Spanning trees in dense directed graphs

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Abstract

In 2001, Komlós, Sárközy and Szemerédi proved that, for each α > 0, there is some c > 0 and n₀ such that, if n ≥ n₀, then every n-vertex graph with minimum degree at least (1/2 + α)n contains a copy of every n-vertex tree with maximum degree at most cn/log n.

We prove the corresponding result for directed graphs. That is, for each α > 0, there is some c > 0 and n₀ such that, if n ≥ n₀, then every n-vertex directed graph with minimum semi-degree at least (1/2 + α)n contains a copy of every n-vertex oriented tree whose underlying maximum degree is at most cn/log n.

As with Komlós, Sárközy and Szemerédi’s theorem, this is tight up to the value of c. Our result improves a recent result of Mycroft and Naia, which requires the oriented trees to have underlying maximum degree at most ∆, for any constant ∆ ∈ N and sufficiently large n. In contrast to the previous work on spanning trees in dense directed or undirected graphs, our methods do not use Szemerédi’s regularity lemma.

1 Introduction

Given two graphs H and G, when may we expect to find a copy of H in G? In general, this decision problem is NP-complete, and therefore we seek simple conditions on G which imply it contains a copy of H. An important early result is Dirac’s theorem from 1952 that, when n ≥ 3, any n-vertex graph with minimum degree at least n/2 contains a cycle through every vertex, that is, a Hamilton cycle. This is a particular instance of the following meta-question, which has seen much subsequent study. Given an n-vertex graph H, what is the lowest minimum degree condition on an n-vertex graph G which guarantees it contains a copy of H? As such a copy of H would contain every vertex in G, we say it is a spanning copy of H.

This question has been studied for many different graphs H, for example when H is a K-factor for some small fixed graph K [8, 14], the k-th power of a Hamilton cycle for any k ≥ 2 [11] and when H has bounded chromatic number and maximum degree, and sublinear bandwith [4]. For more details on these results, and those for other graphs, see the survey by Kühn and Osthus [13]. Here, we will concentrate on the minimum degree required to guarantee different spanning trees.

Komlós, Sárközy and Szemerédi [10] proved in 1995 that, for each α, Δ > 0, there is some n₀ such that, if n ≥ n₀, then every n-vertex graph with minimum degree at least (1/2 + α)n contains a copy of every n-vertex tree with maximum degree at most Δ, thus confirming a conjecture of Bollobás [2]. This result is furthermore notable as one of the earliest applications of the blow-up lemma. In 2001, Komlós, Sárközy and Szemerédi [12] relaxed the maximum degree condition, showing that, for each α > 0, there is some c > 0 and n₀ such that, if n ≥ n₀, then every

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n-vertex graph with minimum degree at least \((1/2 + \alpha)n\) contains a copy of every n-vertex tree with maximum degree at most \(cn/\log n\). This is tight up to the constant \(c\). In this paper, we will prove the corresponding version of this result for directed graphs (digraphs).

The minimum semidegree of a digraph \(D\), denoted by \(\delta^0(D)\), is the smallest in- or out-degree over the vertices in \(D\), that is, \(\delta^0(D) = \min_{v \in V(D)} d^\diamond(v)\). Ghouila-Houri \([7]\) solved the minimum semidegree problem for the directed Hamilton cycle, showing that, if an n-vertex digraph \(D\) has \(\delta^0(D) \geq n/2\), then it contains a directed Hamilton cycle. That is, an n-vertex cycle with the edges oriented in the same direction. DeBiasio, Kühn, Molla, Osthus and Taylor \([5]\) showed that, when \(n\) is sufficiently large, this holds in fact for any n-vertex cycle with any orientations on its edges, except for when the edges change direction at every vertex around the cycle. This latter cycle, known as the anti-directed Hamilton cycle, is only guaranteed to appear if \(\delta^0(D) \geq n/2 + 1\), as shown by DeBiasio and Molla \([6]\).

Recently, Mycroft and Naia \([16, 17]\) gave the first bound on the minimum semidegree required for the appearance of different spanning trees. Here, \(H\) is an oriented n-vertex tree, with some bound on the degree of its underlying (undirected) tree. Mycroft and Naia \([16, 17]\) proved that, for each \(\alpha, \Delta > 0\), there is some \(n_0\) such that, if \(n \geq n_0\), then every n-vertex digraph with minimum semidegree at least \((1/2 + \alpha)n\) contains a copy of every oriented n-vertex tree \(T\) with \(\Delta^\pm(T) \leq \Delta\). Moreover, their result holds for a slightly wider class of trees, allowing them to show that, for each \(\alpha > 0\), almost every labelled oriented n-vertex tree appears in every n-vertex digraph with minimum semidegree at least \((1/2 + \alpha)n\).

In this paper, we introduce new methods to embed oriented trees in digraphs, relaxing the maximum degree condition to give a full directed version of Komlós, Sárközy and Szemerédi’s result, as follows.

**Theorem 1.1.** For each \(\alpha > 0\), there exists \(c > 0\) and \(n_0 \in \mathbb{N}\) such that the following holds for every \(n \geq n_0\). Any n-vertex digraph \(D\) with \(\delta^0(D) \geq (1/2 + \alpha)n\) contains a copy of every oriented n-vertex tree \(T\) with \(\Delta^\pm(T) \leq \Delta\).

We note that the undirected version follows immediately from Theorem 1.1. Indeed, given any n-vertex tree \(T\) and an n-vertex graph \(G\), we can apply Theorem 1.1 to a copy of \(T\) with each edge oriented arbitrarily and a digraph formed from \(G\) by replacing each edge \(uv\) with an edge from \(u\) to \(v\) and an edge from \(v\) to \(u\). This demonstrates that, as with Komlós, Sárközy and Szemerédi’s result, Theorem 1.1 is tight up to the constant \(c\). Furthermore, through Theorem 1.1 we give a new proof of the undirected result without using Szemerédi’s regularity lemma, in contrast to the work of both Komlós, Sárközy and Szemerédi \([10]\), and Mycroft and Naia \([16, 17]\). Key to our result is to use a random embedding of part of the tree using ‘guide sets’ and embedding many leaves (and small subtrees) of the tree using ‘guide graphs’. This replaces the regularity methods of \([10]\) \([16, 17]\), and is sketched in Section 2 where we also outline the rest of this paper.

## 2 Preliminaries

### 2.1 Notation

Let \(D\) be a digraph. We denote by \(V(D)\) and \(E(D)\) the vertex set and edge set of \(D\), respectively, where every element of the edge set of \(D\) is an ordered pair of vertices. We let \(|D| = |V(D)|\), which we call the size of \(D\), and let \(e(D) = |E(D)|\). Letting \(u, v \in V(D)\), if \(uv \in E(D)\), then we say that \(u\) is an in-neighbour of \(v\) and \(v\) is an out-neighbour of \(u\). Denote by \(N_D^-(v)\) and \(N_D^+(v)\), respectively, the set of all in- and out-neighbours of \(v\). We let \(d_D^-(v) = |N_D^-(v)|\) and \(d_D^+(v) = |N_D^+(v)|\), and we refer to these as the in- and out-degree of \(v\), respectively. For each
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2.2.1 Main tools and deduction of Theorem 1.1

Theorem 2.2. In Section 4, we prove Theorem 2.1. Structural decomposition of trees and some simple results on matchings. In Section 3, we prove before deducing Theorem 1.1 from them. In Section 2.2.2, we then discuss in detail the proof of

Suppose that A and B are disjoint subsets of V(D). We write D[A] to mean D induced on the set A, that is, the graph obtained from D by deleting all vertices which are not in A. For each ∈ {+, −}, a -matching from A into B is a set of vertex-disjoint edges such that every edge in the set has one endpoint in A and one endpoint in B, and the endpoint in B is a -neighbour of the endpoint in A, that is, every edge is a -edge from A into B. We say this matching covers A if every vertex of A belongs to some edge in the matching, and we call this a perfect -matching if it covers both A and B. A bare path of length m in a tree is a path with m edges such that each of the internal vertices have degree 2 in the tree. When P is a path in D, we let denote the subgraph of D obtained by removing the internal vertices of P.

For any n ∈ N, we let [n] := {1, . . . , n}. In order to simplify notation, we use hierarchies to state our results. That is, for a, b ∈ (0, 1], whenever we write that a statement holds for a ≪ b (or b ≫ a), we mean that there exists a non-decreasing function f : (0, 1] → (0, 1] such that the statement holds whenever a ≤ f(b). We define similar expressions with multiple variables analogously. We say a random event occurs with high probability if the probability of the event occurring tends to 1 as n tends to infinity. In our proofs, when we have shown that a property holds with high probability, we will implicitly assume that this property holds from that point onwards. For simplicity, we ignore floors and ceilings wherever this does not affect the argument.

2.2 Proof sketch

When 1/n ≪ c ≪ α, we will embed any oriented n-vertex tree T with Δ±(T) ≤ cn/ log n into any n-vertex digraph D with δ+(D) ≥ (1/2 + α)n. We embed T using the absorption method, an approach first introduced in general by Rödl, Ruciński and Szemerédi [18] which has been effective on a range of embedding problems for spanning graphs and digraphs (see, for example, the survey [3]). We first partially embed a subtree T′′ of T into a set A such that, given any subset B ⊆ V(D) with A ⊆ B and |B| = |T′′|, we can complete this embedding of T′′ into D[B] (see Theorem 2.1).

We then use an almost-spanning embedding to embed the vertices in V(T) \ V(T′′) to extend the partial embedding of T′′ (see Theorem 2.2). We will have chosen T′′ so that in this stage a tree, called T′′, is attached to an embedded vertex of T′′. Using the property of the partial embedding of T′′, we then complete the embedding of T′′ with the unused vertices in D. The decomposition of T that we need follows from a simple proposition (Proposition 2.3).

In Section 2.2.1, we state these three results, Theorem 2.1, Theorem 2.2 and Proposition 2.3, before deducing Theorem 1.1 from them. In Section 2.2.2, we then discuss in detail the proof of Theorem 2.2 which is the major challenge overcome by this paper.

In the rest of Section 2, we restate the probabilistic tools we will use, and give a basic structural decomposition of trees and some simple results on matchings. In Section 3, we prove Theorem 2.2. In Section 4, we prove Theorem 2.1.

2.2.1 Main tools and deduction of Theorem 1.1

For Theorem 1.1, we will first find a suitable subtree T′′ ⊆ T and a set A ⊆ V(D) with slightly fewer than |T′′| vertices, so that, given any set B of |T′′| vertices containing A, we can embed

\( \diamond \in \{+, −\} \), we let \( δ^\diamond(D) \) and \( Δ^\diamond(D) \) be, respectively, the minimum and maximum \( \diamond \)-degree of \( D \). For any \( A, B \subseteq V(D) \), and each \( \diamond \in \{+, −\} \), let \( N^\diamond_D(A, B) = \bigcup_{a \in A} (N^\diamond_D(a) \cap B) \), and let \( d^\diamond_D(A, B) = |N^\diamond_D(A, B)| \). We omit the subscript when the graph is clear from context. Note that, for simplicity of notation, we use ‘−’ and ‘in’ interchangeably, and, similarly, we use ‘+’ and ‘out’ interchangeably. We use ‘±’ to represent that a property holds for both ‘−’ and ‘+’.

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$T''$ in $D[B]$. Furthermore, we will ensure that some pre-specified vertex $t \in V(T'')$ is always embedded to some fixed vertex $v \in A$, as follows.

**Theorem 2.1.** Let $1/n \ll c \ll \varepsilon \ll \mu \ll \alpha$. Let $D$ be an $n$-vertex digraph with minimum semidegree at least $(1/2+\alpha)n$. Let $T$ be an oriented tree with $\mu n$ vertices and $\Delta^\pm(T) \leq cn/\log n$, and let $t \in V(T)$.

Then, $V(D)$ contains a vertex set $A$ with size $(\mu - \varepsilon)n$ containing a vertex $v \in A$ such that the following holds. For any set $B \subset V(D)$ with $A \subset B$ and $|B| = \mu n$, $D[B]$ contains a copy of $T$ in which $t$ is copied to $v$.

