Interaction of two circular cylinders in inviscid fluid

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The unsteady problem of two parallel circular cylinders, moving in an inviscid fluid, is analyzed analytically by exploring a conformal mapping. Exact solutions are obtained for the flow induced and the hydrodynamic forces acting on them, as the two circular cylinders, with any radii and at any locations, expand (contract) and translate arbitrarily at time dependent speeds. As the two bodies are far apart, the solutions of the flow and forces reduce to those for a single circular cylinder moving in an unbounded inviscid flow. The force components along the line of centers are inversely proportional to the distance between them, whereas the force components perpendicular to the line of centers are inversely proportional to the square of the distance between them. Numerical analyses are performed for the two circular cylinders deforming and translating at constant speeds and in various ways, as well as a circular cylinder falling to a wall. It has been noticed that they are attracted to each other, as one of them expands and the other contracts, or as they translate perpendicular to the line of centers; whereas they are repelled from each other, as both of them expand, contract, or as they translate along the line of centers. The force on one of them increases with the size and speed of the other, as well as their proximity. [DOI: 10.1063/1.1804536]

I. INTRODUCTION

The hydrodynamic interaction between two bodies poses an interesting theoretical problem as well as one of practical importance, in the context of interactions of marine vessels and high-speed trains. The problem has been studied for a long time by using the potential flow theory, since it provides a good approximation for high Reynolds number flows (cf. Ref. 1). Owing to the difficult mathematical treatment for arbitrarily shaped bodies, theoretical studies have been mainly focused on bodies of simple geometries, such as spheres and circular cylinders. The axisymmetric potential flow of two spheres has been studied by Hicks,[2] Basset,[3] Herman,[4] Lamb,[5] Rouse,[6] Bentwich and Miloh,[7] and Sun and Chwang,[8] among others.

The mutual force between two pulsating spherical bubbles far apart was first studied by Bjerknes[9] and Bjerknes.[10] They observed that this mutual force, later on termed as the secondary Bjerknes force, caused the bubbles to either attract or repel each other depending upon whether the bubble pulsations were in phase or out of phase, respectively. Moreover, the magnitude of the force between the two bubbles, directed along the line connecting their centers, was found to be proportional to the inverse square of the distance between them.

Lagally[11] first considered the problem of the potential flow induced by two stationary circular cylinders in a uniform stream and he obtained the velocity potential of the fluid. The present work extends his theoretical research to the unsteady problem of two circular cylinders expanding and translating arbitrarily. The exact solution of the velocity potential is derived using the conformal mapping and Fourier series. The force acting on any one of the cylinders is obtained in a closed form too. Theoretical analyses are carried out for the asymptotic dynamic behaviors as the two circles are far apart. Numerical analyses are performed for the two circles expanding, contracting, translating along the line of centers, or translating perpendicular to the line, respectively. The falling of a circular cylinder to a wall is also analyzed.

II. EXACT FLOW SOLUTION

Consider two circles C_1 and C_2 with centers O, O_2, radii A_1, A_2, and center distance H. A coordinate system O-xyz fixed with body 1 is chosen with its origin at O, y axis along line O_2O of centers, and z axis perpendicular to the plane of the paper, as shown in Fig. 1(a). Let the two circles expand (contract) with speeds W_1 and W_2, translate with velocities (U_1,V_1) and (U_2,V_2), in an otherwise still fluid.

Assuming the fluid is inviscid and incompressible and the flow irrotational, a velocity potential $\varphi(x,y,t)$ exists in the fluid domain, external to the two bodies. The velocity potential φ satisfies the Neumann boundary value problem of two-dimensional Laplace's equation in a triply connected domain

$$\varphi_{xx}+\varphi_{yy}=0, \tag{1a}$$

$$-\varphi_n = W_1 + V_1 \sin\theta + U_1\cos\theta, \quad \text{on } C_1, \tag{1b}$$

$$-\varphi_n = W_2 + V_2 \sin\theta_2 + U_2\cos\theta_2, \quad \text{on } C_2, \tag{1c}$$

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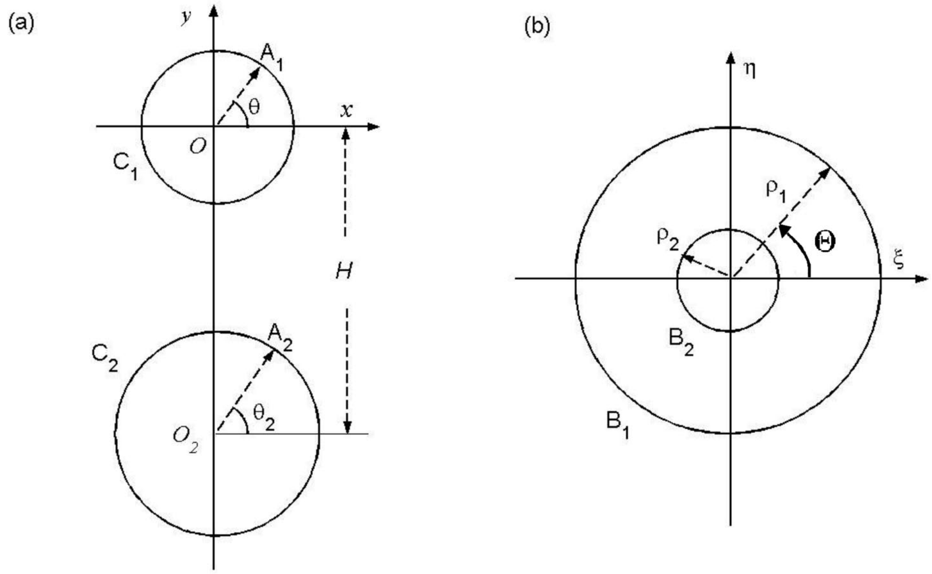

FIG. 1. The conformal mapping of (a) the domain outside two circles C_1 and C_2 in plane $Z=x+iy$ to (b) the domain between two concentric circles B_1 and B_2 in plane $\varsigma=\rho e^{i\Theta}$.

$$\lim_{x\to\infty} 2\pi r\varphi_r = 2\pi(A_1W_1 + A_2W_2), \tag{1d}$$

where $r=\sqrt{x^2+y^2}$, $\theta=tg^{-1}(y/x)$, and $\theta_2=tg^{-1}[(y+H)/x]$. $\mathbf{n}$ is the unit outward normal on the boundaries of the fluid domain. Boundary condition (1d) at infinity is given by conservation of mass.

Note that expanding speeds W_1 and W_2 are prescribed time dependent functions. Translation velocities U_1, U_2, V_1, and V_2 are given time dependent functions for prescribed motions. The dynamic problem of the two cylinders, under the influences of gravity, buoyancy, and hydrodynamic loads, are governed by Newton's second law,

$$U_{it} = \frac{1}{2\pi A_i^2}\frac{F_x^i}{\rho_s^i} + \left(1 - \frac{\rho_f}{\rho_s^i}\right)g\cos\gamma\sin\delta, \quad \text{for } i=1,2, \tag{2a}$$

$$V_{it} = \frac{1}{2\pi A_i^2}\frac{F_y^i}{\rho_s^i} - \left(1 - \frac{\rho_f}{\rho_s^i}\right)g\cos\gamma\cos\delta, \quad \text{for } i=1,2, \tag{2b}$$

where g is the gravitational acceleration; δ and γ are the angles between the gravity direction and y and z axes, respectively; ρ_f is the density of the fluid, and ρ_s^i is the average density of cylinder i; and (F_x^i, F_y^i) are the hydrodynamic forces on the cylinders.

To solve problem (1), we introduce a linear fractional conformal mapping between the physical plane of $Z=x+iy$ and the mapped plane of $\varsigma=\rho e^{i\Theta}$,

$$Z = iC\frac{\varsigma + C}{\varsigma - C} - iC\coth\alpha, \tag{3}$$

where, as usual, "coth" denotes the hyperbolic cotangent, and

$$C = \frac{\sqrt{(A_1^2+A_2^2-H^2)^2 - 4A_1^2A_2^2}}{2H}, \quad \alpha = \operatorname{asinh}(C/A_1). \tag{4}$$

"asinh" in (4) denotes inverse hyperbolic sine. Similarly, "sinh" and "cosh," to be used later on, denote hyperbolic sine and cosine, respectively.

