the following:

$\forall_{n-1}$ forcing with Th(K) gives a model of T.

We may choose K to have the following property:

> If $K \models \exists \bar{x}\theta(\bar{x})$ with $\theta \varepsilon \forall_{n-1}$, then there is some $\lambda(\bar{x},\bar{a}) \ \varepsilon \ \exists_{n-1}$, a condition in the first forcing construction, such that $\lambda(\bar{x},\bar{a}) \Vdash \theta(\bar{x})$ and $K \models \exists \bar{x} \ \lambda(\bar{x},\bar{a})$.

Indeed if $K \models \exists_{n+1}T$ and $K \models \exists \bar{x}\theta(\bar{x})$ then $\exists_{n+1}T \vdash \exists \bar{x}\theta(\bar{x})$, so $\Vdash \exists \bar{x}\theta(\bar{x})$.

It is easy now to see a strategy in the forcing game with $\forall_{n-2}$ conditions over $\forall_n T+\phi(\bar{a})$ to give a model of T (and not just $\exists_{n+1}T$). One just follows the stategy in the $\forall_{n-1}$-forcing game with Th(K) and whenever this stategy dictates one should play $\theta(\bar{x}) \in \forall_{n-1}$, instead play some $\lambda(\bar{x},\bar{a})$ with $\lambda(\bar{x},\bar{a}) \Vdash \theta(\bar{x})$ and play to enforce $\theta(\bar{x})$. Hence $\Vdash T$ as required.

<u>EXAMPLE</u>: Let T be the theory of atomless boolean algebras in the language $L = \{ \leq , = \}$. T is ω-categorical and $\forall_3$ axiomatized, but not $\exists_3$. It follows that $\exists_3 T$ is not finitely axiomatized.

The above proof of 2.1 is based on a proof for $T = I\Sigma_n$ given in [K2], where forcing arguments replace the construction $K^{n+1}(M)$ = all elements of M that are definable by a Σ_{n+1} formula. In fact if $M \models I\Sigma_n^-$ ($= \Sigma_{n+2}(I\Sigma_n)$) then $K^{n+1}(M)$ is the unique enforcable structure (with forcing with Π_n conditions over Th(M).) It follows that

$$M \succ_{\Sigma_{n+1}} K^{n+1}(M) \models I\Sigma_n$$

(see [K2] for details). Thus the arithmetic and the ω-categorical cases both have the following property:

> Forcing with $\forall_n$ (or Π_n) conditions over Th(M) gives a $\exists_{n+1}$ (or Σ_{n+1}) elementary <u>substructure</u> of M.

This raises the following interesting question:

<u>QUESTION 1</u>: What models M (for a countable language L) have the property that a compiled structure K formed by $\forall_n$-forcing with Th(M) ($n \geq 0$) is embedded as a substructure of M?

It is easy to see that the embedding (if it exists) must be $\exists_{n+1}$-elementary. This property holds in the case $M \models T$ for ω-categorical T, and indeed more generally if M is $\exists_{n+1}$-saturated. The arithmetic case works for a different reason. Consider the following property of models M:

$$(\cdot) \quad \begin{cases} \text{Whenever } M \models \exists \bar{x}\theta(\bar{x}) \text{ with } \theta \in \forall_n \text{ there is } \psi(\bar{x}) \in \exists_{n+1} \\ \text{s.t. } \psi(\bar{x}) \supseteq \theta(\bar{x}) \text{ and } M \models \exists \bar{x}\psi(\bar{x}), \text{and for some} \\ k \in \mathbb{N},\ M \models \forall \bar{x}_1, \ldots, \bar{x}_k \left(\bigwedge_i \psi(\bar{x}_i) \rightarrow \bigvee_{i \neq j} (\bar{x}_i = \bar{x}_j) \right) \end{cases}$$

An easy König's lemma argument shows that Question 1 has a positive answer for models satisfying (•). Note too that (•) only depends on $\exists_{n+2}(M)$, so it makes sense to say a theory has (•).

<u>QUESTION 2</u>: If T is an $\forall_{n+2}$ theory satisfying (•), where $n \geq 0$, can $\exists_{n+2}T$ ever be finitely axiomatized?

We have been shifting our attention between theories with very many types (such as arithmetic) and theories with very few types (ω-categoricity) and noted that the common ground was the saturation properties of models. It seems important to understand these properties better. I conclude with a sample observation and a question on these lines for the ω-categorical case. Notice that if $\phi_i(\bar{x}) \in \forall_n$ for each i and $\{\phi_i(\bar{x}) \mid i \in \mathbb{N}\}$ is a type with no parameters over a $\forall_{n+1}$ ω-categorical theory T, then it is contained in a complete type isolated by $\eta(\bar{x}) \in \exists_n$. Then $\exists_{n+1}T \vdash \exists\bar{x}\eta(\bar{x})$ and $\exists_{n+1}T \vdash \forall\bar{x}(\eta(\bar{x}) \rightarrow \phi_i(\bar{x}))$ for each i. Thus every model of $\exists_{n+1}T$ realizes the type $\{\phi_i(\bar{x}) \mid i \in \mathbb{N}\}$.

QUESTION 3: If T is an ω-categorical theory that is $\forall_n$ axiomatized, what saturation properties do models of $\forall_k T$, $\exists_\ell T$ have, for various $k, \ell \in \mathbb{N}$?

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