Theorem 2.1 is proved in Section 4 by randomly embedding most of $T$ and taking $A$ to be the image of this embedding. We then show that the partial embedding of $T$ can be extended using any new vertex in $y \in V(D) \setminus A$ by switching $y$ into the partial embedding in place of some vertex in $A$ that can instead be used to embed a new vertex of $T$. Repeatedly doing this will allow the embedding of $T$ to be completed using any set of $|T| - |A|$ new vertices in $V(D) \setminus A$. This is sketched in more detail at the start of Section 4, before Theorem 2.1 is proved.

We will embed the majority of the tree for Theorem 1.1 using the following almost-spanning embedding.

**Theorem 2.2.** Let $1/n \ll c \ll \varepsilon, \alpha$. Let $D$ be an $n$-vertex digraph with minimum semidegree at least $(1/2+\alpha)n$ and let $v \in V(D)$. Let $T$ be an oriented tree with at most $(1-\varepsilon)n$ vertices and $\Delta^\pm(T) \leq cn/\log n$, and let $t \in V(T)$.

Then, $D$ contains a copy of $T$ in which $t$ is copied to $v$.

Using in addition the following simple proposition (see, for example, [15 Proposition 3.22]), we can now deduce Theorem 1.1.

**Proposition 2.3.** Let $n, m \in \mathbb{N}$ satisfy $1 \leq m \leq n/3$. Given any $n$-vertex tree $T$, there are two edge-disjoint trees $T_1, T_2 \subset T$ such that $E(T_1) \cup E(T_2) = E(T)$ and $m \leq |T_2| \leq 3m$.

**Proof of Theorem 1.1 from Theorems 2.1 and 2.2.** Let $\varepsilon, \mu > 0$ be such that $c \ll \varepsilon \ll \mu \ll \alpha$. Let $D$ be an $n$-vertex digraph with $\delta^0(D) \geq (1/2+\alpha)n$. Let $T$ be an oriented $n$-vertex tree with $\Delta^\pm(T) \leq cn/\log n$.

Using Proposition 2.3 with $m = \mu n$, find edge-disjoint trees $T', T'' \subset T$ such that $E(T') \cup E(T'') = E(T)$ and $\mu n \leq |T''| \leq 3\mu n$. Let $t$ be the vertex which is in both $T'$ and $T''$. By Theorem 2.1 applied with $\mu' = |T''|/n$, there is a set $A \subset V(D)$ such that $|A| = |T''| - \varepsilon n$, and a vertex $v \in A$ such that, for any set $B \subset V(D)$ with $A \subset B$ and $|B| = |T''|$, $D[B]$ contains a copy of $T''$ in which $t$ is copied to $v$.

Let $D' = D - (A \setminus \{v\})$. Let $n' = |D'|$, so that $(1 - 3\mu)n \leq n' \leq n$. Let $\alpha'$ be such that $D'$ has minimum semidegree $(1/2 + \alpha')n'$. Note that $(1/2 + \alpha')n' \geq (1/2 + \alpha')n \geq (1/2 + \alpha - 3\mu)n$, so that $\alpha' \geq \alpha/2$. Furthermore, $n' = n - |T''| + \varepsilon n + 1 = |T'| + \varepsilon n$, and therefore

\[
\frac{|T'|}{n'} = \frac{|T'|}{|T'| + \varepsilon n} \leq \frac{|T'|}{|T'|(1 + \varepsilon)} \leq 1 - \varepsilon/2.
\]

Thus, by Theorem 2.2 we can find a copy, $S'$ say, of $T'$ in $D'$ in which $t$ is copied to $v$. By applying the property of $A$ from Theorem 2.1, we can then find a copy of $T''$ in $D - (V(S') \setminus \{v\})$ in which $t$ is copied to $v$. Together, these give us a copy of $T$. \qed
2.2.2 Proof Sketch of Theorem 2.2

We will embed a \((1 - \varepsilon)n\)-vertex tree \(T\) for Theorem 2.2 by dividing most of \(T\) into a small core forest \(T_0 \subset T\) and a collection of constant-sized subtrees, which are either attached to \(T_0\) by a single edge or by two short paths. It is the trees attached to \(T_0\) by a single edge that will be the most challenging to embed, and so we dedicate most of our attention in the proof sketch to this.

More precisely, we will find a tree \(T' \subset T\), containing a core forest \(T_0 \subset T'\) and vertex-disjoint trees \(S_1, \ldots, S_{\ell} \subset T' - V(T_0)\), for some \(\ell \in \mathbb{N}\), such that \(T'\) is formed from \(T_0\) by, for each \(i \in [\ell]\),

1. either adding \(S_i\) to \(T_0\) to obtain \(T'\),
2. or adding \(S_i\) to \(T_0\) using two bare paths with length 2.

Furthermore, for some \(\mu > 0\) and \(K \in \mathbb{N}\), with \(1/n \ll 1/K, \mu \ll \alpha, \varepsilon\), we will have that

- \(|T_0| \leq \mu n\) (i.e., \(T_0\) is small),
- \(|T'| \geq |T| - \mu n\) (i.e., \(T'\) is most of \(T\)),
- there are at most \(\mu n\) trees \(S_i\) which are in Case (1), and
- each tree \(S_i\) has at most \(K\) vertices.

In Case (1), we say \(S_i\) is added to \(T_0\) as a path, and in Case (2) we say \(S_i\) is added to \(T_0\) as a leaf. The crux of our method is to embed \(T_0\) along with the trees \(S_i\) in Case (2) connected to the embedding of \(T_0\) by the appropriate edge. This is encapsulated in the following lemma, which is proved in Section 3.1.

**Lemma 2.4.** Let \(1/n \ll c \ll \mu \ll \alpha, \varepsilon\), let \(c \ll 1/K\) and let \(\ell \in \mathbb{N}\). Suppose \(D\) is an \(n\)-vertex digraph with \(\delta^0(D) \geq (1/2 + \alpha)n\) and \(v \in V(D)\).

Suppose that \(T\) is an oriented tree with \(|T| \leq (1 - \varepsilon)n\) and \(\Delta^\pm(T) \leq cn/\log n\). Suppose that \(T', S_1, \ldots, S_{\ell} \subset T\) are vertex-disjoint subtrees with \(|T'| \leq \mu n\), and \(|S_i| \leq K\) for each \(i \in [\ell]\). Suppose that \(T\) is formed from \(T'\) by attaching each \(S_i, i \in [\ell]\), to \(T'\) by an edge. Finally, let \(t \in V(T')\).

Then, \(D\) contains a copy of \(T\) in which \(t\) is copied to \(v\).

We will now briefly sketch how Theorem 2.2 can be proved from Lemma 2.4. Let \(m\) be the total number of vertices that appear in the trees \(S_i\) in Case (1) above. To embed these trees, we use the fact that two random sets in \(D\) of the same (linear) size are likely to have a perfect matching from one to the other (see Proposition 2.12). Taking \(p \gg 1/n\) and \(Kp \leq 1\), we can, with high probability, find \(pn\) copies of an oriented tree with \(K\) vertices in a random set of \(Kpm\) vertices in \(D\) by taking randomly \(K\) disjoint subsets within this set of size \(pn\) and finding appropriate matchings between them (see Section 2.5). Collecting isomorphic trees \(S_i\) together, and applying this to each of the constantly many (depending on \(K\)) isomorphism classes, allows us to embed the trees \(S_i\) in Case (1) with high probability in a random set with size \(m + \varepsilon n/4\). Here, the extra \(\varepsilon n/4\) vertices allow us to find a linear number of trees in each isomorphism class by finding some additional trees if required.

Thus, in a partition of \(V(D)\) into sets \(V_1 \cup V_2 \cup V_3 \cup V_4\) chosen uniformly at random so that \(|V_1| = n - m - 3\varepsilon n/4, |V_2| = m + \varepsilon n/4, |V_3| = |V_4| = \varepsilon n/4\), with high probability, the following occur.

- \(\delta^\pm(D[V_1]) \geq (1/2 + \alpha/2)|V_1|\), so that, applying Lemma 2.4 we can embed \(T_0\) along with the trees \(S_i\) in Case (2) connected to the embedding of \(T_0\) by the appropriate edge.
• We can embed the trees $S_i$ in Case [1] in $D[V_2]$.

• Then, using that there are at most $\mu n$ trees in Case [1], we can greedily attach them to the embedding of $T_0$ using two paths with length 2 whose interior vertex is an unused vertex in $V_3$ (see Section 3.2).

• Finally, as $|T| - |T'| \leq \mu n$, we can greedily extend the resulting embedding of $T'$ to one of $T$, by adding a sequence of leaves using vertices in $V_4$ (see Section 3.3).

Here, the last two steps are (with high probability) possible using the semi-degree condition of $D$. Note that, as $\mu \ll \varepsilon$, we only embed a small proportion of vertices into $V_3$ and $V_4$.

We will now give a detailed proof sketch of Lemma 2.4.

**Proof sketch of Lemma 2.4**

To simplify our discussion, let us assume that each tree $S_i$ in Lemma 2.4 consists of only a single vertex, which is an out-neighbour in the tree $T$ of a vertex of $T_0$, and that every vertex in $T_0$ is attached to exactly one such tree. That is, $T$ consists of $T_0$ with an out-matching attached.

Our embedding of $T_0$ is randomised, which will allow the methods described to be used to find matchings attached from different subsets of the image of the embedding of $T_0$ to different random sets. This will allow the embedding below for $T_0$ to be used for the general case.

Let us detail the example situation precisely. Suppose we have a $\mu n$-vertex tree $T_0$ and choose two disjoint random sets $V_0, V_1 \subset V(D)$ with size $p_0 n$ and $p_1 n$ respectively, where $p_0 \gg \mu$ and $p_1 = (1 + o(1))\mu$. We will randomly embed $T_0$ into $V_0$, so that there is an out-matching from the vertex set of the embedding of $T_0$ into $V_1$. Note that there are many spare vertices in $V_0$, possible as in the general case we embed $T_0$ once. However, as we then find (potentially) many different matchings, we need to do this with few spare vertices, and therefore use most of the vertices in $V_1$ (as $p_1$ is only a little larger than $\mu$).

We will choose $T_0$ vertex-by-vertex, say in order $t_1, \ldots, t_t$, so that each new vertex is embedded as an in- or out-leaf of the previously embedded subtree. Having chosen the random sets $V_0, V_1$, and before beginning the embedding, we will find guide sets $A_{\gamma \beta} \subset N_D^\gamma(v, V_0)$, $v \in V_0$ and $\diamond \in \{+, -\}$, which we use to guide the random embedding. We then start the random embedding, under the rule that if, for some $v \in V_0$ and $\diamond \in \{+, -\}$, we are attaching a $\diamond$-edge as a leaf to $v$, then we choose this leaf uniformly at random from the unused vertices in $A_{\gamma \beta}$.

The guide sets ensure that, with high probability, there will be a matching from the embedding of $T_0$ into $V_1$. These guide sets are found using Lemma 3.5 and they exist (with high probability for the choice of $V_0, V_1$) due to the semi-degree condition in $D$. Essentially, for some constants $\beta, \gamma$, we find, for each $v \in V(D)$ and $\diamond \in \{+, -\}$, a set $A_{\gamma \beta} \subset N_D^\beta(v, V_0)$ with size $\beta n$ and bipartite digraphs $H_{\gamma \beta} \subset D^\beta[A_{\gamma \beta}, V_1]$, $\diamond \in \{+, -\}$, so that in $H_{\gamma \beta}$ each vertex in $A_{\gamma \beta}$ has around $\gamma \beta n$ $\diamond$-neighbours in $V_1$, and each vertex in $V_1$ has around $\gamma \beta n$ $\diamond$-edges leading into it. That is, $H_{\gamma \beta}$ is approximately regular on each side with edge density approximately $\gamma$.

The guide graphs $H_{\gamma \beta}$ can be used to find the matching from the embedding of $T_0$ as follows. When a vertex $t_i$ is embedded using a guide set $A_{\gamma \beta}$, to some vertex $s_i$ say, we add the edges in $H_{\gamma \beta}$ adjacent to $s_i$ to an auxiliary graph $K$ — note that approximately $\gamma \beta n$ edges are added next to $s_i$. Note further that, as most of the vertices in $A_{\gamma \beta}$ will be unused, each $w \in V_1$ will have an edge added from $s_i$ to $w$ with probability approximately

$$
\frac{d^-_{H_{\gamma \beta}}(w)}{|A_{\gamma \beta}|} \approx \frac{\gamma \beta n}{\beta n} = \gamma.
$$

(1)
When this is complete, $K$ is a bipartite digraph with vertex classes $\{s_1, \ldots, s_\ell\}$ and $V_1$. Each vertex $s_i$ will have out-degree approximately $\gamma p_1 n$, and, due to the randomness of the embedding and (1), each vertex in $V_1$ will have in-degree which is approximately $\gamma \ell = |T_0| \approx \gamma p_1 n$.