The conformal mapping maps the domain outside two circles C_1, $|Z|=A_1$, and C_2, $|Z-Hi|=A_2$, in the plane Z to the domain between two concentric circles B_1, $|\varsigma|=\rho_1$, and B_2, $|\varsigma|=\rho_2$, in the plane ς, as sketched in Fig. 1(b). ρ_1 and ρ_2 are

$$\rho_1 = Ce^{\alpha}, \quad \rho_2 = Ce^{-\beta}, \quad \beta = \operatorname{asinh}(C/A_2). \tag{5}$$

To transform the boundary value problem of (1) in the physical plane Z to the mapped plane ζ, the following variables are needed:

$$\cos\theta|_{C_1} = 2e^{\alpha}\frac{\sinh\alpha\sin\Theta}{\Delta}, \qquad \sin\theta|_{C_1} = -2e^{\alpha}\frac{1-\cosh\alpha\cos\Theta}{\Delta}, \tag{6a}$$

$$\cos\theta_2|_{C_2} = 2e^{\beta}\frac{\sinh\beta\sin\Theta}{\Delta_2}, \qquad \sin\theta_2|_{C_2} = 2e^{\beta}\frac{1-\cosh\beta\cos\Theta}{\Delta_2}, \tag{6b}$$

$$J = \left|\frac{dZ}{d\varsigma}\right|_{C_1} = \frac{2}{\Delta}, \quad J_2 = \left|\frac{dZ}{d\varsigma}\right|_{C_2} = \frac{2}{\Delta_2}, \tag{6c}$$

where

$$\Delta = e^{2\alpha} - 2e^{\alpha}\cos\Theta + 1, \quad \Delta_2 = e^{2\beta} - 2e^{\beta}\cos\Theta + 1. \tag{7}$$

Point $\varsigma=C$ in the mapped plane corresponds to the point $Z=\infty$ in the physical plane, where it is a singular point source with the strength of $-2\pi(A_1W_1+A_2W_2)$, according to (1d). To simplify problem (1), we remove the contribution of the point source from φ. The volume flux of the fluid across the circular C_2 in the physical plane is at the value of $2\pi A_2W_2$, which remains at the same value in the mapped plane ζ. The point $\varsigma=0$ inside B_2 in the mapped plane corresponds to the point $Z=-iC(1+\coth\alpha)$ inside the circle C_2 in the physical plane. We further remove the potential due to a point source at $\varsigma=0$ with the strength of $2\pi A_2W_2$ from φ, which provides the volume flux of $2\pi A_2W_2$ across B_2 associated with the deforming of the circle C_2,

$$\varphi = A_2W_2 \ln \rho - (A_1W_1 + A_2W_2)\ln|\varsigma - C| + \phi. \tag{8}$$

The boundary problem of ϕ then becomes

$$\phi_{\rho\rho} + \frac{1}{\rho^2}\phi_{\Theta\Theta} = 0, \tag{9a}$$

$$\frac{\partial\phi}{\partial\rho} = \frac{A_1W_1 - A_2W_2}{Ce^{\alpha}}\frac{1 - e^{\alpha}\cos\Theta}{\Delta} + \frac{4V_1e^{\alpha}(1 - \cosh\alpha\cos\Theta)}{\Delta^2} - \frac{4U_1e^{\alpha}\sinh\alpha\sin\Theta}{\Delta^2}, \quad \text{on } B_1, \tag{9b}$$

$$\frac{\partial\phi}{\partial\rho} = \frac{A_1W_1 - A_2W_2}{Ce^{-\beta}}\frac{1 - e^{\beta}\cos\Theta}{\Delta_2} + \frac{4V_2e^{3\beta}(1 - \cosh\alpha\cos\Theta)}{\Delta_2^2} + \frac{4U_2e^{3\beta}\sinh\alpha\sin\Theta}{\Delta_2^2}, \quad \text{on } B_2. \tag{9c}$$

The right-hand sides of (9b) and (9c) can be further expanded as the Fourier series in Θ as follows:

$$\frac{\partial\phi}{\partial\rho} = -\frac{A_1W_1 - A_2W_2}{C}\sum_{n=1}^{\infty}\frac{\cos(n\Theta)}{e^{(n+1)\alpha}} - V_1\sum_{n=1}^{\infty} n\frac{\cos(n\Theta)}{e^{(n+1)\alpha}} - 2U_1\sum_{n=1}^{\infty} n\frac{\sin(n\Theta)}{e^{(n+1)\alpha}}, \quad \text{on } B_1, \tag{10a}$$

$$\frac{\partial\phi}{\partial\rho} = -\frac{A_1W_1 - A_2W_2}{C}\sum_{n=1}^{\infty}\frac{\cos(n\Theta)}{e^{(n-1)\beta}} - V_2\sum_{n=1}^{\infty} n\frac{\cos(n\Theta)}{e^{(n-1)\beta}} + 2U_2\sum_{n=1}^{\infty} n\frac{\sin(n\Theta)}{e^{(n-1)\beta}}, \quad \text{on } B_2, \tag{10b}$$

by using the following integration formulas:

$$\int_0^{2\pi}\frac{\cos n\Theta d\Theta}{a^2 - 2a\cos\Theta + 1} = \frac{2\pi}{a^n(a^2-1)}, \quad a > 1, \tag{11a}$$

$$\int_0^{2\pi}\frac{\cos n\Theta d\Theta}{(a^2 - 2a\cos\Theta + 1)^2} = \frac{2\pi}{a^n(a^2-1)^3}[2 + (n+1)(a^2 - 1)], \quad a > 1. \tag{11b}$$

Assume the solution of (9) takes the form

$$\phi = \sum_{n=1}^{\infty}(A_n\rho^n + B_n\rho^{-n})\cos(n\Theta) + (C_n\rho^n + D_n\rho^{-n})\sin(n\Theta).$$

Determining its coefficients with (9b), (9c), and (10), and substituting it into (8), we obtain

$$\varphi = \varphi_1 + \varphi_2 + \varphi_3, \tag{12a}$$

$$\varphi_1 = A_2W_2\ln\rho - (A_1W_1 + A_2W_2)\ln(\rho^2 - 2\rho C\cos\Theta + C^2) - (A_1W_1 - A_2W_2)\sum_{n=1}^{\infty}\frac{1}{n}\frac{\cos(n\Theta)}{\sinh(n\alpha + n\beta)} \times\left[\sinh(n\beta)\left(\frac{\rho}{\rho_1}\right)^n - \sinh(n\alpha)\left(\frac{\rho_2}{\rho}\right)^n\right], \tag{12b}$$

$$\varphi_2 = -2C\sum_{n=1}^{\infty}\frac{\cos(n\Theta)}{1 - e^{-2n(\alpha+\beta)}}\left[e^{-n\alpha}(V_1 - V_2e^{-2n\beta})\left(\frac{\rho}{\rho_1}\right)^n - e^{-n\beta}(V_2 - V_1e^{-2n\alpha})\left(\frac{\rho_2}{\rho}\right)^n\right], \tag{12c}$$

$$\varphi_3 = -2C\sum_{n=1}^{\infty}\frac{\sin(n\Theta)}{1 - e^{-2n(\alpha+\beta)}}\left[e^{-n\alpha}(U_1 + U_2e^{-2n\beta})\left(\frac{\rho}{\rho_1}\right)^n + e^{-n\beta}(U_2 + U_1e^{-2n\alpha})\left(\frac{\rho_2}{\rho}\right)^n\right]. \tag{12d}$$

Solutions φ_1, φ_2, and φ_3 are the potentials for two circular cylinders expanding at speeds W_1 and W_2, translating along the line of centers at velocities V_1 and V_2, and translating perpendicular to the line at velocities U_1 and U_2, respectively, in an otherwise still fluid. The three series in (12) are absolutely convergent in the whole flow domain corresponding to $\rho_2 \leqslant \rho \leqslant \rho_1$ and $0 \leqslant \Theta \leqslant 2\pi$.

We further consider a special case where the two circles and their motions are identical. As $A_1=A_2$, and $W_1=W_2$, (12b) is simplified as follows:

$$\begin{aligned}\varphi_1 &= A_2W_2\ln\rho - 0.5(A_1W_1 + A_2W_2)\times\ln(\rho^2 - 2\rho C\cos\Theta + C^2)\\ &= 0.5A_2W_2\ln\frac{\rho^2}{\rho^2 - 2\rho C\cos\Theta + C^2} - 0.5A_1W_1\ln(\rho^2 - 2\rho C\cos\Theta + C^2).\end{aligned} \tag{13}$$

Using the conformal mapping of (4), we have

$$|Z + iC(\coth\alpha - 1)| = \left|iC\left(\frac{\varsigma + C}{\varsigma - C} - 1\right)\right| = \left|iC\frac{2C}{\varsigma - C}\right| = \frac{2C^2}{\sqrt{\rho^2 - 2\rho C\cos\Theta + C^2}}, \tag{14a}$$

$$|Z + iC(\coth\alpha + 1)| = \left|iC\left(\frac{\varsigma + C}{\varsigma - C} + 1\right)\right| = \left|iC\frac{2\varsigma}{\varsigma - C}\right| = \frac{2C\rho}{\sqrt{\rho^2 - 2\rho C\cos\Theta + C^2}}. \tag{14b}$$

Substituting the above formula into (13), we have

$$\varphi_1 = A_1W_1\ln|Z + iC(\coth\alpha - 1)| + A_2W_2\ln|Z + iC(\coth\alpha + 1)| + \text{const}. \tag{15}$$

Thus, the velocity potentials for the two identical circular cylinders deforming at the same speed can be expressed in terms of two point sources located at $Z_{1,2}=-iC(\coth\alpha\mp 1)$, and at the same strength of $2\pi A_1W_1$. It is easy to verify using (12b) that, in general, (15) stands for two circles at different radii satisfying $A_1W_1=A_2W_2$.

