Theorem 10.16 *If* $\mathbf{A}$ *is any upper triangular* $n \times n$ *matrix with entries from* $\mathbb{R}$ *or* $\mathbb{C}$*, and* $\lambda_1, \lambda_2, \dots, \lambda_n$ *are the diagonal entries of* $\mathbf{A}$ *including repetitions, then the matrix*

$$(\mathbf{A} - \lambda_1 \mathbf{I})(\mathbf{A} - \lambda_2 \mathbf{I}) \dots (\mathbf{A} - \lambda_n \mathbf{I})$$

is the zero matrix.

Proof If $n = 1$, this is obvious. We prove the general statement using induction on n.

Given an upper triangular $n \times n$ matrix, take the standard basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ of the underlying vector space $\mathbb{R}^n$ or $\mathbb{C}^n$. Observe first that

$$(\mathbf{A} - \lambda_1 \mathbf{I}) \dots (\mathbf{A} - \lambda_{n-1} \mathbf{I})(\mathbf{A} - \lambda_n \mathbf{I}) = (\mathbf{A} - \lambda_n \mathbf{I})(\mathbf{A} - \lambda_1 \mathbf{I}) \dots (\mathbf{A} - \lambda_{n-1} \mathbf{I})$$

from Section 9.2 in the last chapter.

Now $(\mathbf{A} - \lambda_n \mathbf{I})\mathbf{e}_n$ is a linear combination of $\mathbf{e}_1, \dots, \mathbf{e}_{n-1}$. Also, the linear transformation given by $\mathbf{A}$ on the subspace $\operatorname{span}(\mathbf{e}_1, \dots, \mathbf{e}_{n-1})$ has upper triangular matrix with respect to this basis, with diagonal entries $\lambda_1, \dots, \lambda_{n-1}$, so by the induction hypothesis we have

$$(\mathbf{A} - \lambda_1 \mathbf{I}) \dots (\mathbf{A} - \lambda_{n-1} \mathbf{I})\mathbf{e}_i = \mathbf{0}$$

for all $i < n$. Thus

$$(\mathbf{A} - \lambda_n \mathbf{I})(\mathbf{A} - \lambda_1 \mathbf{I}) \dots (\mathbf{A} - \lambda_{n-1} \mathbf{I})\mathbf{e}_i = \mathbf{0}$$

for $i < n$. Moreover, $(\mathbf{A} - \lambda_1 \mathbf{I}) \dots (\mathbf{A} - \lambda_{n-1} \mathbf{I})(\mathbf{A} - \lambda_n \mathbf{I})\mathbf{e}_n$ is a linear combination of the $(\mathbf{A} - \lambda_1 \mathbf{I}) \dots (\mathbf{A} - \lambda_{n-1} \mathbf{I})\mathbf{e}_i$ for $i < n$, all of which are $\mathbf{0}$. Therefore

$$(\mathbf{A} - \lambda_1 \mathbf{I}) \dots (\mathbf{A} - \lambda_{n-1} \mathbf{I})(\mathbf{A} - \lambda_n \mathbf{I})\mathbf{e}_i = \mathbf{0}$$

for all $i \leq n$.

We have shown that the matrix

$$(\mathbf{A} - \lambda_1 \mathbf{I}) \dots (\mathbf{A} - \lambda_{n-1} \mathbf{I})(\mathbf{A} - \lambda_n \mathbf{I})$$

represents the zero transformation, since multiplying $\mathbf{e}_i$ by it gives $\mathbf{0}$ for all $i \leq n$. Therefore this matrix is zero, as required. □