Thus, $K$ will be a bipartite graph with the in-degrees in one vertex class approximately equal to the out-degrees in the other. Via Hall’s matching criterion, an out-matching will exist from $\{s_1, \ldots, s_\ell\}$ to $V_1$ which covers most of the vertices in $\{s_1, \ldots, s_\ell\}$. By ensuring that $V_1$ is likely to be a little larger than $\ell$, we in fact will get with high probability that such an out-matching can cover $\{s_1, \ldots, s_\ell\}$.

Note that, in the sketch above, we do not use the graph $H_{v,\circ}$. However, in practice, we find such guide sets and guide graphs with $V_1 = V(D) \setminus V_0$ (see Lemma 3.3), before taking random subsets of $V_1$. We will find out-matchings into some of these random sets, and in-matchings into some others. Therefore, it is important to have both guide graphs $H_{v,\circ}$ and $H_{v,\circ}^+$, and, furthermore, that the same set $A_{v,\circ}$ is used for both graphs.

Finally, let us note where the condition $\Delta^\pm(T) \leq cn/\log n$ is used in our proof of Lemma 2.4. In the sketch above the set $V_1$ will always have size which is linear in $n$, but we may need to attach the trees in Lemma 2.4 together comprise linearly (in $n$) many vertices in $T$, then they are attached to at least $C\log n$ different vertices, for some large constant $C$, which gives us sufficient probability concentrations when these vertices are randomly embedded for the corresponding versions of Hall’s criterion to hold (see the proof of Claim 3.7).

### 2.3 Probabilistic tools

Let $n, m, k \in \mathbb{N}$ be such that $\max\{m, k\} \leq n$. Let $A$ be a set of size $n$, and $B \subseteq A$ be such that $|B| = m$. Let $A'$ be a uniformly random subset of $A$ of size $k$. Then the random variable $X = |A' \cap B|$ is said to have hypergeometric distribution with parameters $n, m$ and $k$, which we denote by $X \sim \text{Hyp}(n, m, k)$. We will use the following Chernoff-type bound.

**Lemma 2.5** (see, for example, [9]). Suppose $X \sim \text{Hyp}(n, m, k)$. Then for any $0 < \alpha < 3/2$, we have

$$
P[|X - \mathbb{E}[X]| \geq \alpha \mathbb{E}[X]] \leq 2 \exp\left(-\alpha^2 \mathbb{E}[X]/3\right).
$$

A sequence of random variables $(X_i)_{i \geq 0}$ is a martingale if $\mathbb{E}[X_{i+1} | X_0, \ldots, X_i] = X_i$ for each $i \geq 0$. We will use the following Azuma-type bound for martingales.

**Lemma 2.6** (see, for example, [1]). Let $(X_i)_{i \geq 0}$ be a martingale and let $c_i > 0$ for each $i \geq 1$. If $|X_i - X_{i-1}| < c_i$ for each $i \geq 1$, then, for each $n \geq 1$,

$$
P[|X_n - X_0| \geq t] \leq 2 \exp\left(-\frac{t^2}{\sum_{i=1}^n c_i^2}\right).
$$

We will use this bound for supermartingales and submartingales. A sequence of random variables $(X_i)_{i \geq 0}$ is a supermartingale if $\mathbb{E}[X_{i+1} | X_0, \ldots, X_i] \leq X_i$ for each $i \geq 0$, and a submartingale if $\mathbb{E}[X_{i+1} | X_0, \ldots, X_i] \geq X_i$ for each $i \geq 0$. The bound on the upper tail in Lemma 2.6 holds for supermartingales, while the bound on the lower tail holds for submartingales.

### 2.4 Structural lemmas

In this section we decompose undirected trees. Note that we will later apply this to directed trees as the edge directions do not affect the decompositions. We will use the following simple but useful lemma (see [15 Lemma 4.1]) which tells us that either a tree has many leaves, or it has many bare paths.
Lemma 2.7. Let $t, m \geq 2$, and suppose that $T$ is a tree with at most $t$ leaves. Then there is some $s$ and some vertex-disjoint bare paths $P_i$, $i \in [s]$, in $T$ with length $m$ so that $|T - P_1 - \cdots - P_s| \leq 6mt + 2|T|/(m + 1)$.

We can now prove the following key lemma, in which we decompose a tree for our embedding.

Lemma 2.8. Let $0 \ll 1/n \ll 1/K \ll 1/k \ll \eta$. Let $T$ be a tree on $n$ vertices with $t \in V(T)$. Then, $T$ contains forests $T_0 \subset T_1 \subset T_2 \subset T_3 = T$, such that $T_2$ is a tree, and the following hold.

- **P1** $|T_0| \leq \eta n$ and $t \in V(T_0)$.
- **P2** $T_1$ is formed from $T_0$ by the vertex-disjoint addition of trees, $S_v, v \in V(T_0)$, so that, for each $v \in V(T_0)$, $S_v - v$ is a forest of trees with size at most $K$.
- **P3** $T_2$ is formed from $T_1$ by the addition of trees with size at least $k$ and at most $K$ attached to $T_1$ with exactly two bare paths of length 2.
- **P4** $|T_3| - |T_2| \leq \eta n$.

Proof. Let $\varepsilon$ satisfy $1/K \ll \varepsilon \ll 1/k$, and let $S_0 = T$. Do the following for $i = 0, 1, 2, \ldots$ as far as possible, where a set of independent leaves is a set of leaves which pairwise have no common neighbours in the tree. If $S_i$ has at least $\varepsilon n$ independent leaves, the set $L_i$ say, then remove $L_i \setminus \{t\}$ from $S_i$ to get the tree $S_{i+1}$. Suppose this finishes with $S_t$, which does not have at least $\varepsilon n$ independent leaves. Note that $\ell \leq 1/\varepsilon + 1$. We will show the following claim.

Claim 2.9. To get from $T$ to $S_t$, for each $v \in V(S_t)$, there is a tree removed from $v$ which has at most $2^\ell$ vertices.

Proof of Claim 2.9. We will show by induction on $i = 0, 1, \ldots, \ell$, that, to get from $S_{t-i}$ to $S_t$, for each $v \in V(S_t)$, there is a tree removed from $v$ which has at most $2^i$ vertices. Thus the claim follows when $i = \ell$. Note that this is trivially true for $i = 0$ and label the tree removed from $v \in V(S_t)$ to get from $S_{t-i}$ to $S_t$ as $T_{v,i}$.

Now, let $0 \leq i < \ell$, and assume that $|T_{v,i}| \leq 2^i$ for each $v \in V(S_t)$. As we remove a set of independent leaves from $S_{t-i-1}$ to get to $S_{t-i}$, for each $v \in V(S_t)$, we remove a set of independent leaves of $T_{v,i+1}$ to get $T_{v,i}$. Therefore, for each $v \in V(S_t)$, $|T_{v,i+1}| \leq 2|T_{v,i}| \leq 2^{i+1}$, as required.

Let $L(S_t)$ be the set of leaves of $S_t$. Remove $L(S_t) \setminus \{t\}$ and call the resulting tree $S'$. Note that, as $S_t$ does not have at least $\varepsilon n$ independent leaves, $S'$ does not have at least $\varepsilon n$ leaves. Thus, by Lemma 2.7, for some $m \leq n/(k + 1)$, $S'$ contains vertex disjoint bare paths $P_1, \ldots, P_m$ with length $k$ such that $t \notin V(P_i)$ for each $i \in [k]$ and

$$|S' - P_1 - \cdots - P_m| \leq 6k \cdot \varepsilon n + 2n/(k + 1) + k + 1 \leq \eta n/4. \quad (2)$$

For each path $P_i$, $i \in [m]$, if possible, find within $P_i$ a path $P_i'$ with length at least $k - 2\eta^3 k$, such that, labelling its endvertices $x_i$ and $y_i$ the following hold.

(i) Each of $x_i$ and $y_i$ had a tree with size at most $\eta k/4$ removed from them in $T$ to reach $S'$.

(ii) Letting $Q_i$ be the component of $T - \{x_i, y_i\}$ containing $P_i' - \{x_i, y_i\}$, we have $|Q_i| \leq K$. 

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Say, with relabelling, these paths are $P_1', \ldots, P_m'$. We will show that $m' \geq m - \eta n/2k$. Note first that the number of $i \in [m]$ with no vertices $x_i$ and $y_i \in V(P_i)$ respectively within $\eta^3 k$ of the two endvertices of $P_i$, so that each of $x_i$ and $y_i$ had a tree with at most $\eta k/4$ vertices deleted from them, is at most $n/(\eta^3 k \cdot \eta k/4) \leq \eta n/4k$. Note further that the number of $i \in [m]$ with at least $K$ vertices in $V(P_i)$ or in a component of $T - E(P_i)$ containing an interior vertex of $P_i$ is at most $n/K \leq \eta n/4k$. Therefore, we can find such a path $P_i'$ for all but at most $\eta n/2k$ values of $i \in [m]$, so that $m' \geq m - \eta n/2k$.

Letting $T_0 = S' - P_1' - \ldots - P_m'$, we will now show that $|T_0| \leq \eta n$. Note that, for each $i \in [m']$, $|V(P_i') \setminus V(P_i)| \leq 2\eta^3 k$. Therefore, as $m \leq n/k$,

$$|T_0| \leq |S' - P_1 - \ldots - P_m| + k \cdot \eta n/2k + m \cdot 2\eta^3 k \leq \eta n.$$ 

Furthermore, clearly $t \in V(T_0)$, and thus $P_1$ holds.

Note that, by Claim 2.9, for each $i \in [m']$, $|Q_i| \leq k2^\ell \leq K$. For each $v \in V(T_0)$, let $R_v$ be the tree containing $v$ in $T[(V(T) \setminus V(T_0)) \cup \{v\}]$, without any of the $x_i$, $y_i$ as neighbours. Now, by Claim 2.9, $R_v - v$ consists of trees with at most $2^\ell \leq K$ vertices. Let $T_1 = T_0 \cup (\bigcup_{v \in V(T_0)} R_v)$. Thus, $P_2$ holds.

Let $T_3 = T$ and let $T_2$ be $T[V(T_1) \cup (\bigcup_{i \in [m']} \{x_i, y_i\} \cup Q_i)]$. Note that $P_3$ holds by construction, and as $|Q_i| \leq K$ for each $i \in [m']$. Furthermore, the only missing vertices from $T$ are those in $R_v - v$, for each $v \in \{x_i, y_i : i \in [m']\}$, and thus $T_2$ is a tree. For each such $v$, $|R_v| \leq \eta k/4$ by (i). Therefore, $|T_3| - |T_2| \leq (n/k)(2\eta k/4) \leq \eta n$, and hence $P_4$ holds.

### 2.5 Matchings between random sets

With high probability, any random subset of vertices in the digraph in Theorem 1 holds satisfies a similar minimum semidegree condition, as follows.

**Lemma 2.10.** Let $1/n \ll c, \alpha$, and suppose $D$ is an $n$-vertex digraph with $\delta^0(D) \geq (1/2 + \alpha)n$. Let $A \subseteq V(D)$ be chosen uniformly at random subject to $|A| = cn$. Then, with high probability, for every vertex $v \in V(D)$, we have $|N^\pm_D(v, A)| \geq (1/2 + \alpha/2)|A|$. 