As $A_1=A_2$, $U_1=U_2$, and $V_1=V_2$, (12c) and (12d) are simplified as

$$\varphi_2=-CV_1\sum_{n=1}^{\infty}\frac{\cos(n\Theta)}{\cosh(n\alpha)}\left[\left(\frac{\rho}{\rho_1}\right)^n-\left(\frac{\rho_2}{\rho}\right)^n\right], \tag{16a}$$

$$\varphi_3=-CU_1\sum_{n=1}^{\infty}\frac{\sin(n\Theta)}{\sinh(n\alpha)}\left[\left(\frac{\rho}{\rho_1}\right)^n+\left(\frac{\rho_2}{\rho}\right)^n\right]. \tag{16b}$$

When the two circles are far apart, $H\rightarrow\infty$, the exact solutions of the velocity potential of (12b)–(12d), in the flow field not far away from the circle C_1, can be simplified as follows (cf. Appendix A):

$$\lim_{H\rightarrow\infty}\varphi_1=A_1W_1\ln|Z|, \tag{17a}$$

$$\lim_{H\rightarrow\infty}\varphi_2=-\frac{A_1^2V_1y}{r^2}, \tag{17b}$$

$$\lim_{H\rightarrow\infty}\varphi_3=-\frac{A_1^2U_1x}{r^2}. \tag{17c}$$

As expected, the limits of (17a)–(17c) are the velocity potentials of a single circular cylinder deforming at the speed of W_1, and moving at the velocities of V_1 and U_1 along y and x axes, respectively, in an unbounded inviscid fluid.

III. FORCE FORMULA

The pressure field can be obtained from the Bernoulli equation

$$\frac{p}{\rho_f}=-\varphi_t-\frac{1}{2}(\varphi_x^2+\varphi_y^2). \tag{18}$$

Only the force acting on body 1 is considered, as the force on body 2 can be obtained simply by rotating the parameters between them. The force components F_x and F_y on body 1 are

$$\frac{F_x}{\rho_f}=-\oint_{c_1}p\cos\theta\,dl=\left(\frac{d}{dt}-\frac{W_1}{A_1}\right)\oint_{c_1}\varphi\cos\theta\,dl+\frac{1}{2}\oint_{c_1}(\varphi_x^2+\varphi_y^2)\cos\theta\,dl, \tag{19a}$$

$$\frac{F_y}{\rho_f}=-\oint_{c_1}p\sin\theta\,dl=\left(\frac{d}{dt}-\frac{W_1}{A_1}\right)\oint_{c_1}\varphi\ \sin\theta\,dl+\frac{1}{2}\oint_{c_1}(\varphi_x^2+\varphi_y^2)\sin\theta\,dl. \tag{19b}$$

Notice that the time dependency of C_1 has been considered in the first terms on the right-hand side of (19a) and (19b). Taking the first term in (19a) as an example, we have

$$\begin{aligned}\oint_{c_1}\varphi_t\cos\theta\,dl&=\int_0^{2\pi}\varphi_t\cos\theta A_1d\theta\\&=\int_0^{2\pi}[(A_1\varphi)_t-A_{1t}\varphi]\cos\theta\,d\theta\\&=\left(\frac{d}{dt}-\frac{A_{1t}}{A_1}\right)\int_0^{2\pi}A_1\varphi\cos\theta\,d\theta\\&=\left(\frac{d}{dt}-\frac{W_1}{A_1}\right)\oint_{c_1}\varphi\cos\theta\,dl.\end{aligned} \tag{20}$$

Denoting the four integrals in (19a) and (19b) as $I_1(t)$, $I_2(t)$, $J_1(t)$, and $J_2(t)$,

$$I_1(t)=\oint_{c_1}\varphi\cos\theta\,dl,\ \ I_2(t)=\frac{1}{2}\oint_{c_1}(\varphi_x^2+\varphi_y^2)\cos\theta\,dl, \tag{21a}$$

(19a) and (19b) become

$$\frac{F_x}{\rho_f}=I_{1t}-\frac{W_1}{A_1}I_1+I_2, \tag{22a}$$

$$\frac{F_y}{\rho_f}=J_{1t}-\frac{W_1}{A_1}J_1+J_2. \tag{22b}$$

$I_1(t)$, $I_2(t)$, $J_1(t)$, and $J_2(t)$ can be integrated analytically with the following results (cf. Appendix B):

$$I_1=4\pi C\sum_{n=1}^{\infty}nb_ne^{-2n\alpha}, \tag{23a}$$

$$I_2=\frac{\pi}{\rho_1}\sinh\alpha\left(2c_0d_1e^{-\alpha}+\sum_{n=1}^{\infty}f_ne^{-2n\alpha}\right), \tag{23b}$$

$$J_1=4\pi C\sum_{n=1}^{\infty}na_ne^{-2n\alpha}, \tag{23c}$$

$$J_2=\frac{\pi}{C}\left(-c_0^2+2c_0c_1e^{-\alpha}\cosh\alpha_1+2\sum_{n=1}^{\infty}g_ne^{-2n\alpha}\right), \tag{23d}$$

where

$$a_n = \frac{A_1W_1 + A_2W_2}{2n} - \frac{A_1W_1 - A_2W_2}{2n}\frac{1 - 2e^{-2n\beta} + e^{-2n(\alpha+\beta)}}{1 - e^{-2n(\alpha+\beta)}} - C\frac{V_1 - 2V_2e^{-2n\beta} + V_1e^{-2n(\alpha+\beta)}}{1 - e^{-2n(\alpha+\beta)}}, \tag{24a}$$

$$b_n = -C\frac{U_1 + 2U_2e^{-2n\beta} + U_1e^{-2n\alpha-2n\beta}}{1 - e^{-2n(\alpha+\beta)}}, \tag{24b}$$

$$c_n = -(A_1W_1 + nCV_1), \tag{24c}$$

$$d_n = -nCU_1, \tag{24d}$$

$$f_n = -n(n+1)(a_{n+1}b_n - a_nb_{n+1}) + (c_nd_{n+1} - c_{n+1}d_n), \tag{24e}$$

$$g_n = e^{-\alpha}\cosh\alpha[n(n+1)(a_na_{n+1} + b_nb_{n+1}) + c_nc_{n+1}] - [n^2(a_n^2 + b_n^2) + c_n^2 + d_n^2]. \tag{24f}$$

I_{1t} and J_{1t}, needed in (22a) and (22b), can be obtained from (23a) and (23b):

$$I_{1t} = 4\pi\sum_{n=1}^{\infty} n(C_tb_n + Cb_{nt} - 2nC\alpha_tb_n)e^{-2n\alpha}, \tag{25a}$$

$$J_{1t} = 4\pi\sum_{n=1}^{\infty} n(C_ta_n + Ca_{nt} - 2nC\alpha_ta_n)e^{-2n\alpha}. \tag{25b}$$

C_t, α_t needed in (25) and β_t can be obtained from (4) and (5),

$$C_t = -\frac{CH_t}{H} + \frac{(A_1^2 + A_2^2 - H^2)(A_1W_1 + A_2W_2 - HH_t) - 2A_1W_1A_2^2 - 2A_2W_2A_1^2}{2H^2C}, \tag{26a}$$

$$\alpha_t = \frac{C_tA_1 - CW_1}{A_1^2\cosh\alpha}, \quad \beta_t = \frac{C_tA_2 - CW_2}{A_2^2\cosh\beta}, \tag{26b}$$

where $H_t = V_1 - V_2$. a_{nt} and b_{nt} can then be obtained straightforwardly from (24a) and (24b), which contains the time derivatives of U_{1t}, V_{1t}, W_{1t}, U_{2t}, V_{2t}, and W_{2t}. As an illustration, b_{nt} is

$$b_{nt} = -C_t\frac{U_1 + 2U_2e^{-2n\beta} + U_1e^{-2n\alpha-2n\beta}}{1 - e^{-2n(\alpha+\beta)}} - 2nC(\alpha_t + \beta_t)e^{-2n(\alpha+\beta)}\frac{U_1 + 2U_2e^{-2n\beta} + U_1e^{-2n\alpha-2n\beta}}{(1 - e^{-2n(\alpha+\beta)})^2} - C\frac{U_{1t} + 2U_{2t}e^{-2n\beta} - 4n\beta_tU_2e^{-2n\beta} + U_{1t}e^{-2n\alpha-2n\beta} - 2n(\alpha_t + \beta_t)U_1e^{-2n\alpha-2n\beta}}{1 - e^{-2n(\alpha+\beta)}}. \tag{27}$$

The six series in (23a)–(23d) and (25a) and (25b) are absolutely convergent.

As $U_1 = U_2 = V_1 = V_2 = 0$, and $A_1W_1 = A_2W_2$, the force acting on circle 1 can be simplified as follows from (22)–(24):

$$\frac{F_y}{\rho_f} = 2\pi\frac{d(A_1^2W_1e^{-\alpha})}{dt} - \frac{\pi}{C}(A_1W_1)^2. \tag{28}$$

As two moving circles are far apart, the asymptotic behaviors of the force acting on the circle C_1 have been derived in Appendix C as follows:

$$\lim_{H\to\infty}\frac{F_x}{\rho_f} = \left(\frac{d}{dt} - \frac{W_1}{A_1}\right)\lim_{H\to\infty} I_1 + \lim_{H\to\infty} I_2 = -\pi A_1^2U_{1t} + \left\{2\pi\left(\frac{W_1}{A_1}U_2A_1^2A_2^2 - \frac{d}{dt}(U_2A_1^2A_2^2)\right) + \frac{\pi}{2}A_1^2U_1(A_1W_1 - A_2W_2)\right\}\frac{1}{H^2} + O(H^{-3}), \tag{29a}$$

$$\lim_{H\to\infty}\frac{F_y}{\rho_f} = \left(\frac{\partial}{\partial t} - \frac{W_1}{A_1}\right)\lim_{H\to\infty} J_1 + \lim_{H\to\infty} J_2 = -\pi A_1^2V_{1t} + 2\pi\left(\frac{d}{dt}(A_1^2A_2W_2) - \frac{W_1}{A_1}A_1^2A_2W_2\right)\frac{1}{H} + \left\{2\pi\left(\frac{d}{dt}(A_1^2A_2^2V_2) - \frac{W_1}{A_1}A_1^2A_2^2V_2 - A_1^2A_2W_2H_t\right) + \pi U_1A_1^2(-A_1W_1 + A_2W_2)\right\}\frac{1}{H^2} + O(H^{-3}). \tag{29b}$$

A few conclusions can be drawn from the above formula. First, as the two circles are far apart, the limits of the forces reduce to that acting on a single circular cylinder accelerating in an unbounded inviscid fluid:

$$F_x = -\pi\rho_fA_1^2U_{1t}, \quad F_y = -\pi\rho_fA_1^2V_{1t}. \tag{30}$$

Second, the dynamic influences of the circle C_2 far away to the circle C_1 are the first order of magnitude $O(H^{-1})$, in the direction along the line of centers, and are the second order of magnitude $O(H^{-2})$, in the direction perpendicular to the line of centers. The first-order dynamic forces are due to the deformations of the two circles, and the second-order forces are due to the nonlinear coupling of their deformations and translations. As two circular cylinders far apart are only in translations, the forces between them are the third-order small quantities of $O(H^{-3})$.