**Proof.** Let $v$ be an arbitrary vertex of $D$ and let $A \subseteq V(D)$ be a uniformly random subset with $|A| = cn$. For $\diamond \in \{+, -\}$, we let $Z^\diamond_v$ be the random variable which measures $|N^\diamond(v) \cap A|$. Then $Z^\diamond_v$ has hypergeometric distribution with expectation

$$\mathbb{E}[Z^\diamond_v] = \frac{|N^\diamond(v)| |A|}{n} \geq \left(\frac{1}{2} + \alpha\right) cn.$$ 

Therefore, by Lemma 2.5, we have

$$\mathbb{P}\left[|Z^\diamond_v - \mathbb{E}[Z^\diamond_v]| > \frac{\alpha/2}{1/2 + \alpha}(1/2 + \alpha)cn\right] \leq 2 \exp\left(-\left(\frac{\alpha/2}{1/2 + \alpha}\right)^2 \frac{(1/2 + \alpha)cn}{3}\right) = 2 \exp\left(-\frac{\alpha^2 cn}{6 + 12\alpha}\right).$$

Then, applying a union bound, with probability at least $1 - 2n \exp\left(-\frac{\alpha^2 cn}{6 + 12\alpha}\right) = 1 - o(1)$, we have that $Z^\diamond_v \geq (1/2 + \alpha/2)|A|$ for each $\diamond \in \{+, -\}$ and $v \in V(D)$. 

The following digraph version of Hall’s matching criterion implies a matching exists, as follows directly from the same result for undirected graphs.
Lemma 2.11. Let $D$ be a bipartite digraph with vertex classes $A$ and $B$, and let $\diamond \in \{+,-\}$. Suppose that for every $S \subset A$, $|N^\diamond_D(S,B)| \geq |S|$. Then there is a $\diamond$-matching from $A$ into $B$ which covers $A$.

We will refer to the condition in Lemma 2.11 as Hall’s criterion. In combination with Lemma 2.10, Lemma 2.11 shows that with high probability there is a perfect matching between a large random pair of disjoint equal-sized vertex subsets in the digraph, as follows.

Proposition 2.12. Let $1/n \ll p, \alpha$, and suppose $D$ is an $n$-vertex digraph with $\delta^0(D) \geq (1/2 + \alpha)n$. Let $A, B$ be chosen uniformly at random from all disjoint pairs of subsets of $V(D)$, each with size $pn$, and let $\diamond \in \{+,-\}$. Then, with high probability, there is a perfect $\diamond$-matching from $A$ into $B$.

Proof. By Lemma 2.10, with high probability we can assume the following. For all $v \in A$, we have $|N^\pm(v,B)| \geq (1/2 + \alpha/2)|B|$, and, for all $v \in B$, we have $|N^\pm(v,A)| \geq (1/2 + \alpha/2)|A|$. We will now show that Hall’s criterion holds.

Let $S \subset A$, such that $S \neq \emptyset$ and $|S| \leq (1/2 + \alpha/2)pn$, and let $x \in S$. Then, $|N^\diamond(x,B)| \geq |N^\diamond(x,S)| \geq (1/2 + \alpha/2)pn \geq |S|$, so Hall’s condition is trivially satisfied. Now take $S \subset A$, $|S| > (1/2 + \alpha/2)pn$, and assume for a contradiction that $|N^\diamond(S,B)| < |S|$. Then in particular, $B \setminus N^\diamond(S,B) \neq \emptyset$. Take $b \in B \setminus N^\diamond(S,B)$, and let $\diamond \in \{+, -\}$ be such that $\diamond \neq \diamond$. We have $|N^\diamond(b,A)| \geq (1/2 + \alpha/2)pn$. However, since $b \notin N^\diamond(S,B)$, we have $N^\diamond(b,A) \cap S = \emptyset$. So,

$$pn = |A| \geq |N^\diamond(b,A)| + |S| \geq (1/2 + \alpha/2)pn + (1/2 + \alpha/2)pn = (1 + \alpha)pn > pn,$$

giving a contradiction. Thus, Hall’s criterion is satisfied for all $S \subset A$ and so, since $|A| = |B|$, by Lemma 2.11 there is a perfect $\diamond$-matching from $A$ into $B$. \qed

We use Proposition 2.12 to embed many vertex disjoint small trees, via the following two lemmas. In Lemma 2.13, we embed linearly many copies of a given constant-sized tree into specified subsets of our digraph. In Lemma 2.14, we embed a forest of constant-sized trees covering almost all the vertices in our digraph.

Lemma 2.13. Let $1/n \ll 1/K, p, \alpha$ with $pK \leq 1$. Suppose $T$ is an oriented $K$-vertex tree containing $t \in V(T)$. Let $D$ be an $n$-vertex digraph with $\delta^0(D) \geq (1/2 + \alpha)n$. Let $V_1, V_2$ be vertex disjoint subsets of $V(D)$ chosen uniformly at random subject to $|V_1| = pn$ and $|V_2| = (K - 1)pn$.

Then, with high probability, $D[V_1 \cup V_2]$ contains $pn$ vertex disjoint copies of $T$, in which $t$ is copied into $V_i$ in each copy of $T$.

Proof. Let $V_1 = U_1$, and let $U_2 \cup \cdots \cup U_K$ be a partition of $V_2$ chosen uniformly at random so that $|U_i| = pn$ for each $i \in \{2, \ldots, K\}$. Note that the distribution of any pair of sets $U_i, U_j$ with $1 \leq i < j \leq K$ is that of two disjoint vertex sets with size $pn$ in $V(D)$, uniformly at random chosen from all such pairs.

Label the vertices of $T$ by $t_1, \ldots, t_K$ so that $t_1 = t$ and $T[{t_1, \ldots, t_i}]$ is a tree for each $i \in \{1, \ldots, K\}$. For each $i \in \{2, \ldots, K\}$, let $j_i \in \{1, \ldots, i - 1\}$ be such that $t_{j_i}$ is the in- or out-neighbour in $T[{t_1, \ldots, t_{j_i - 1}]}$ of the vertex $t_i$, and let $\diamond_i \in \{+, -\}$ be such that $t_i \in N^\diamond_i(t_{j_i})$.

Now by Proposition 2.12, for each $i \in \{2, \ldots, K\}$, with high probability, we can find a $\diamond_i$-matching from $U_{j_i}$ into $U_i$. By applying a union bound, we see that, with high probability, for every $i \in \{2, \ldots, K\}$, there is a $\diamond_i$-matching, $M_i$ say, from $U_{j_i}$ into $U_i$.

Note that the union of these matchings, $\bigcup_{2 \leq i \leq K} M_i \subset D[V_1 \cup V_2]$ is the disjoint union of $pn$ copies of $T$, in which, for each $i \in [K]$, the copy of $t_i$ is in $V_i$. Thus, in each of these $pn$ copies of $T$, $t = t_1$ is copied into $V_1 = U_1$, as required. \qed
Lemma 2.14. Let $1/n \ll 1/K, \varepsilon$, and suppose $F$ is a digraph with at most $(1 - \varepsilon)n$ vertices which is the disjoint union of trees with size at most $K$. Let $D$ be an $n$-vertex digraph with $\delta^0(D) \geq (1/2 + \alpha)n$. Then, with high probability, $D$ contains a copy of $F$.

Proof. Arrange the components of $F$ into isomorphic classes of trees $\mathcal{R}_1, \ldots, \mathcal{R}_\ell$, noting that we may take $\ell \leq (2K)^{K-1}$. For each $i \in [\ell]$, let $t_i = |\mathcal{R}_i|$ and let $s_i$ be the size of each component in $\mathcal{R}_i$. Uniformly at random, take, in $V(D)$, disjoint subsets $V_{i,1}$ and $V_{i,2}$, $i \in [\ell]$, with $|V_{i,1}| = p_i n$ and $|V_{i,2}| = (s_i - 1)p_i n$, where $p_i = t_i/n + \varepsilon/\ell s_i$, for each $i \in [\ell]$. Note that this is possible, since

$$\sum_{i=1}^\ell s_i p_i n = \sum_{i=1}^\ell \left( s_it_i + \frac{\varepsilon n}{\ell} \right) \leq n.$$

For each $i \in [\ell]$, we can apply Lemma 2.13 to show that, with high probability, there are $p_i n$ copies of the underlying tree of $\mathcal{R}_i$ in $D_i = D[V_{i,1} \cup V_{i,2}]$. Since $p_i n \geq t_i$, this implies that with high probability, we can find a copy of $\mathcal{R}_i$ in $D_i$ for each $i \in [\ell]$. By applying a union bound and using that $1/n \ll 1/\ell$, we have, with high probability, that there is a copy of $F$ in $D$. \qed

3 Almost-spanning trees

The key aim of this section is to prove Theorem 2.2, that is, to prove we can embed an almost-spanning tree $T$ in our digraph. By Lemma 2.8, we can find $T_0 \subset T_1 \subset T_2 \subset T_3 = T$, satisfying $\textbf{P1}$ to $\textbf{P3}$ in Section 3.1, we show that we can embed $T_1$. In Section 3.2, we show that we can embed $T_2 \setminus T_1$, and $T_3 \setminus T_2$. We conclude in Section 3.3 by combining this to obtain an embedding of $T$.

3.1 Embedding constant-sized trees as stars

As sketched in Section 2.2, we will embed $T_0$ randomly, leaf by leaf, using a guide set to embed each new vertex. Each guide set has an accompanying guide graph, which we later use to find a matching. The property of the guide graph that we use to find the matching is that it is skew-bounded, as follows.

Definition 3.1. A digraph $D$ with vertex sets $A, B \subset V(D)$ is $(a, b, \circ)$-skew-bounded on $(A, B)$ if $d_D^\circ(v, B) \geq a$ for each $v \in A$ and $d_D^b(v, A) \leq b$ for each $v \in B$, where $\circ \in \{+, -\}$ and $\circ \neq \circ$.

This property can imply a matching exists via Hall’s criterion, as follows.

Proposition 3.2. Let $a \geq b$ and $\circ \in \{+, -\}$. Suppose $D$ is a digraph containing disjoint vertex sets $A, B \subset V(D)$, such that $D$ is $(a, b, \circ)$-skew-bounded on $(A, B)$. Then, there is a $\circ$-matching from $A$ into $B$ in $D$.

Proof. Let $U \subset A$. As $D$ is $(a, b, \circ)$-skew-bounded on $(A, B)$, there are at least $a|U|$ and at most $b|N_D^\circ(U, B)|$ $\circ$-edges from $U$ to $N_D^\circ(U, B)$. Thus, $|N_D^\circ(U, B)| \geq a|U|/b \geq |U|$. Therefore, by Lemma 2.11, there is a $\circ$-matching from $A$ into $B$. \qed

In the following lemmas, we find our guide sets and guide graphs. We start by finding in $D$, for each $v \in V(D)$ and $\circ \in \{+, -\}$, a guide set $A$ and guide graphs which are skew-bounded on $(A, V(D))$. 

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Lemma 3.3. Let $1/n \ll \varepsilon \ll \alpha, \eta \leq 1$ and $1/n \ll \mu \leq \alpha^2/2$. Let $D$ be an $n$-vertex digraph with $\delta^0(D) \geq (1/2 + \alpha)n$, let $v \in V(D)$ and let $\circ \in \{+, -\}$.

Then, there is a set $A \subset N^D_0(v)$ with size $\mu n$ and digraphs $H^+, H^- \subset D$ such that, for each $\circ \in \{+, -\}$, $H^\circ$ is $(\varepsilon n, (1 + \eta)\mu \varepsilon n, \circ)$-skew-bounded on $(A, V(D))$.

Proof. We start by showing that we can label the vertices of $V(D)$ as $V(D) = \{x_1, \ldots, x_n\} = \{y_1, \ldots, y_n\}$ so that, for each $i \in [n]$,

$$|N^-_D(x_i) \cap N^+_D(v) \cap N^+_D(y_i)| \geq \alpha^2 n. \quad (3)$$

To do this, create an auxiliary graph, as follows. For each $w \in V(D)$, create distinct new vertices $w^-$ and $w^+$, and let $V^+ = \{w^+: w \in V(D)\}$ and $V^- = \{w^-: w \in V(D)\}$. Consider the auxiliary bipartite graph $H$ with vertex set $V^+ \cup V^-$, where for each $x, y \in V(D)$, there is an edge between $x^+$ and $y^-$ if and only if $|N^+_D(x) \cap N^+_D(v) \cap N^+_D(y)| \geq \alpha^2 n$.

Claim 3.4. $\delta(H) \geq (1/2 + \alpha/2)n$.

Proof of Claim 3.4. Let $x \in V(D)$. We have $|N^-_D(x) \cap N^+_D(v)| \geq n - (n - d^-_D(x)) - (n - d^+_D(v)) \geq 2\alpha n$. Let $B = N^-_D(x) \cap N^+_D(v)$ and $Y = \{y \in V(D) : |N^+_D(v) \cap B| \geq \alpha^2 n\}$, and note that $d_H(x^+) = |Y|$.

For each $u \in B$, we have $|N^-_D(u)| \geq (1/2 + \alpha)n$, and thus $e_D(V(D), B) \geq (1/2 + \alpha)|B|n$. By the choice of $Y$, we have $e_D(V(D), B) \leq |Y||B| + \alpha^2 n^2$. Therefore, as, in addition, $2\alpha n \leq |B|$, we have

$$(1/2 + \alpha)|B|n \leq |Y||B| + \alpha^2 n^2 \leq |Y||B| + \alpha|B|n/2.$$ 

Thus, $(1/2 + \alpha/2)|B|n \leq |Y||B|$, so that $|Y| \geq (1/2 + \alpha/2)n$. Therefore, $d_H(x^+) = |Y| \geq (1/2 + \alpha/2)n$.

A similar argument, with the signs reversed, shows that $d_H(y^-) \geq (1/2 + \alpha/2)n$ for each $y \in V(D)$, completing the proof of the claim. \qed

As in the proof of Proposition 2.12, Claim 3.4 easily implies that Hall’s criterion is satisfied, so that there is a matching from $V^+$ to $V^-$ in $H$. That is, we can label the vertices of $V(D)$ as $V(D) = \{x_1, \ldots, x_n\} = \{y_1, \ldots, y_n\}$ so that, for each $i \in [n]$, (3) holds.

We will now show by induction that, for each $0 \leq i \leq \mu n$, there is a set $A_i \subset N^D_0(v)$ with size $i$ and graphs $H^+_{i}, H^-_{i} \subset D$ such that, for each $\circ \in \{+, -\}$, $H^\circ_i$ is $(\varepsilon n, (1 + \eta)\mu n, \circ)$-skew-bounded on $(A_i, V(D))$, $e(H^+_{i}) = i \varepsilon n$, and, for each $j \in [n]$, $d^+_H(x_j) = d^+_H(y_j)$.

Note that if $A_0 = \emptyset$ and if $H^+_{0}, H^-_{0}$ have no edges and vertex set $V(D)$, then the conditions hold, so assume that $0 \leq i < \mu n$ and we have $A_i \subset N^D_0(v)$ and $H^+_{i}, H^-_{i} \subset D$ as described.

Let $J_i \subset [n]$ be the set of $j \in [n]$ for which $d^-_H(x_j) = d^-_H(y_j) \leq (1 + \eta/2)\mu n$. Note that, as $e(H^+_{i}) = e(H^-_{i}) = i \varepsilon n \leq \mu n^2$, we have

$$(n - |J_i|)(1 + \eta/2)\mu n \leq \mu n^2.$$ 

Thus, as $\eta \leq 1$, $(n - |J_i|) \leq n/(1 + \eta/2) \leq n(1 - \eta/4)$, so that $|J_i| \geq \eta n/4$.

For each $j \in J_i$, let $W_{i,j} = (N^-_0(x_j) \cap N^+_0(v) \cap N^+_0(y_j)) \setminus A_i$, noting that, by (3), $|W_{i,j}| \geq \alpha^2 n - i > \alpha^2 n - \mu n \geq \alpha^2 n/2$. By averaging, choose some $w_i \in V(D)$ such that

$$|\{j \in J_i : w_i \in W_{i,j}\}| \geq \sum_{j \in J_i} |W_{i,j}| \geq \frac{\eta n/4 \cdot \alpha^2 n/2}{n} \geq \varepsilon n,$$

using that $\alpha, \eta \gg \varepsilon$. Choose a set $J'_i \subset \{j \in J_i : w_i \in W_{i,j}\}$ with size $\varepsilon n$. Let $A_{i+1} = A_i \cup \{w_i\}$. Let $H^+_{i+1}$ be the digraph $H^+_{i}$ with edges $w_i x_j, j \in J'_i$, added. Note that, as $d^+_H(x_j) \leq (1 + \eta/2)\mu n$
for each \( j \in J'_i \), \( H^+_{i,j} \) is \((\varepsilon n, (1 + \eta)\mu \varepsilon n, +)\)-skew-bounded on \((A_{i+1}, V(D))\). Furthermore, by the definition of \( W_{i,j} \), the edges added to \( H^+_{i,j} \) are in \( D \), and therefore \( H^+_{i+1} \subset D \).

Let \( H^-_{i+1} \) be the digraph \( H^- \) with the edges \( y_jw_j, j \in J'_i \), added. Note that, similarly, \( H^-_{i+1} \) is \((\varepsilon n, (1 + \eta)\mu \varepsilon n, -)\)-skew-bounded on \((A_{i+1}, V(D))\). Finally, noting that \( A_{i+1} \) has size \( i + 1 \), that \( e(H^+_{i+1}) = e(H^-_{i+1}) = (i + 1)\varepsilon n \) and that, for each \( j \in [n] \), \( d^+_H(x_j) = d^-_H(y_j) \), completes the inductive step, and hence the proof.

We now show that the guide sets and guide graphs found by Lemma 3.3 have a similar skew-bounded property when restricted to random vertex subsets, as follows.

**Lemma 3.5.** Let \( 1/n \ll \varepsilon \ll \alpha, \eta \leq 1 \) and \( 1/n \ll 1/k, p_0, p_1, \ldots, p_k \leq 1 \). Let \( \mu = \alpha^2 p_0/4 \). Let \( D \) be an \( n \)-vertex digraph with \( \delta^0(D) \geq (1/2 + \alpha)n \). Let \( V_0, V_1, \ldots, V_k \subset V(D) \) be disjoint random sets chosen uniformly at random subject to \( |V_i| = p_n \) for each \( i \in \{0, \ldots, k\} \).

Then, with high probability, for each \( v \in V(D) \) and \( \diamond \in \{+, -\} \), there is a set \( A_{v,\diamond} \subset N^\diamond_D(v) \) with size \( \mu n \) and digraphs \( H^\diamond_{v,\diamond} \subset D \), \( \diamond \in \{+, -\} \), such that, for each \( \diamond \in \{+, -\} \) and \( i \in [k] \), \( H^\diamond_{v,\diamond} \) is \((\varepsilon p_n, (1 + \eta)\varepsilon \mu n, \diamond)\)-skew-bounded on \((A_{v,\diamond}, V_i)\).

**Proof.** By Lemma 3.3 applied with \( \varepsilon' = (1 + \eta/4)\varepsilon, \eta' = \eta/4 \) and \( \mu' = (1 + \eta/4)\alpha^2/4 \), for each \( v \in V(D) \) and \( \diamond \in \{+, -\} \), there is a set \( A_{v,\diamond} \subset N^\diamond_D(v) \) with size \((1 + \eta/4)\alpha^2 n/4 \) and digraphs \( H^\diamond_{v,\diamond} \subset D \) such that, for each \( \diamond \in \{+, -\} \), \( H^\diamond_{v,\diamond} \) is \(((1 + \eta/4)\varepsilon n, (1 + \eta/4)^3\varepsilon \alpha^2 n/4, \diamond)\)-skew-bounded on \((A_{v,\diamond}, V(D))\).

Select \( V_0, V_1, \ldots, V_k \subset V(D) \) according to the distribution in the lemma. Using Lemma 2.5 and a union bound, we have that, with high probability, the following hold.

**Q1** For each \( v \in V(D) \) and \( \diamond \in \{+, -\} \), \( |A_{v,\diamond} \cap V_0| \geq \alpha^2 p_0 n/4 = \mu n \).

**Q2** For each \( v \in V(D) \), \( \diamond, \circ \in \{+, -\} \), and \( w \in A_{v,\diamond} \), \( |N^\circ_{H^\circ_{v,\diamond}}(w, V_i)| \geq \varepsilon p_n \).

**Q3** For each \( v \in V(D) \), \( \diamond, \circ \in \{+, -\} \), and \( w \in V(D) \), \( |N^\circ_{H^\circ_{v,\diamond}}(w, A_{v,\diamond}) \cap V_0| \leq (1 + \eta)\varepsilon^2 p_0 n/4 = (1 + \eta)\varepsilon \mu n \), where \( \diamond \in \{+, -\} \) is such that \( \circ \neq \diamond \).

Indeed, by Lemma 2.5 as \( \varepsilon, \eta, \alpha, p_0, p_1, \ldots, p_k \gg 1/n \), for any instance of \( v \in V(D) \), \( \diamond, \circ \in \{+, -\} \), and \( w \in V(D) \), the property **Q1** above holds with probability \( 1 - \exp(-\Omega(n)) \), and the same is true for **Q2** and **Q3**. Therefore, by a union bound, with high probability, the properties **Q1, Q2, and Q3** hold.

Now, for each \( v \in V(D) \) and \( \diamond \in \{+, -\} \), using **Q1** choose \( A_{v,\diamond} \subset A_{v,\diamond} \cap V_0 \) with \( |A_{v,\diamond}| = \mu n \). By **Q2** and **Q3** we have, for each \( \diamond \in \{+, -\} \) and \( i \in [k] \), that \( H^\diamond_{v,\diamond} \) is \((\varepsilon p_n, (1 + \eta)\varepsilon \mu n, \diamond)\)-skew-bounded on \((A_{v,\diamond}, V_i)\), as required.

We will now use the guide sets produced by Lemma 3.5 to randomly embed \( T_0 \), the small core of the original tree, and then use the guide graphs to find matchings from certain subsets of the image of the embedding to other random sets, as follows.

**Lemma 3.6.** Let \( 1/n \ll c < \beta \ll \varepsilon, q, \alpha \leq 1 \) and \( 1/n \ll c < p < 1/m \). Let \( D \) be an \( n \)-vertex digraph with \( \delta^0(D) \geq (1/2 + \alpha)n \).

Let \( T \) be an oriented tree with \( \Delta^\pm(T) \leq cn/\log n \) consisting of a subtree \( T_0 \subset T \) with \( |T_0| \leq \beta n \), such that every vertex in \( V(T) \setminus V(T_0) \) is attached as a leaf to \( T_0 \). Let \( t \in V(T_0) \). Let \( U_0 = V(T_0) \) and let \( U_1 \cup \ldots \cup U_m \) be a partition of \( V(T) \setminus V(T_0) \) such that, for each \( i \in [m] \), either \( e_T(V(T_0), U_i) = 0 \) or \( e_T(U_i, V(T_0)) = 0 \). Let \( V_0, V_1, \ldots, V_m \subset V(D) \) be disjoint random sets chosen uniformly at random subject to \( |V_0| = qn \), and, for each \( i \in [m] \), \( |V_i| = |(1 + \varepsilon)|U_i| + pn \).

Then, with high probability, for each \( s \in V_0 \), there is an embedding of \( T \) into \( D \) such that \( t \) is embedded to \( s \), and, for each \( i \in \{0, 1, \ldots, m\} \), \( U_i \) is embedded into \( V_i \).
Proof. Choose $\mu$ such that $c, \beta \ll \mu \ll \epsilon, q, \alpha$. For each $j \in [m]$, let $p_j = \left(\left(1 + \epsilon\right)|U_i|/n\right) + p$. Choose $V_0, V_1, \ldots, V_m$ according to the distribution in the lemma. By Lemma 3.3 with $p_0 = q$, with high probability, for each $v \in V(D)$ and $o \in \{+, -\}$, there is

R1 a set $A_{v,o} \subset N^2_0(v) \cap V_0$ with size $qa^2n/4$, and

R2 digraphs $H^o_{v,o} \subset D$, $o \in \{+, -\}$, such that, for each $j \in [m]$ and $o \in \{+, -\}$, $H^o_{v,o}$ is $(\mu p_j n, (1 + \epsilon/2)\mu qa^2n/4, o)$-skew-bounded on $(A_{v,o}, V_j)$.

We will now show that, given only R1 and R2 we can embed $T$ as required in the lemma for each $s \in V_0$. Let then $s \in V_0$. We will randomly embed $T_j$ into $D[V_0]$, as follows, before showing that, with positive probability, it can be extended into the required copy of $T$. Let $\ell = |T_0|$ and label $V(T_0) = \{t_1, \ldots, t_{\ell}\}$, so that $t_1 = t$ and $T_0([t_1, \ldots, t_{\ell}])$ is a tree for each $i \in [\ell]$. Let $s_1 = s$ and embed $t_1$ to $s_1$. For each $i \in \{2, \ldots, \ell\}$ in turn, let $j_i \in \{1, \ldots, \ell - 1\}$ be such that $t_{j_i}$ is the in- or out-neighbour of $t_i$ in $T_0([t_1, \ldots, t_{\ell}])$ and let $o_i \in \{+, -\}$ be such $t_i \in N^2_{T_0}(t_{j_i})$, and, uniformly at random, embed $t_i$ to $s_i \in A_{s_{j_i}, o_i} \setminus \{s_1, \ldots, s_{i-1}\}$.