As the centers of the two circles are at fixed locations, the force on circle 1 due to the deforming of circle 2 is simplified as

$$F_y = 2\pi\rho_f\left(\frac{d}{dt}(A_1^2A_2W_2) - A_1A_2W_1W_2\right)\frac{1}{H} + O(H^{-3}). \quad (31)$$

When the two circles undergo small-amplitude radical harmonic pulsations at the same frequency, i.e.,

$$A_i(t) = A_{i0}[1 + \delta_i \cos(\omega t + \psi_i)], \quad i = 1,2, \quad (32)$$

with $\delta_i \ll 1$, where ω is the pulsation frequency, and ψ_1 and ψ_2 are their initial phases. Substituting (32) into (31), and averaging over one cycle, one can obtain the averaging force as follows:

$$\langle F_y \rangle = -\frac{2\pi\rho_f\omega^2A_{10}A_{20}}{H}\delta_1\delta_2\cos(\psi_1 - \psi_2) + O(H^{-3}). \quad (33)$$

It can be seen from (33) that the two circles attract each other when their pulsations are in phase, and vice versa the two circles repel each other when their pulsations are out of phase. This is a further affirmation of the presence of a Bjerknes-type effect.

Blake, Taib, and Doherty[12] developed the Kelvin impulse criterion for bubbles using the concept of the Bjerknes force, which has been widely used in predicting the directions of the motion of bubbles and the liquid jets formed in bubbles collapsing (cf. Refs. 13–17). The Bjerknes effect has also been developed and applied widely in predicting the coupled pulsation and translation of two gas bubbles in an acoustic wave.[18–22] Those studies are based on the approximate analytical solution of the velocity potential for two oscillating spheres far apart. The exact solutions of the velocity potential for two moving circles obtained here can be used to provide a more accurate Kelvin impulse criterion for cylindrical-type bubbles, as well as an accurate Bjerknes force for predicting the coupled pulsation and translation of two cylindrical-type bubbles in an acoustic wave.

IV. DYNAMIC ANALYSES

The forces on the circular cylinders given in Sec. III are calculated using the MATLAB. The series in (23) and (25) converge rapidly, since their terms decay exponentially. The calculations have been performed at a very high accuracy with the series truncated when the terms are at $O(10^{-8})$, since the CPU time needed is minimal. The computational results are given in terms of the scaled distance of centers H^* and scaled force F_y^* as follows:

$$H^* = \frac{H}{A}, \quad F_y^* = \frac{F_y}{\rho_f V^2 A^2}, \quad V = \sqrt{U_1^2 + V_1^2 + W_1^2}. \quad (34)$$

We first consider the case where only one of the two circles is moving. Figure 2 shows the scaled force on circle 1 versus the distance of centers as one of the two circles is (a) expanding (contracting), (b) translating along the line of centers, and (c) translating perpendicular to the line. As circle 1 is stationary, the referenced velocity V in (23) is chosen as $V=\sqrt{U_2^2+V_2^2+W_2^2}$. The ratio of their radii is $A_2/A_1=0.5$, 1.0, 2.0. Note that circle 1 at $A_2/A_1=0.5$, 1.0, 2.0 corresponds to circle 2 at $A_2/A_1=2.0$, 1.0, 0.5, respectively. Circle 1 experiences repellence as it alone deforms [Fig. 2(a)]. As circle 2 alone deforms, circle 1 experiences attraction as the clearance between them is not small; however, it experiences repellence as the clearance is small. They are repelled from each other as one of them translates along the line of centers [Fig. 2(b)], whereas they are attracted to each other as one of them translates perpendicular to the line [Fig. 2(c)]. The forces on the two circles increase with their proximity. The force on the stationary circle appears larger than that on the moving one, but the difference becomes small when the clearance between them is small. This is consistent with the attraction (repellency) phenomenon for a vessel passing by (approaching to) a stationary vehicle observed in experiments by Dand.[23] He also noticed that the force on the stationary body is larger than that on the moving one.

We next consider the case where both of the two circles are moving. Figure 3 shows the scaled force on circle 1 versus the distance of centers as the two circles deform at $W_2/W_1=-2$, -1, 0, 1, and 2. They are attracted to each other as one expands and the other contracts (dashed line), and conversely they are repelled from each other as both of them expand or contract (solid line). The force on circle 1 increases with the deforming speed of circle 2 and their proximity. The mechanism underlying the phenomenon can be explained as follows. Being an axisymmetric case, the flow velocity along the axis of symmetry is along the axis. As both of them expand (contract), the flow velocity along the axis of symmetry changes direction from one circle to the other. A stagnation point thus occurs on the axis of symmetry between the two circles, and a high-pressure region is thus formed over there, which repels both of the two circles. When one circle expands and the other contracts, the fluid flow between the two circles is pushed by the expanding circle and attracted by the contracting one, and is consequently faster than the flow beside them, which is mainly produced by one of them only.

Figure 4 shows the force on circle 1 versus the distance of centers as they both either expand or contract at $W_2/W_1=1$ and $A_2/A_1=0.5$, 1.0, 2.0 (dashed line), and as one expands and the other contracts at $W_2/W_1=-1$ and $A_2/A_1=0.5$, 1.0, 2.0 (solid line). As expected, the magnitude of the force on body 1 increases with the size of body 2.

Figure 5(a) shows the scaled force on circle 1 versus the distance of centers as the two circles translate along the line of centers at $V_2/V_1=-2$, -1, 0, 1, and 2. The force is always repellence, no matter if they approach to $(V_2/V_1<0)$ or depart from $(V_2/V_1>0)$ each other. The repellence increases with the relative velocities between them and their proximity. This phenomenon reduces head-on collisions between floating bodies. To interpret this phenomenon, we choose an in-