Claim 3.7. For each $j \in [m]$, with high probability, the embedding of $T_0$ can be extended to an embedding of $T[V(T_0) \cup U_j]$ by embedding $U_j$ into $V_j$.

As $p \gg 1/n$, and $m \leq 1/p$, we can take a union bound over all $j \in [m]$, to show that, with positive probability, for each $j \in [m]$, the embedding of $T_0$ can be extended to $T[V(T_0) \cup U_j]$ by embedding $U_j$ into $V_j$, and hence $T$ can be embedded as required in the lemma. Therefore, there is some choice of the embedding of $T_0$ for which this can be done. It is left then to prove Claim 3.7.

Proof of Claim 3.7. Let $j \in [m]$ and let $o_j \in \{+, -\}$ be such that all the edges from $V(T_0)$ to $U_j$ in $T$ are $o_j$-edges. For each $i \in [\ell]$, let $d_{j,i}$ be the number of $o_j$-edges from $T_0$ to $U_j$ with $o_j = o_i$. For each $i \in [\ell]$, take $d_{j,i}$ new vertices and call them $w_{j,i, i'}, i' \in [d_{j,i}]$. Let $W_j = \{w_{j,i, i'} : i \in [\ell], i' \in [d_{j,i}]\}$. Let $K_j$ be the directed graph with vertex set $W_j \cup V_j$, containing only $o_j$-edges from $W_j$ to $V_j$, and where, for each $i \in [\ell], i' \in [d_{j,i}]$ and $v \in V_j$, there is an $o_j$-edge from $w_{j,i, i'}$ to $v$ in $K_j$ if, and only if, $s_i v \in E(H^o_{s_{j_i}, o_i})$.

We will show that, with high probability, $K_j$ is $(\mu p_j n, \mu p_j n, o_j)$-skew-bounded on $(W_j, V_j)$. This is enough to prove the claim, as, by Proposition 3.2, there is a perfect $o_j$-matching from $W_j$ into $V_j$ in $K_j$. Thus, we can label distinct vertices $s_{j,i, i'}$, $i \in [\ell], i' \in [d_{j,i}]$ in $V_j$ so that $w_{j,i, i'} \rightarrow v_{j,i, i'}$, $i \in [\ell]$ and $i' \in [d_{j,i}]$, is a matching in $K_j$. For each $i \in [\ell]$, use the vertices $v_{j,i, i'}$, $i' \in [d_{j,i}]$, to embed $d_{j,i}$ $o_j$-neighbours of $t_i$ in $U_j$ into $V_j$. This is possible as, by the definition of $K_j$ and $H^o_{s_{j_i}, o_i}$, $s_{j,i, i'}$ is an $o_j$-edge in $D$. Therefore, this extends the embedding of $T_0$ to an embedding of $T_0 \cup T[U_j]$ with $U_j$ embedded into $V_j$, as required.

Thus, it is sufficient to prove that, with high probability, $K_j$ is $(\mu p_j n, \mu p_j n, o_j)$-skew-bounded on $(W_j, V_j)$. Now, for each $i \in [\ell]$, $s_i \in A_{s_{j_i}, o_i}$, and therefore $s_i$ has at least $\mu p_j n$ $o_j$-neighbours in $V_j$ in $H^o_{s_{j_i}, o_i}$ by R2. Therefore, for each $i \in [\ell]$ and $i' \in [d_{j,i}]$, $w_{j,i, i'}$ has at least $\mu p_j n$ $o_j$-neighbours in $K_j$. Thus, each $v \in W_j$ has at least $\mu p_j n$ $o_j$-neighbours in $K_j$. Thus, letting $\overline{o_j} \in \{+, -\}$ with $\overline{o_j} \neq o_j$, it is sufficient to prove that, for each $v \in V_j$, with probability $1 - o(n^{-1})$, $d^{|j, i, \overline{o_j}}_{K_j} (v, W_j) \leq \mu p_j n$.

Let then $v \in V_j$. For each $i \in [\ell]$, let

$$X^{|j, i, v)} = \begin{cases} d_{j,i} & \text{if } s_i v \in E(H^o_{s_{j_i}, o_i}) \\ 0 & \text{otherwise,} \end{cases}$$

so that $d^{|j, i, v)}_{K_j} (v, W_j) = \sum_{i \in [\ell]} X^{|j, i, v)}$. Note that, when $s_i \in A_{s_{j_i}, o_i} \setminus \{s_1, \ldots, s_{i-1}\}$ is chosen uniformly at random, by R1 and R2 and as $\beta \ll \epsilon, \alpha, q$ and $i \leq \ell \leq \beta n$, if $d_{j,i} > 0$, then $X_i = d_{j,i}$ with
probability at most

\[
\frac{d^\ell_{H_{s_i}}(v)}{|A_{s_i}| \setminus \{s_1, \ldots, s_{i-1}\}} \leq \frac{(1 + \varepsilon/2)\mu q\alpha^2 n/4}{q\alpha^2 n/4 - (i - 1)} \leq (1 + \varepsilon)\mu.
\]

Let \( \gamma = (1 + \varepsilon)\mu \). Then, for each \( i \in [\ell] \), \( E(X_i^{j,v} | X_1^{j,v}, \ldots, X_{i-1}^{j,v}) \leq \gamma \cdot d_{j,i} \).

Let \( Y_i^{j,v} = 0 \) and, for each \( i \in [\ell] \), let \( Y_i^{j,v} = \sum_{1 \leq i' \leq i} (X_{i'}^{j,v} - \gamma d_{j,i}) \). Then, \( Y_0^{j,v}, Y_1^{j,v}, \ldots, Y_{i}^{j,v} \) is a supermartingale, since \( E[Y_{i+1}^{j,v} | Y_1, \ldots, Y_i] = E[X_i^{j,v} | Y_1, \ldots, Y_i] - \gamma d_{j,i+1} + Y_i^{j,v} \leq \gamma d_{j,i+1} - \gamma d_{j,i+1} + Y_i^{j,v} \) for each \( i \geq 0 \). Note further that \( |Y_{i+1}^{j,v} - Y_i^{j,v}| = |X_i^{j,v} - \gamma d_{j,i+1}| \leq d_{j,i+1} \) for each \( j \in [\ell - 1] \). Furthermore, as \( d_{j,i} \leq cn/\log n \) for each \( i \in [\ell] \), and \( \sum_{i \in [\ell]} d_{j,i} \leq |U_j| \leq n \), we have \( \sum_{i \in [\ell]} d_{j,i} \leq cn/\log n \). Therefore, by Azuma’s inequality (Lemma 2.6) with \( t = pn/3 \), and using that \( c \ll p \),

\[
\mathbb{P}(Y_{i}^{j,v} \geq pn/3) \leq 2 \exp(-p^2 n^2 \log n/9cn^2) = o(n^{-1}).
\]

Thus, with probability \( 1 - o(n^{-1}) \), we have \( Y_{i}^{j,v} < pn/3 \), so that

\[
d^\ell_{K_j}(v, W_j) = \sum_{i \in [\ell]} X_i^{j,v} = \gamma \cdot \left( \sum_{i \in [\ell]} d_{j,i} \right) + Y_{i}^{j,v} = \gamma |U_j| + Y_{i}^{j,v} < \gamma |U_j| + pn/3 \leq \gamma p_j n/(1 + \varepsilon) = (1 + \varepsilon)\mu p_j n/(1 + \varepsilon) = \mu p_j n,
\]

completing the proof of the claim, and hence the lemma.

Finally, by combining Lemma 3.6 and Lemma 2.13 we can prove Lemma 2.4.

**Proof of Lemma 2.4.** Let \( p \) satisfy \( 1/n \ll c \ll p \ll 1/K \). For each \( j \in [\ell] \), let \( s_j \) be the vertex of \( S_j \) with an in- or out-neighbour in \( V(T') \) in \( T \). Let \( \mathcal{R} \) be a maximal set of pairs \((R, r)\) for which \( R \) is a directed tree with at most \( K \) edges and \( r \in V(R) \), such that the pairs \((R, r)\) are unique up to isomorphism. Let \( m = |\mathcal{R}| \) and enumerate \( \mathcal{R} \) as \((R_1, r_1), \ldots, (R_m, r_m)\). Note that \( p \ll 1/m \).

Let \( T'' = T[V(T') \cup N^+_T(V(T'))] \cup N^-_T(V(T')) \). For each \( i \in [m] \) and \( \diamond \in \{+, -\} \), let \( U_{i,\diamond} \subset V(T'') \) be the set of vertices \( s_j, j \in [\ell] \), for which \((S_j, s_j)\) is isomorphic to \((R_i, r_i)\) and the edge from \( V(T') \) to \( s_j \) in \( T \) is a \( \diamond \)-edge.

In \( V(D) \), take disjoint random sets \( V_0 \) and \( V_{i,\diamond} \), \( i \in [m], \diamond \in \{+, -\} \) and \( j \in \{1, 2\} \), uniformly at random subject to the following.

- \( |V_0| = \varepsilon n/2 \).
- For each \( i \in [m] \) and \( \diamond \in \{+, -\} \), we have that \( |V_{i,\diamond,1}| = |(1 + \varepsilon/6)|U_{i,\diamond}| + pn \) and \( |V_{i,\diamond,2}| = |(1 + \varepsilon/6)|U_{i,\diamond}| + pn |(R_i - 1)\).

Note that this is possible, as

\[
|V_0| + \sum_{i \in [m], \diamond \in \{+, -\}} (|V_{i,\diamond,1}| + |V_{i,\diamond,2}|) = |V_0| + \sum_{i \in [m], \diamond \in \{+, -\}} (|(1 + \varepsilon/6)|U_{i,\diamond}| + pn |R_i| - 1).
\]

\[
\leq \varepsilon n/2 + (1 + \varepsilon/6) \sum_{j \in [\ell]} |S_j| + \sum_{i \in [m]} 2pn \cdot |R_i| \leq \varepsilon n/2 + (1 + \varepsilon/6)|T| + (2pn) \cdot m \cdot K \leq n.
\]
Now, with probability \( \varepsilon/2 \), \( v \in V_0 \). By Lemma 3.6, with high probability, if \( v \in V_0 \), then there is an embedding of \( T'' \) into \( D \) such that \( t \) is embedded to \( v, V(T') \subset V_0 \), and, for each \( i \in [m] \) and \( \phi \in \{+,-\}, U_{i,\phi} \) is embedded into \( V_{i,\phi} \). By Lemma 2.13 for each \( i \in [m] \) and \( \phi \in \{+,-\}, D[V_{i,\phi}1 \cup V_{i,\phi}2] \) contains \( V_{i,\phi}1 \) vertex disjoint copies of \( R_i \), in which \( r_i \) is copied into \( V_{i,\phi}1 \). For each \( i \in [m] \) and \( \phi \in \{+,-\}, \) add each copy of \( R_i \) containing an embedded vertex of \( U_{i,\phi} \) to the embedding of \( T'' \). Note that this results in a copy of \( T \).

3.2 Embedding constant-sized trees as paths

Given our decomposition \( T_0 \subset T_1 \subset T_2 \subset T_3 = T \), we have now embedded \( T_1 \). We now embed the vertices from \( V(T_2) \setminus V(T_1) \), recalling that we obtain \( T_2 \) from \( T_1 \) by adding constant-sized trees, where each tree is attached to \( T_1 \) by exactly two bare paths of length 2.

In the following lemma, we embed \( T_2 \setminus T_1 \) so that the vertices in \( V(T_2) \cap V(T_1) \) are embedded to preselected vertices (labelled \( a_i,b_i \), \( i \in [\ell] \)). This allows us to extend our embedding of \( T_1 \) to one of \( T_2 \).

**Lemma 3.8.** Let \( 1/n \ll 1/K \ll 1/k \ll \alpha, \varepsilon \). Suppose \( T \) is a forest formed of vertex-disjoint oriented trees \( T_i, i \in [\ell] \), with at most \( (1-\varepsilon)n \) vertices in total, and so that \( k \leq |T_i| \leq K \), for each \( i \in [\ell] \), and each tree \( T_i \) contains distinct vertices \( r_i \) and \( s_i \) which are leaves in \( T_i \) whose neighbour has total in- and out-degree 2.

Suppose \( D \) is an \( n \)-vertex digraph with \( \delta^0(D) \geq (1/2 + \alpha)n \), containing the distinct vertices \( a_i, b_i, i \in [\ell] \). Then, \( D \) contains a copy of \( T \) in which, for each \( i \in [\ell] \), \( r_i \) is embedded to \( a_i \) and \( s_i \) is embedded to \( b_i \).

**Proof.** Let \( \beta \) be such that \( 1/k \ll \beta \ll \alpha, \varepsilon \). For each \( i \in [\ell] \), let \( r_i' \) and \( s_i' \) be the neighbours in \( T_i \) of \( r_i \) and \( s_i \), respectively, and let \( T_i' = T_i - \{r_i', s_i', r_i, s_i\} \). Let \( T' \) be the forest composed of connected components \( T_i', i \in [\ell] \), so that \( |T'| \leq (1-\varepsilon)n \). Let \( A = \{a_i, b_i : i \in [\ell]\} \). Then \( |A| = 2\ell \leq 2n/k \). Let \( B \subset V(D) \setminus A \) be a random subset of vertices with \( |B| = \beta n \).

Let \( D' = D - A - B \). As \( 1/k, \beta \ll \alpha, \varepsilon \), we have \( |D'| \geq (1-\varepsilon/4)n \) and \( \delta^0(D') \geq (1/2 + \alpha/2)|D'| \).

Since
\[
|T'| \leq (1-\varepsilon)n \leq \frac{(1-\varepsilon)}{(1-\varepsilon/4)}|D'| \leq (1-\varepsilon/2)|D'|
\]
we have, by Lemma 2.14, with high probability we can find a copy, \( S' \), say, of \( T' \) inside \( D' \).

Let \( r_i'' \) and \( s_i'' \) be the neighbours in \( T' \) of \( r_i' \) and \( s_i' \), respectively, for each \( i \in [\ell] \), and let \( a_i'' \) and \( b_i'' \) be the copy of \( r_i'' \) and \( s_i'' \) in \( S' \), respectively.

**Claim 3.9.** The following holds with high probability. For any pair of vertices \( u, v \in V(D) \) and \( \phi, \psi \in \{+,-\}, \) we have that \( |N^\phi(u) \cap N^\psi(v) \cap B| \geq \alpha \beta n \).

**Proof of Claim 3.9.** Let \( u, v \in V(D) \) and \( \phi, \psi \in \{+,-\} \). Note that, by the semi-degree condition on \( D, |N^\phi(u) \cap N^\psi(v)| \geq 2an \), and hence \( |N^\phi(u) \cap N^\psi(v) \cap B| \) has a hypergeometric distribution with \( E |N^\phi(u) \cap N^\psi(v) \cap B| \geq 2\alpha \beta n \). By Lemma 2.5 and a union bound over all pairs \( u, v \in D \) and \( \phi, \psi \in \{+,-\}, \) the statement in the claim thus holds with probability \( 1 - o(1) \).

Thus, with high probability, we can assume the property in the claim holds. Now, for each \( i \in [\ell] \), embed \( a_i \) and \( b_i \) to \( r_i \) and \( s_i \), respectively. Let \( \phi_i, \phi_i', \phi_i'' \in \{+,-\} \) be such that \( r_i' \in N^\phi_i(r_i) \cap N^{\phi_i'}(r_i'') \), and \( s_i' \in N^{\phi_i}(s_i) \cap N^{\phi_i''}(s_i'') \). Greedily and disjointly, for each \( i \in [r] \), embed \( r_i' \) to a vertex in \( N^\phi_i(a_i) \cap N^{\phi_i'}(a_i'') \cap B \) and embed \( s_i' \) to a vertex in \( N^{\phi_i}(b_i) \cap N^{\phi_i''}(b_i'') \cap B \).

Note that this is possible, since, from the property in the claim we have, for each \( i \in [r] \)
\[
|N^\phi_i(a_i) \cap N^{\phi_i'}(a_i'') \cap B|, |N^{\phi_i}(b_i) \cap N^{\phi_i''}(b_i'') \cap B| \geq \alpha \beta n \geq \frac{2n}{k} \geq 2r.
\]
This completes the embedding of \( T \) with the property required in the lemma.

\[\square\]
3.3 Proof of Theorem 2.2

We now combine Lemma 2.4 and Lemma 3.8 to find a copy of any almost-spanning tree.

Proof of Theorem 2.2. Take $K,k$ and $\eta$ so that $1/n \ll 1/K \ll 1/k \ll \eta \ll \varepsilon, \alpha$. Let $D$ be an $n$-vertex graph with $\delta^0(D) \geq (1/2 + \alpha)n$. Let $T$ be an oriented tree on at most $(1 - \varepsilon)n$ vertices with $\Delta^+(T) \leq cn/\log n$. By Lemma 2.8 we can find forests $T_0 \subset T_1 \subset T_2 \subset T_3$ satisfying $\textbf{P1}$ to $\textbf{P3}$. Randomly partition $V(D)$ into three parts, $V(D) = V_1 \cup V_2 \cup V_3$ so that $|V_1| = |T_1| + \varepsilon n/3$, $|V_2| = |T_2| - |T_1| + \varepsilon n/3$, and $|V_3| = |T| - |T_2| + \varepsilon n/3$. Note that, with probability at least $\varepsilon/3$, we have $v \in V_1$.

By applying Lemma 2.10 with $A = V_1$, with high probability we have $\delta^0(D[V_1]) \geq (1/2 + \alpha/2)|V_1|$. Thus, by applying Lemma 2.4 to $D = D[V_1]$ and $T = T_1$, we can find a copy of $T_1$ in $V_1$ in which $t$ is copied to $v$. By $\textbf{P3}$ for some $\ell \in \mathbb{N}$, $T_2$ is formed from $T_1$ by the addition of trees $F_i$, $i \in [\ell]$, where $k \leq |F_i| \leq K$, which are each attached to $T_1$ by exactly two bare paths of length 2, $P_i$ and $Q_i$ say. For each $i \in [\ell]$, let $p_i$ and $q_i$ be the endpoint of $P_i$ and $Q_i$, respectively, which belongs to $T_1$. Let $a_i$ and $b_i$ be the embedding in $V_1$ of $p_i$ and $q_i$, respectively, and let $A = \{a_i, b_i : i \in [\ell]\}$.

By Lemma 2.10 again, we have, with high probability, $\delta^0(D[A \cup V_2]) \geq (1/2 + \alpha/2)|A \cup V_2|$. Applying Lemma 3.8 to $D[A \cup V_2]$ with $T_i = F_i \cup P_i \cup Q_i$, $r_i = p_i$, and $s_i = q_i$, for each $i \in [\ell]$, we can find a copy of $T_2$ in $D[V_1 \cup V_2]$. Now since $T_2$ is a tree, any vertex in $T_2 \setminus T_1$ can have at most one neighbour in $T_2$. Note that, by Lemma 2.10, we know that with high probability every vertex in $D$ has at least $(1/2 + \alpha/2)|V_3| \geq \eta n$ in-neighbours in $V_3$ and at least $(1/2 + \alpha/2)|V_3| \geq \eta n$ out-neighbours in $V_3$. Let $j = |T_3| - |T_2| \leq \eta n$ and order the vertices of $T_3 \setminus T_2$ by $u_1, \ldots, u_j$, so that $T[V(T_2) \cup \{u_1, \ldots, u_j\}]$ is a tree for each $i \in [j]$. Embed the vertices $u_1, \ldots, u_j$ greedily into $V_3$, to complete the copy of $T$ in $D$. Noting that this embedding was successful with probability at least $\varepsilon/3 - o(1) > 0$, there must always be such a copy of $T$. $\square$

4 Absorption from switching

The aim of this section is to prove Theorem 2.1. The main idea is as follows. Given a small tree $T$, we split it into two trees $T'$ and $T''$ and randomly embed $T'$ vertex by vertex. With positive probability, the resulting tree is such that, given the right number of other vertices in the graph, we can embed $T''$ to extend this into a copy of $T$ while making some small modifications to the copy of $T'$. Essentially, we show that, for each vertex $y$, there are many vertices in the embedding of $T'$ which we can switch with $y$ and still get a copy of $T$. We then embed $T''$ vertex-by-vertex, at each step switching an unused vertex into the copy of $T'$ in place of a vertex which we can instead use to extend the (partial) embedding of $T''$.

Proof of Theorem 2.1. Take $\lambda$ such that $\varepsilon \ll \lambda \ll \mu$. Using Proposition 2.3 let $T = T' \cup T''$, where $t \in V(T')$ and $\varepsilon n < |T''| \leq 3\varepsilon n$. Let $\ell = |T'|$, and label $V(T')$ as $t_1, \ldots, t_\ell$ so that $t_1 = t$, $T'[t_1, \ldots, t_\ell]$ is a tree for each $i \in [\ell]$, and the leaves of $T'$ appear last in this order (except for $t$) and in any bare path of length 6 the middle 3 vertices appear consecutively. For each $i \in [\ell]$, let $T_i = T'[\{t_1, \ldots, t_i\}]$.

Pick an arbitrary vertex $v \in V(D)$, and let $R_1$ be the graph with only the vertex $v$. For each $i = 2, \ldots, \ell$, do the following. Let $\phi_i \in \{+, -\}$ be such that $N^\phi_i(t_i)$ is non-empty (and thus contains exactly one vertex. Let $\phi_i \in \{+, -\}$ with $\phi_i \neq \phi_i$. Take $R_{i-1}$, which is a copy of $T_{i-1}$, and let $w_i$ be the copy of the sole vertex in $N^\phi_i(t_i)$ in $R_{i-1}$. Pick a vertex $v_i$ independently at random from $N^\phi_i(w_i) \setminus V(R_{i-1})$. Embed $t_i$ to $v_i$ to get $R_i$, a copy of $T_i$. 17
Note that this process always ends with a copy of $T'$, as $N^c_{D}(w_i) \setminus V(R_{i-1})$ always has size at least $d^c_{D}(w_i) - |T| \geq (1/2 + \alpha)n - \mu n$ and $\mu \ll \alpha$. Let $R = R_{\ell}$, so that $R$ is a copy of $T'$. We will show that, with positive probability the following property holds.

For each distinct $x, y \in V(D)$ and $\phi \in \{+,-\}$,

$$|\{i \in [\ell] : v_i \in N^c_{D}(x) \text{ and } N_{R}^{\phi}(v_i) \subset N^c_{D}(y)\}| \geq \gamma n.$$ 

Noting $|R| = |T'| \leq |T| - |T''| + 1 \leq (\mu - \epsilon)n$, let $A \subset V(D)$ contain $V(R)$ so that $|A| = (\mu - \epsilon)n$, and let $v$ be the copy of $t$. We will show in two claims that, with positive probability $S$ holds, and that, if $S$ holds, then $A$ and $v$ satisfy the property in the theorem. Thus, the theorem follows from these two claims.

**Claim 4.1.** With positive probability, $S$ holds.

**Proof of Claim 4.1.** Fix $x, y \in V(D)$ and $\phi \in \{+,-\}$ with $x \neq y$. We will show that $S$ holds for $x, y$ and $\phi$ with probability at least $1 - 1/4n^2$, so that the result follows by a union bound.

For convenience, let us take two cases. Either $T'$ has $2\mu^2 n$ leaves (Case I) or $\mu^2 n$ vertex-disjoint bare paths with length 6 (Case II). One of these cases must hold, as, suppose that Case I does not hold and thus $T'$ has fewer than $2\mu^2 n$ leaves. Then, by Lemma 2.7, we know that there is some $s$ and some vertex-disjoint bare paths $P_i \in [s]$, in $T'$ of length 6 so that $|T' - P_1 - \cdots - P_s| \leq 72\mu^2 n + 2\ell/7.\!$ Removing the internal vertices of each path $P_i \in [s]$, from $T'$ removes 5 vertices, and $|T'| = \ell$, so that $\ell - 5s \leq 72\mu^2 n + 2\ell/7$, and therefore

$$s \geq (\ell - 2\ell/7)/5 - 72\mu^2 n/5 \geq \ell/7 - 15\mu^2 n \geq (\mu - 3\epsilon)n/7 - 15\mu^2 n \geq \mu^2 n,$$

where the final inequality holds since $\epsilon \ll \mu$.