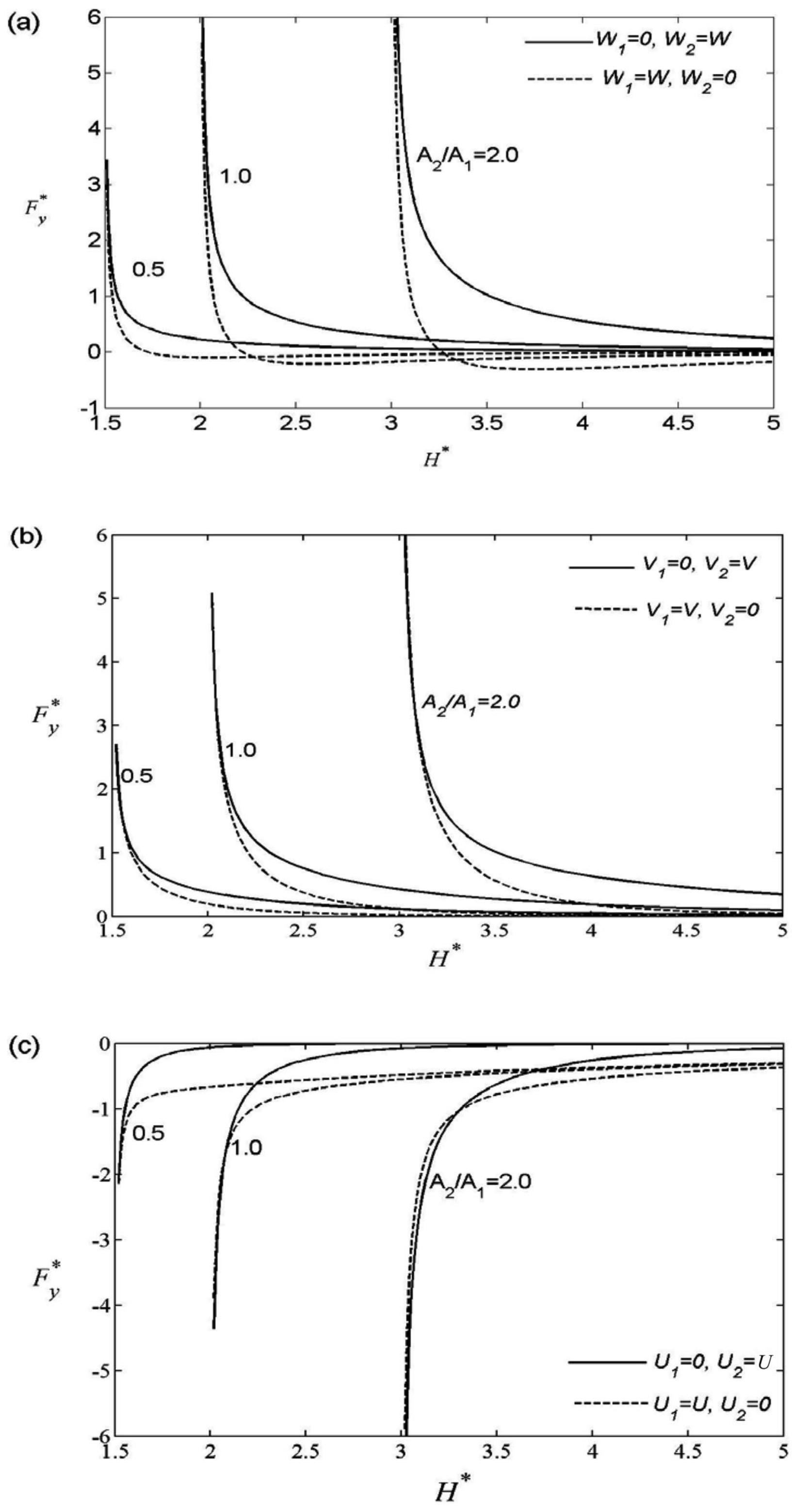

FIG. 2. The force on circle 1 vs the distance of centers as one of the two circles (a) expands, (b) translates along the line of centers, and (c) translates perpendicular to the line of centers. The ratio of their radii is $A_2/A_1 = 0.5$, 1.0, 2.0.

ertial coordinate system moving at the speed of $(V_1+V_2)/2$, assuming the two circles are moving at constant speeds. In this system, the two circles move in contrary direction, at the velocities of $(V_1-V_2)/2$ and $-(V_1-V_2)/2$, respectively, and the flow velocity along the axis of symmetry changes direction from one circle to the other. A stagnation point is thus formed on the axis of symmetry between the two circles. Since the Bernoulli equation stands in any inertial coordinate system, a high-pressure region occurs near the stagnation point, which repels both the circles.

Figure 5(b) shows the scaled force on circle 1 versus the distance of centers as the two circles translate perpendicular to the line of centers at $U_2/U_1=-2, -1, 0, 1$, and 2. The force is attraction, which increases with their proximity. This is because the flow passing through the two circles is restricted by both of them and thus moves faster than the flow beside

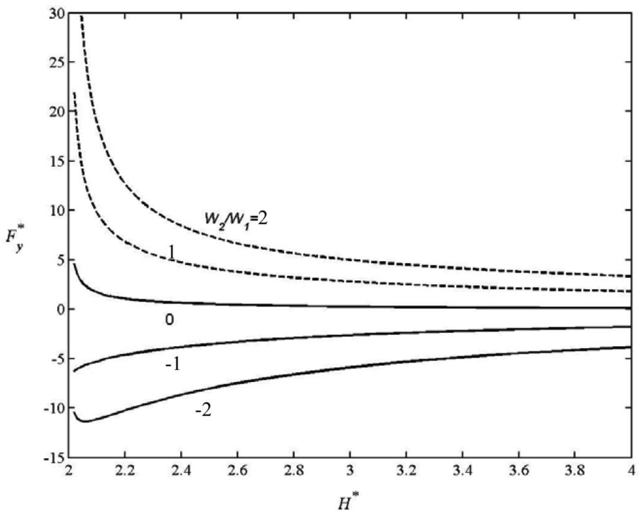

FIG. 3. The force on circle 1 vs the distance of centers as the two circles expand (contract) at $W_2/W_1=-2$, -1, 0, 1, and 2.

them restricted only by one of them. The attraction is larger as the two circles translate in the same direction than in opposite directions.

At last, we consider a circular cylinder falling to a wall, under gravity, buoyancy, and hydrodynamic loads, which can be analyzed with the exact solutions obtained here using the image method. Falling velocity V_1 is obtained by integrating (2b), with $\delta=\gamma=0$. Figures 6(a) and 6(b) show the scaled velocity and center height versus time for the fall at $\rho_s^1/\rho_f=1.05$, and started with its center at a height of $2A_1$ from the wall at zero velocity. The length scale is chosen as A_1 and the time scale as $\sqrt{A_1/g}$. We also depict the corresponding results for the fall without the hydrodynamic load for comparison. The falling cylinder is slowed down apparently by the repellence of the wall at the later stage, and the moment of the cylinder at impact to the wall is reduced by about one-third by the hydrodynamic load. Unlike a body moving in an unbounded inviscid fluid with the added mass being a constant, the added mass in this problem increases with time. Similar phenomena were observed for a sphere falling to a wall by Milne-Thomson,[24] and a two-dimensional flat plate falling to a wall by Yih.[25]

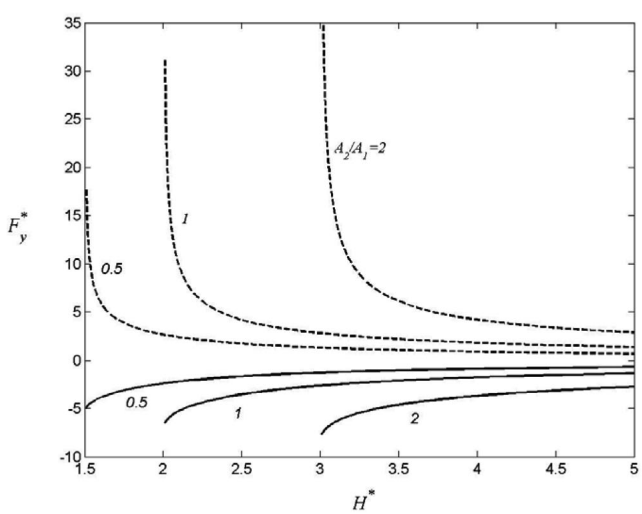

FIG. 4. The force on circle 1 vs the distance of centers as the two circles expand (contract) at $W_2/W_1=1$ and $A_2/A_1=0.5$, 1.0, 2.0 (dashed line), and at $W_2/W_1=-1$ and $A_2/A_1=0.5$, 1.0, 2.0 (solid line).

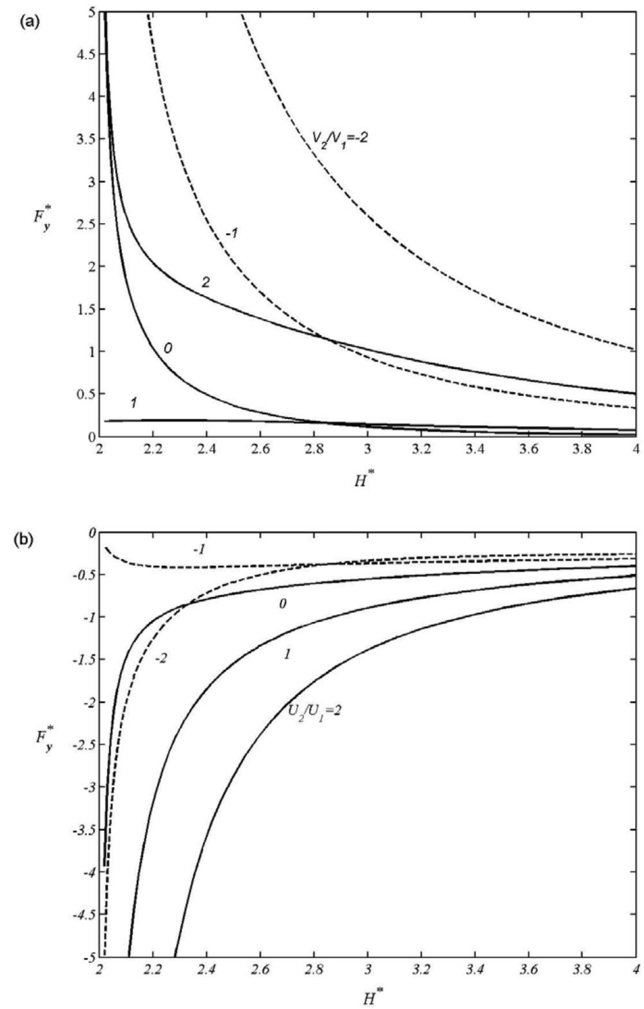

FIG. 5. The force on circle 1 vs the distance of centers as (a) the two circles translate along the line of centers at $V_2/V_1=-2$, -1, 0, 1, and 2; (b) the two circles translate perpendicular to the line of centers at $U_2/U_1=-2$, -1, 0, 1, and 2.

V. CONCLUSIONS

The unsteady problem of two parallel circular cylinders moving and deforming in an inviscid fluid is analyzed analytically, by using a linear fractional conformal mapping that transforms two arbitrary circles to concentric circles. Exact solutions are obtained for the flow induced and the hydrodynamic forces acting on them, as the two circular cylinders, with any radii and at any locations, expand (contract) and translate arbitrarily at time dependent speeds.

As the two bodies are far apart, the center distance $H\to\infty$, the exact solutions of the flow and forces obtained reduce to those for a single circular cylinder in an unbounded inviscid fluid. The dynamic forces on the two bodies far apart are the first order of magnitude $O(H^{-1})$ in the direction along the line of centers, and are the second order of magni-

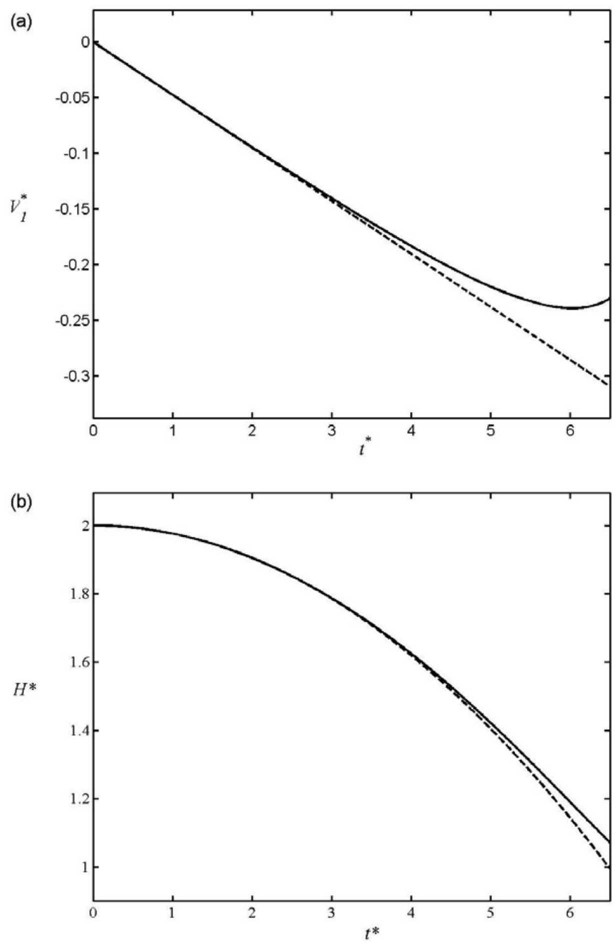

FIG. 6. The scaled velocity and center height vs time for fall of a circular cylinder to a wall at $\rho_s/\rho_f=1.05$, compared to the results without dynamic loads (dashed line).

tude $O(H^{-2})$ in the direction perpendicular to the line of centers. The first-order dynamic forces are due to the deformations of the two circles, and the second-order forces are due to the nonlinear coupling of their deformations and translations. As two circular cylinders far apart are in translations, the forces between them are the third-order small quantities of $O(H^{-3})$.

As the two circular cylinders at fixed positions oscillate at the same frequency, the averaging force cause them to either attract or repel each other depending upon whether their pulsations are in phase or out of phase, respectively. This is analogous to the Bjerknes-type force between the two pulsating spherical bubbles noticed by Bjerknes[9] and Bjerknes.[10] The force between the two oscillating spherical bubbles is inversely proportional to the square of the distance between them; whereas the force between the two oscillating circular cylinders is inversely proportional to the distance between them.

Numerical analyses are performed for the two circular cylinders deforming and translating at constant speeds and in various ways, as well as a circular cylinder falling to a wall. It has been noticed that the two circular cylinders are attracted to each other, as one of them expands and the other contracts, or as they translate perpendicular to the line of centers; whereas they are repelled from each other, as both of them expand, contract, or as they translate along the line of centers. The repellence is larger as they translate in opposite directions along the line of centers than that in the same direction. Conversely, the attraction is larger as they translate in the same direction perpendicular to the line of centers than in opposite direction. The force on one of them increases with the size and speed of the other, as well as their proximity. It has also been noticed that a circular cylinder falling to a wall is slowed down significantly by the presence of the wall.

The exact solutions obtained here can be applied to analyze two nearly parallel slender bodies of revolution in close interaction using the method of matched asymptotic expansions, which is similar to the two-dimensional cross-flow solutions deployed in the classical slender body theory. The solutions can be used to provide an accurate Kelvin impulse criterion for cylindrical-type bubbles in predicting the directions of the motion of bubbles and the liquid jets formed in bubbles collapsing, refering Blake, Taib, and Doherty.[12] The solutions can also be used to provide an accurate Bjerknes force for predicting the coupled pulsation and translation of two cylindrical-type bubbles in an acoustic refering, Harkin, Kaper, and Nadin,[21] etc.

ACKNOWLEDGMENTS

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APPENDIX A: THE LIMIT OF THE VELOCITY POTENTIAL SOLUTIONS AS THE TWO CIRCLES ARE FAR APART

In this appendix, we calculate the limits of the velocity potential solutions of (12) as the two circles are far apart. It can be obtained from the conformal mapping (4) and (5) that, as $H\to\infty$, we have

$$C = 0.5H + O(H^{-1}), \quad \alpha = \ln\frac{H}{A_1} + O(H^{-2}),$$
$$\beta = \ln\frac{H}{A_2} + O(H^{-2}), \tag{A1}$$

$$\rho_1 = 0.5H^2/A_1 + O(1), \quad \rho_2 = 0.5A_2 + O(1). \tag{A2}$$

As $H\to\infty$, and $|Z|=O(A_1)$, i.e., for the flow field not far away from the circle C_1, the conformal mapping (3) becomes

$$W \to 0.5i\frac{H^2}{Z} \quad \text{or}$$
$$\rho(\cos\Theta + i\sin\Theta) \to 0.5\frac{H^2}{r}(\sin\theta + i\cos\theta). \tag{A3}$$

Therefore,

$$\rho \cos \Theta \to 0.5\frac{H^2}{r} \sin \theta, \quad \rho \sin \Theta \to 0.5\frac{H^2}{r} \cos \theta, \tag{A4}$$

$$O(\rho) = \frac{H^2}{r}. \tag{A5}$$

Using (A1), (A2), and (A5), we have the following estimation:

$$\begin{aligned} \phi_0 &= \lim_{H\to\infty}\sum_{n=1}^{\infty}\frac{1}{n}\frac{\cos(n\Theta)}{\sinh(n\alpha+n\beta)} \\ &\quad\times\left[\sinh(n\beta)\left(\frac{\rho}{\rho_1}\right)^n - \sinh(n\alpha)\left(\frac{\rho_2}{\rho}\right)^n\right] \\ &= \lim_{H\to\infty}\sum_{n=1}^{\infty}\frac{1}{n}\cos(n\Theta)e^{-n\alpha}\left(\frac{\rho}{\rho_1}\right)^n. \end{aligned} \tag{A6}$$

Introduce

$$b = e^{-\alpha}\frac{\rho}{\rho_1}. \tag{A7}$$

It is easy to check that $0 \leqslant b < 1$, as $H \to \infty$ and $|Z| = O(A_1)$. Using the following formulas:

$$\begin{aligned} &\sum_{n=1}^{\infty}\frac{1}{n}b^n \cos(n\Theta) \\ &\quad = -0.5 \ln(b^2 - 2b \cos \Theta + 1), \quad \text{as } 0 \leqslant b < 1, \end{aligned} \tag{A8}$$

(A6) becomes

$$\begin{aligned} \phi_0 &= \lim_{H\to\infty}\sum_{n=1}^{\infty}\frac{1}{n}\cos(n\Theta)e^{-n\alpha}\left(\frac{\rho}{\rho_1}\right)^n \\ &= \lim_{H\to\infty}\sum_{n=1}^{\infty}\frac{1}{n}b^n \cos(n\Theta) \\ &= -0.5\lim_{H\to\infty} \ln(b^2 - 2b \cos \Theta + 1) = 0, \end{aligned} \tag{A9}$$

since $b \to 0$, as $H \to \infty$ and $|Z| = O(A_1)$.

Substituting (A9) into (12b), one has

$$\begin{aligned} \lim_{H\to\infty} \varphi_1 &= \lim_{H\to\infty}[A_2W_2 \ln \rho - 0.5(A_1W_1 + A_2W_2) \\ &\qquad \times\ln(\rho^2 - 2\rho C \cos \Theta + C^2)] \\ &= \lim_{H\to\infty}\left(0.5A_2W_2 \ln \frac{\rho^2}{\rho^2 - 2\rho C \cos \Theta + C^2} \right.\\ &\qquad \left. - 0.5A_1W_1 \ln(\rho^2 - 2\rho C \cos \Theta + C^2)\right). \end{aligned} \tag{A10}$$

Using (A1), (A5), and (A10) becomes

$$\begin{aligned} \lim_{H\to\infty} \varphi_1 &= -0.5A_1W_1 \lim_{H\to\infty} \ln(\rho^2 - 2\rho C \cos \Theta + C^2) \\ &= -A_1W_1 \lim_{H\to\infty} \ln \rho = A_1W_1 \ln|r|. \end{aligned} \tag{A11}$$

(12c) can be estimated as follows using (A1), (A2), and (A5):

$$\lim_{H\to\infty} \varphi_2 = -2CV_1 \lim_{H\to\infty}\sum_{n=1}^{\infty} b^n \cos(n\Theta). \tag{A12}$$

By using the following formulas:

$$\frac{1-b^2}{b^2 - 2b \cos \Theta + 1} = 1 + 2\sum_{n=1}^{\infty} b^n \cos(n\Theta), \quad 0 \leqslant b < 1, \tag{A13}$$

one can obtain

$$\begin{aligned} \lim_{H\to\infty} \varphi_2 &= 2 \lim_{H\to\infty} CV_1\frac{b^2 - b \cos \Theta}{b^2 - 2b \cos \Theta + 1} \\ &= -2V_1 \lim_{H\to\infty} Cb \cos \Theta. \end{aligned} \tag{A14}$$