**Case I.** Assume that at least $\mu^2 n$ leaves of $T'$ are out-leaves, where the proof whenever $T'$ has at least $\mu^2 n$ in-leaves follows similarly. Let $\ell'$ be the smallest integer such that, for each $i > \ell'$, $t_i$ is a leaf of $T'$. We will analyse the embedding of $T'$ in two stages. First, for the embedding of $t_1, \ldots, t_{\ell'}$, we show that with high probability there will be plenty of these vertices which are adjacent to out-leaves in $t_{\ell'+1}, \ldots, t_{\ell}$ that are embedded to in-neighbours of $y$. Then, we will analyse the embedding of $t_{\ell'+1}, \ldots, t_{\ell}$, and show that plenty of these vertices whose in-neighbor in $t_1, \ldots, t_{\ell}$ was embedded to an in-neighbour of $y$ are themselves embedded to a $\diamond$-neighbour of $x$.

For each $i \in [\ell']$, let $c_i$ be the number of out-leaves of $t_i$ in $T'$. For each $i \in [\ell']$, let $X_i$ be the random variable which takes value $c_i$ if $v_i \in N^c_{D}(y)$, and 0 otherwise. Note that, for each $i \in [\ell]$, if $c_i > 0$, then, when the process selects $v_i$, having chosen $v_1, \ldots, v_{i-1}$, $X_i = c_i$ with probability at least

$$\frac{|(N^c_{D}(w_i) \setminus V(R_{i})) \cap N_{D}(y)|}{n} \geq \frac{|(N^c_{D}(w_i)) \cap N_{D}(y)| - |R_i|}{n} \geq \frac{2\alpha n - \mu n}{n} \geq \alpha,$$

as $\alpha \gg \mu$. Thus, for each $i \in [\ell']$, $\mathbb{E}[X_i | X_1, \ldots, X_{i-1}] \geq \alpha c_i$.

Note that $\sum_{i \in [\ell']} c_i$ is the number of out-leaves of $T'$, so that $\sum_{i \in [\ell']} c_i \geq \mu^2 n$ and, as $\Delta(T) \leq cn/\log{n}$, $\sum_{i \in [\ell']} c^2_i \leq cn^2 / \log{n}$. Let $Z_0 = 0$ and, for each $i \in [\ell']$, let $Z_i = \sum_{j \leq i} (X_j - \alpha c_j)$. Then, $(Z_i)_{i \geq 0}$ is a submartingale, since $\mathbb{E}[Z_{i+1} | Z_1, \ldots, Z_i] = Z_i + \mathbb{E}[X_{i+1} - \alpha c_{i+1} | X_1, \ldots, X_i] \geq Z_i$, for each $i \in [\ell']$. Furthermore, for each $i \in [\ell']$, we have $|Z_i - Z_{i-1}| = |X_i - \alpha c_i| \leq c_i$. Therefore, by Azuma’s inequality (Lemma 2.6) with $t = \alpha \mu^2 n/2$, we have

$$\mathbb{P} \left[ \sum_{i \in [\ell']} (X_i - \alpha c_i) \leq -t \right] \leq 2 \exp \left( \frac{-t^2}{\sum_{i \in [\ell']} c^2_i} \right) \leq 2 \exp \left( \frac{-t^2 \log{n}}{cn^2} \right) \leq \frac{1}{8n^2}.$$
Here, the final inequality holds because $c \ll \mu, \alpha$. Therefore, with probability at least $1 - 1/8n^2$, we have $\sum_{i \in [\ell]} X_i \geq \sum_{i \in [\ell]} \alpha c_i - \alpha \mu^2 n / 2 \geq \alpha \mu^2 n / 2$.

Let $m = \sum_{i \in [\ell]} X_i \geq \alpha \mu^2 n / 2$. Consider now the embedding of $t_{v_{i+1}}, \ldots, t_{l}$. Let $j_1, \ldots, j_m \in \{\ell + 1, \ldots, \ell\}$ be such that $t_{j_i}$ is an out-leaf of $T'$ and the image of $N_{T'}(t_{j_i})$ is an in-neighbour of $y$ for each $i \in [m]$. For each $i \in [m]$, let $Y_i$ be the random variable which takes value 1 if $v_{j_i}$ is in $N_D^-(x)$, and 0 otherwise. Note that, similarly to the calculation in (4), $\mathbb{E}[Y_i \mid Y_1, \ldots, Y_{i-1}] \geq \alpha$ for each $i \in [m]$. Let $Z_0 = 0$ and, for each $i \in [m]$, let $Z_i = \sum_{j \leq i} (Y_i - \alpha)$. Then, $(Z_i)_{i \geq 0}$ is a submartingale, since $\mathbb{E}[Z_{i+1} \mid Z_1, \ldots, Z_i] = Z_i + \mathbb{E}[Y_i - \alpha \mid Y_1, \ldots, Y_{i-1}] \geq Z_i$ for each $i \in [m]$. Furthermore, $|Z_i - Z_{i-1}| = |Y_i - \alpha| \leq 1$ for each $i \in [m]$. Therefore, by Azuma’s inequality (Lemma 2.6) with $t = \alpha m / 2$, we see that

$$\mathbb{P} \left[ \sum_{i \in [m]} Y_i - \alpha \leq -t \right] \leq 2 \exp \left( -\frac{t^2}{(1-\alpha)^2 m} \right) \leq \frac{1}{8n^2},$$

where the final inequality holds because $1/n \ll \mu, \alpha$. Hence, with probability at least $1 - 1/8n^2$, we have $\sum_{i \in [m]} Y_i \geq \alpha m / 2$. Note that $\{i \in [\ell] : v_i \in N_D^-(x) \text{ and } N_R^+(v_i) \subset N_D^+(y)\} \geq \sum_{i \in [m]} Y_i$.

Thus, by taking a simple union bound over the events in (5) and (6) and using $\lambda \ll \alpha, \mu$, we see that in total, with probability at least $1 - 1/4n^2$,

$$\{i \in [\ell] : v_i \in N_D^-(x) \text{ and } N_R^+(v_i) \subset N_D^+(y)\} \geq \alpha m / 2 \geq \lambda n.$$

Taking a union bound over all possible $x, y \in V(D)$ and $\phi \in \{+, -, \}$, we see that in this case holds with probability at least $1/2$.

**Case II.** Let $m = \mu^2 n$. Let $P_1, \ldots, P_m$ be vertex disjoint paths of length 6 in $T$, so that, if, for each $i \in [m]$, $j_i$ is such that $t_{j_i}$ is the middle vertex of $P_i$, then the vertices $t_{j_i}$ appear in order in $t_1, \ldots, t_\ell$.

For each $i \in [m]$, let $X_i$ be the random variable taking value 1 if

$$v_{j_i} \in N_D^-(x) \text{ and } N_R^+(v_{j_i}) \subset N_D^+(y)$$

and 0 otherwise. Note that, by virtue of the labelling of the $t_1, \ldots, t_\ell$, the vertices that appear in $N_R^+(v_{j_i})$ are exactly the vertices $v_{j_i-1}$ and $v_{j_i+1}$. When we choose each of $v_{j_i-1}, v_{j_i}, v_{j_i+1}$, the probability that it satisfies its condition in (7) (however the previous vertices $v_{j'}$ are chosen) is at least $\alpha$, in a calculation similar to (4). Therefore, we have, for each $i \in [m]$, that $\mathbb{E}[X_i \mid X_1, \ldots, X_{i-1}] \geq \alpha^3$.

Now, let $Z_0 = 0$ and, for each $i \in [m]$, let $Z_i = \sum_{j \leq i} (X_j - \alpha^3)$. Then, $\mathbb{E}[Z_{i+1} \mid Z_1, \ldots, Z_i] = Z_i + \mathbb{E}[X_{i+1} \mid X_1, \ldots, X_i] - \alpha^3 \geq Z_i$ for each $i \in [m]$, and thus $(Z_i)_{i \geq 0}$ is a submartingale. Furthermore, $|Z_i - Z_{i-1}| = |X_i - \alpha^3| \leq 1$ for each $i \in [m]$. Thus, by Azuma’s inequality (Lemma 2.6), letting $t = \alpha^3 m / 2$, we have

$$\mathbb{P}[Z_m \leq -t] \leq 2 \exp \left( -\frac{t^2}{m} \right) = 2 \exp \left( -\frac{\alpha^6 m}{4} \right) \leq \frac{1}{4n^2},$$

as $1/n \ll \alpha, \mu$. Therefore, with probability at least $1 - 1/4n^2$, we have $Z_m > -t$, so that, as $\lambda \ll \mu, \alpha,$

$$\{i \in [\ell] : v_i \in N_D^-(x) \text{ and } N_R^+(v_i) \subset N_D^+(y)\} \geq \{i \in [m] : v_{j_i} \in N_D^-(x) \text{ and } N_R^+(v_{j_i}) \subset N_D^+(y)\}$$

$$= \sum_{i \in [m]} X_i = Z_m + \alpha^3 m \geq \alpha^3 m - t \geq \lambda n.$$
Taking a union bound over all possible \( x, y \in V(D) \) and \( \diamond \in \{+,-\} \), we see that in this case holds with probability at least \( 1/2 \).

**Claim 4.2.** If \( \mathcal{S} \) holds then \( A \) and \( v \) satisfy the property in the theorem.

**Proof of Claim 4.2.** Let \( B \subset V(D) \) with \( A \subset B \) and \( |B| = \mu n \). Let \( k = |T'| - 1 \leq 3\varepsilon n \) and label the vertices of \( V(T') \setminus V(T) \) as \( s_1, \ldots, s_k \), so that, for each \( i \in [k] \), \( T'_i := T' \cup T'' \setminus \{s_1, \ldots, s_i\} \) is a tree. Note that \( |B \setminus V(R)| = k \) and label the vertices of \( B \setminus V(R) \) as \( y_1, \ldots, y_k \).

Let \( S_0 = R \). Now, for each \( i = 1, \ldots, k \) in turn, do the following. Let \( x_i \in V(S_{i-1}) \) and \( \alpha_i \in \{+,-\} \) be such that we need to add a \( \alpha_i \)-neighbour to \( x_i \) as a leaf to get a copy of \( T'_i \). Choose some \( j'_i \in \{\ell \} \setminus \{1, j'_1, \ldots, j'_{i-1}\} \) such that

\[
v_{j'_i} \in N^\pm_D(x_i) \quad \text{and} \quad N^\pm_{S_{i-1}}(v_{j'_i-1}) \subset N^\pm_D(y_i) \quad \text{and} \quad d^+_{S_{i-1}}(v_{j'_i}) + d^-_{S_{i-1}}(v_{j'_i}) \leq 4/\lambda.
\]

Replace \( v_{j'_i} \) with \( y_i \) in \( S_{i-1} \) and add \( v_{j'_i} \) as a \( \alpha_i \)-neighbour of \( x_i \) to get \( S_i \), a copy of \( T'_i \) with vertex sets \( V(S_{i-1}) \cup \{y_i\} \).

We need only show that there is such a vertex \( v_{j'_i} \) in each case, as if this process finds \( S_k \), then we have a copy of \( T'_k = T \). Fix then \( i \in [k] \). Note that there are at most \( (4/\lambda) \cdot 3\varepsilon n \leq \lambda n/4 \) vertices next to the vertices \( v_{j'_i}, \ldots, v_{j'_i} \) in \( R_{i-1} \), and therefore \( N^+_R(v_{j'_i}) = N^+_R(v_{j'_i}) \) and \( N^-_R(v_{j'_i}) = N^-_R(v_{j'_i}) \) for all but at most \( \lambda n/4 \) values of \( i' \in [\ell] \). Furthermore, as \( \sum_{i' \in [\ell]} (d^+_R(t_{v_{j'_i}}) + d^-_R(t_{v_{j'_i}})) \leq 2n \), at most \( \lambda n/2 \) values of \( i \in [k] \) can have \( d^+_{S_{i-1}}(v_{j'_i}) + d^-_{S_{i-1}}(v_{j'_i}) \leq 4/\lambda \). Thus, such an \( j'_i \) will always exist.

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**References**