Substituting (A2), (A4), and (A7) into (A14), we have

$$\begin{aligned} \lim_{H\to\infty} \varphi_2 &= -2V_1 \lim_{H\to\infty} bC \cos \Theta \\ &= -2V_1 \lim_{H\to\infty} e^{-\alpha}\frac{\rho}{\rho_1}C \cos \Theta \\ &= -V_1 \lim_{H\to\infty} e^{-2\alpha}\rho \cos \Theta \\ &= -2V_1 \lim_{H\to\infty}\left(\frac{A}{H}\right)^2 0.5\frac{H^2}{r} \sin \theta \\ &= -\frac{A_1^2V_1}{r}\sin \theta = -\frac{A_1^2V_1y}{r^2}. \end{aligned} \tag{A15}$$

Similarly, using (A1), (A2), and (A5), the limit of φ_3 of (12c) can be expressed as

$$\lim_{H\to\infty} \varphi_3 = -2CU_1 \lim_{H\to\infty}\sum_{n=1}^{\infty} b^n \sin(n\Theta). \tag{A16}$$

By using the following formulas:

$$\frac{\sin \Theta}{b^2 - 2b \cos \Theta + 1} = \frac{1}{b}\sum_{n=1}^{\infty} b^n \sin(n\Theta), \quad 0 \leqslant b < 1, \tag{A17}$$

one can obtain

$$\lim_{H\to\infty} \varphi_3 = -2U_1 \lim_{H\to\infty} bC \sin \Theta = -\frac{A_1^2U_1}{r} \cos \theta = -\frac{A_1^2U_1x}{r^2}. \tag{A18}$$

As expected, the limits of the velocity potentials of (12a)–(12c) given in (A11), (A15), and (A18) are the velocity potentials of a single circular cylinder deforming at the speed of W_1, and moving at the velocities V_1 and U_1 along y and x axes, respectively, in an unbounded inviscid fluid.

APPENDIX B: THE DERIVATION OF (23)

We first transform the four integrals in (21) along the circle C_1 in the physical plane Z to that along the circle B_1, $|s|=\rho_1$, in the transformed plane ζ:

$$I_1 = \oint_{C_1} \varphi \cos\theta\, dl = \oint_{B_1} \varphi \cos\theta J\, dl, \tag{B1a}$$

$$I_2 = \frac{1}{2}\oint_{C_1} (\varphi_x^2 + \varphi_y^2)\cos\theta\, dl = \frac{1}{2}\oint_{B_1}\left(\varphi_\rho^2 + \frac{1}{\rho^2}\varphi_\Theta^2\right)\cos\theta J^{-1} dl, \tag{B1b}$$

$$J_1 = \oint_{C_1} \varphi \sin\theta dl = \oint_{B_1} \varphi \sin\theta J\, dl, \tag{B1c}$$

$$J_2 = \frac{1}{2}\oint_{C_1} (\varphi_x^2 + \varphi_y^2)\sin\theta\, dl = \frac{1}{2}\oint_{B_1}\left(\varphi_\rho^2 + \frac{1}{\rho^2}\varphi_\Theta^2\right)\sin\theta J^{-1} dl. \tag{B1d}$$

We next expand φ, φ_Θ, and $\rho\varphi_\rho$ on B_1 in terms of the Fourier series in Θ using (12),

$$\varphi|_{B_1} = -A_1 W_1 \ln\rho_1 + 2\sum_{n=1}^{\infty}[a_n\cos(n\Theta) + b_n \sin(n\Theta)]e^{-n\alpha}, \tag{B2a}$$

$$\varphi_\Theta|_{B_1} = 2\sum_{n=1}^{\infty}[-na_n\sin(n\Theta) + nb_n\cos(n\Theta)]e^{-n\alpha}, \tag{B2b}$$

$$\rho\varphi_\rho|_{B_1} = c_0 + 2\sum_{n=1}^{\infty}[c_n\cos(n\Theta) + d_n\sin(n\Theta)]e^{-n\alpha}, \tag{B2c}$$

where a_n, b_n, c_n, and d_n, are given in (24a)–(24c).

To calculate I_1 and J_1, we express $J\cos\theta$ and $J\sin\theta$ on B_1 as

$$(J\cos\theta)|_{B_1} = \frac{2e^{2\alpha}\sinh\alpha\sin\theta}{(e^{2\alpha} - 2e^{\alpha}\cos\Theta + 1)^2} = \sum_{n=1}^{\infty} n\frac{\sin(n\Theta)}{e^{n\alpha}}, \tag{B3a}$$

$$(J\sin\theta)|_{B_1} = -\frac{2e^{\alpha} - (e^{2\alpha}+1)\cos\Theta}{(e^{2\alpha} - 2e^{\alpha}\cos\Theta + 1)^2} = \frac{1}{e^{\alpha}}\sum_{n=1}^{\infty} n\frac{\cos(n\Theta)}{e^{n\alpha}}, \tag{B3b}$$

by using (6), (7), and (11b). Substituting (B2a) and (B3a) into (B1a), we have

$$\begin{aligned} I_1 &= \oint_{B_1}\varphi J\cos\theta\, dl \\ &= 2e^{-\alpha}\rho_1\int_0^{2\pi}\left(\sum_{n=1}^{\infty} ne^{-n\alpha}\sin(n\Theta)\right) \\ &\quad\times\left(-A_1W_1\ln\rho_1 + 2\sum_{n=1}^{\infty}[a_ne^{-n\alpha}\cos(n\Theta) + b_ne^{-n\alpha}\sin(n\Theta)]\right)d\Theta \\ &= 4C\int_0^{2\pi}\left(\sum_{n=1}^{\infty} nb_ne^{-2n\alpha}\sin^2(n\Theta)\right)d\Theta \\ &= 4\pi C\sum_{n=1}^{\infty} nb_ne^{-2n\alpha}. \end{aligned} \tag{B4}$$

In the above equation, we have used the following integral formula:

$$\begin{aligned} &\int_0^{2\pi}\sin(n\Theta)\cos(m\Theta)d\Theta = 0, \\ &\int_0^{2\pi}\sin(n\Theta)\sin(m\Theta)d\Theta = \pi\delta_{nm}, \quad \text{for } n,m \geqslant 1, \end{aligned} \tag{B5}$$

where δ_{nm} is the Kronecker delta, i.e., $\delta_{\mathrm{nm}}=0$ as $n\neq m$, and $\delta_{\mathrm{nm}}=1$ as $n=m$.

Similarly, substituting (B2a) and (B3b) into (B1a), one can obtain J_1 as follows:

$$\begin{aligned} J_1 &= \oint_{B_1}\varphi J\sin\theta\, dl \\ &= 2C\int_0^{2\pi}\left(\sum_{n=1}^{\infty} ne^{-n\alpha}\cos(n\Theta)\right)\left(-A_1W_1\ln\rho_1 + 2\sum_{n=1}^{\infty}[a_ne^{-n\alpha}\cos(n\Theta) + b_ne^{-n\alpha}\sin(n\Theta)]\right)d\Theta \\ &= 4\pi C\sum_{n=1}^{\infty} na_ne^{-2n\alpha}. \end{aligned} \tag{B6}$$

To calculate I_2 and J_2, we need to calculate their integrants. Using (6) and (7), we have

$$(J^{-1}\cos\theta)|_{B_1} = e^{\alpha}\sinh\alpha\sin\Theta, \tag{B7a}$$

$$(J^{-1}\sin\theta)|_{B_1} = -e^{\alpha}(1 - \cosh\alpha\cos\Theta). \tag{B7b}$$

Using (B2b) and (B2c), we have

$$\begin{aligned}(\rho_1^2\varphi_\rho^2+\varphi_\Theta^2)|_{B_1} &= c_0^2+4c_0c_1e^{-\alpha}\cos\Theta+4c_0d_1e^{-\alpha}\sin\Theta\\ &\quad+2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}e^{-(n+m)\alpha}\{(-nma_nb_m+c_md_n)\\ &\quad\times[\sin((n+m)\Theta)+\sin((n-m)\Theta)]\\ &\quad+(nmb_nb_m+c_nc_m)[\cos((n+m)\Theta)\\ &\quad+\cos((n-m)\Theta)]+(nma_na_m+d_nd_m)\\ &\quad\times[\cos((n-m)\Theta)-\cos((n+m)\Theta)]\}.\end{aligned} \tag{B8}$$

Substituting (B7) and (B8) into (B1b) and (B1c), we obtain

$$\begin{aligned}I_2 &= \frac{1}{2}\oint_{B_1}\left(\varphi_\rho^2+\frac{1}{\rho^2}+\varphi_\Theta^2\right)\cos\theta J^{-1}dl\\ &= \frac{1}{2\rho_1}\int_0^{2\pi}(\rho_1^2\varphi_\rho^2+\varphi_\Theta^2)\cos\theta J^{-1}d\Theta\\ &= \frac{1}{2\rho_1}\int_0^{2\pi}e^{\alpha}\sinh\alpha\sin\Theta\Bigg(4c_0d_1e^{-\alpha}\sin\Theta\\ &\quad+2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-nma_nb_m+c_md_n)\\ &\quad\times e^{-(m+n)\alpha}\sin((n-m)\Theta)\Bigg)d\Theta\\ &= \frac{1}{\rho_1}\int_0^{2\pi}e^{\alpha}\sinh\alpha\Bigg(2c_0d_1e^{-\alpha}\sin^2\Theta\\ &\quad+\sin^2\Theta e^{-\alpha}\sum_{n=1}^{\infty}f_ne^{-2n\alpha}\Bigg)d\Theta\\ &= \frac{\pi}{\rho_1}\sinh\alpha\left(2c_0d_1+\sum_{n=1}^{\infty}f_ne^{-2n\alpha}\right),\end{aligned} \tag{B9a}$$

$$\begin{aligned}J_2 &= \frac{1}{2}\oint_{B_1}\left(\varphi_\rho^2+\frac{1}{\rho^2}+\varphi_\Theta^2\right)\sin\theta J^{-1}dl\\ &= \frac{1}{2\rho_1}\int_0^{2\pi}(\rho_1^2\varphi_\rho^2+\varphi_\Theta^2)\sin\theta J^{-1}d\Theta\\ &= \frac{1}{2\rho_1}\int_0^{2\pi}e^{\alpha}(-1+\cosh\alpha\cos\Theta)\\ &\quad\times\Bigg(c_0^2+4c_0c_1e^{-\alpha}\cos\Theta+2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(nma_na_m\\ &\quad+nmb_nb_m+c_nc_m+d_nd_m)\\ &\quad\times e^{-(m+n)\alpha}\cos((n-m)\Theta)\Bigg)d\Theta\\ &= \frac{\pi}{C}\left[-c_0^2+2c_0c_1e^{-\alpha}\cosh\alpha+2\sum_{n=1}^{\infty}g_ne^{-2n\alpha}\right],\end{aligned} \tag{B9b}$$

where f_n and g_n are given in (24e) and (24f). The following integral formula have been used in the above equations:

$$\int_0^{2\pi}\sin\Theta\cos((n\pm m)\Theta)d\Theta=0,\qquad \int_0^{2\pi}\sin\Theta\sin((n+m)\Theta)d\Theta=0, \tag{B10a}$$

$$\int_0^{2\pi}\cos\Theta\sin((n\pm m)\Theta)d\Theta=0,\qquad \int_0^{2\pi}\cos\Theta\cos((n+m)\Theta)d\Theta=0, \tag{B10b}$$

$$\int_0^{2\pi}\sin\Theta\sin((n-m)\Theta)d\Theta=\begin{cases}0, & \text{as } |n-m|\neq 1\\ \pi, & \text{as } n=m+1\\ -\pi, & \text{as } n=m-1,\end{cases} \tag{B10c}$$

$$\int_0^{2\pi}\cos\Theta\cos((n-m)\Theta)d\Theta=\begin{cases}0, & \text{as } |n-m|\neq 1\\ \pi, & \text{as } n=m\pm 1,\end{cases} \tag{B10d}$$

for $n,m\geqslant 1$.

APPENDIX C: THE ASYMPTOTIC BEHAVIORS OF THE FORCES AS THE TWO MOVING CIRCLES FAR APART

In this section, we consider the asymptotic behaviors of the forces between two moving circles far apart. As $H\to\infty$, using (A1) and (A2), a_n, b_n, c_n, d_n in (24) are reduced to

$$\lim_{H\to\infty}a_n=\frac{A_2W_2}{n}-CV_1+V_2\frac{A_2^2}{H}+O(H^{-2}), \tag{C1}$$

$$\lim_{H\to\infty}b_n=-CU_1-U_2\frac{A_2^2}{H}+O(H^{-2}), \tag{C2}$$

$$c_n=-(A_1W_1+nCV_1),\quad d_n=-nCU_1. \tag{C3}$$

Substituting (C1)–(C3) into (24e) and (24f), one can obtain

$$\lim_{H\to\infty}f_n=CU_1(A_1W_1-A_2W_2)+O(1), \tag{C4}$$

$$\lim_{H\to\infty}g_n=0.5CU_1(-A_1W_1+A_2W_2)+O(1). \tag{C5}$$

Substituting (C1)–(C5) and (A1) into (23a)–(23d), one has

$$\lim_{H\to\infty} I_1 = 4\pi \lim_{H\to\infty} C\sum_{n=1}^{\infty} nb_n e^{-2n\alpha}$$
$$= 4\pi \lim_{H\to\infty} C\left(-CU_1 - U_2\frac{A_2^2}{H} + O(H^{-2})\right)e^{-2\alpha}$$
$$= -4\pi \lim_{H\to\infty} C^2 e^{-2\alpha}U_1 - 2\pi U_2\frac{A_1^2A_2^2}{H^2} + O(H^{-3})$$
$$= -\pi A_1^2U_1 - 2\pi U_2\frac{A_1^2A_2^2}{H^2} + O(H^{-3}), \quad \text{(C6)}$$

$$\lim_{H\to\infty} I_2 = \lim_{H\to\infty}\frac{\pi}{2C}\left(2A_1W_1CU_1 + CU_1(A_1W_1 - A_2W_2)\sum_{n=1}^{\infty} e^{-2n\alpha}\right) + O(H^{-3})$$
$$= \lim_{H\to\infty}\frac{\pi}{2C}\left(2A_1W_1CU_1 + CU_1(A_1W_1 - A_2W_2)\frac{A_1^2}{H^2}\right) + O(H^{-3})$$
$$= \pi A_1W_1U_1 + \frac{\pi}{2}U_1(A_1W_1 - A_2W_2)\frac{A_1^2}{H^2} + O(H^{-3}), \quad \text{(C7)}$$

$$\lim_{H\to\infty} J_1 = 4\pi \lim_{H\to\infty} C\sum_{n=1}^{\infty} na_n e^{-2n\alpha}$$
$$= 4\pi \lim_{H\to\infty} n\left(\frac{A_2W_2}{n} - CV_1 + V_2\frac{A_2^2}{H} + O(H^{-2})\right)e^{-2n\alpha}$$
$$= 4\pi \lim_{H\to\infty} C\left(A_2W_2 - CV_1 + V_2\frac{A_2^2}{H}\right)e^{-2\alpha} + O(H^{-3})$$
$$= 4\pi A_2W_2\lim_{H\to\infty} Ce^{-2\alpha} - 4\pi V_1\lim_{H\to\infty} C^2e^{-2\alpha} + 2\pi V_2\frac{A_1^2A_2^2}{H^2} + O(H^{-3})$$
$$= -\pi A_1^2V_1 + 2\pi A_2W_2\frac{A_1^2}{H} + 2\pi V_2\frac{A_1^2A_2^2}{H^2} + O(H^{-3}), \quad \text{(C8)}$$

$$\lim_{H\to\infty} J_2 = \lim_{H\to\infty}\frac{\pi}{C}\left(-c_0^2 + 2c_0c_1e^{-\alpha}\cosh\alpha + 2\sum_{n=1}^{\infty} g_n e^{-2n\alpha}\right)$$
$$= \lim_{H\to\infty}\frac{\pi}{C}\left(-(A_1W_1)^2 + A_1W_1(A_1W_1 + CV_1) + CU_1(-A_1W_1 + A_2W_2)\frac{A_1^2}{H^2}\right) + O(H^{-3})$$
$$= \pi A_1W_1V_1 + \pi U_1(-A_1W_1 + A_2W_2)\frac{A_1^2}{H^2} + O(H^{-3}). \quad \text{(C9)}$$

Substituting (C6)–(C9) into the force formula of (22), one can derive the asymptotic behaviors of the forces acting on the circle C_1 up to the second-order magnitude $O(H^2)$:

$$\lim_{H\to\infty}\frac{F_x}{\rho_f} = \left(\frac{\partial}{\partial t} - \frac{W_1}{A_1}\right)\lim_{H\to\infty} I_1 + \lim_{H\to\infty} I_2$$
$$= -\left(\frac{\partial}{\partial t} - \frac{W_1}{A_1}\right)\left(\pi A_1^2U_1 + 2\pi U_2\frac{A_1^2A_2^2}{H^2}\right) + \pi A_1W_1U_1 + \frac{\pi}{2}U_1(A_1W_1 - A_2W_2)\frac{A_1^2}{H^2} + O(H^{-3})$$
$$= -\pi A_1^2U_{1t} + \left\{2\pi\left(\frac{W_1}{A_1}U_2A_1^2A_2^2 - \frac{d}{dt}(U_2A_1^2A_2^2)\right) + \frac{\pi}{2}A_1^2U_1(A_1W_1 - A_2W_2)\right\}\frac{1}{H^2} + O(H^{-3}), \quad \text{(C10)}$$

$$\lim_{H\to\infty}\frac{F_y}{\rho_f} = \left(\frac{\partial}{\partial t} - \frac{W_1}{A_1}\right)\lim_{H\to\infty} J_1 + \lim_{H\to\infty} J_2$$
$$= -\pi A_1^2V_{1t} + 2\pi\left(\frac{d}{dt}(A_1^2A_2W_2) - \frac{W_1}{A_1}A_1^2A_2W_2\right)\frac{1}{H} + \left\{2\pi\left(\frac{d}{dt}(A_1^2A_2^2V_2) - \frac{W_1}{A_1}A_1^2A_2^2V_2 - A_1^2A_2W_2H_t\right) + \pi U_1A_1^2(-A_1W_1 + A_2W_2)\right\}\frac{1}{H^2} + O(H^{-3}). \quad \text{(C11)}$$

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