Interaction of two circular cylinders in inviscid fluid

Qian Xi Wang

Division of Environmental & Water Resources Engineering, Maritime Research Centre, School of Environmental & Civil Engineering, Nanyang Technological University, 50 Nanyang Avenue, Singapore 630798, Singapore

(Received 7 April 2004; accepted 17 August 2004; published online 1 November 2004)

The unsteady problem of two parallel circular cylinders, moving in an inviscid fluid, is analyzed analytically by exploring a conformal mapping. Exact solutions are obtained for the flow induced and the hydrodynamic forces acting on them, as the two circular cylinders, with any radii and at any locations, expand (contract) and translate arbitrarily at time dependent speeds. As the two bodies are far apart, the solutions of the flow and forces reduce to those for a single circular cylinder moving in an unbounded inviscid flow. The force components along the line of centers are inversely proportional to the distance between them, whereas the force components perpendicular to the line of centers are inversely proportional to the square of the distance between them. Numerical analyses are performed for the two circular cylinders deforming and translating at constant speeds and in various ways, as well as a circular cylinder falling to a wall. It has been noticed that they are attracted to each other, as one of them expands and the other contracts, or as they translate perpendicular to the line of centers; whereas they are repelled from each other, as both of them expand, contract, or as they translate along the line of centers. The force on one of them increases with the size and speed of the other, as well as their proximity.

I. INTRODUCTION

The hydrodynamic interaction between two bodies poses an interesting theoretical problem as well as one of practical importance, in the context of interactions of marine vessels and high-speed trains. The problem has been studied for a long time by using the potential flow theory, since it provides a good approximation for high Reynolds number flows (cf. Ref. 1). Owing to the difficult mathematical treatment for arbitrarily shaped bodies, theoretical studies have been mainly focused on bodies of simple geometries, such as spheres and circular cylinders. The axisymmetric potential flow of two spheres has been studied by Hicks, Basset, Herman, Lamb, Rouse, Bentwich and Miloh, and Sun and Chwang, among others.

The mutual force between two pulsating spherical bubbles far apart was first studied by Bjerknes and Bjerknes. They observed that this mutual force, later on termed as the secondary Bjerknes force, caused the bubbles to either attract or repel each other depending upon whether the bubble pulsations were in phase or out of phase, respectively. Moreover, the magnitude of the force between the two bubbles, directed along the line connecting their centers, was found to be proportional to the inverse square of the distance between them.

Lagally first considered the problem of the potential flow induced by two stationary circular cylinders in a uniform stream and he obtained the velocity potential of the fluid. The present work extends his theoretical research to the unsteady problem of two circular cylinders expanding and translating arbitrarily. The exact solution of the velocity potential is derived using the conformal mapping and Fourier series. The force acting on any one of the cylinders is obtained in a closed form too. Theoretical analyses are carried out for the asymptotic dynamic behaviors as the two circles are far apart. Numerical analyses are performed for the two circles expanding, contracting, translating along the line of centers, or translating perpendicular to the line, respectively. The falling of a circular cylinder to a wall is also analyzed.

II. EXACT FLOW SOLUTION

Consider two circles $C_1$ and $C_2$ with centers $O_1$, $O_2$, radii $A_1$, $A_2$, and center distance $H$. A coordinate system $O$-xyz fixed with body 1 is chosen with its origin at $O$, y axis along line $O_2O$ of centers, and z axis perpendicular to the plane of the paper, as shown in Fig. 1(a). Let the two circles expand (contract) with speeds $W_1$ and $W_2$, translate with velocities $(U_1, V_1)$ and $(U_2, V_2)$, in an otherwise still fluid.

Assuming the fluid is inviscid and incompressible and the flow irrotational, a velocity potential $\varphi(x, y, t)$ exists in the fluid domain, external to the two bodies. The velocity potential $\varphi$ satisfies the Neumann boundary value problem of two-dimensional Laplace’s equation in a triply connected domain

$$\varphi_{xx} + \varphi_{yy} = 0. \quad (1a)$$

$$- \varphi_n = W_1 + V_1 \sin \theta + U_1 \cos \theta, \quad \text{on } C_1, \quad (1b)$$

$$- \varphi_n = W_2 + V_2 \sin \theta_a + U_2 \cos \theta_2, \quad \text{on } C_2. \quad (1c)$$
\[ \lim_{x \to \infty} 2\pi r \varphi, = 2\pi (A_1 W_1 + A_2 W_2). \]  

where \( r = \sqrt{x^2 + y^2} \), \( \theta = \tan^{-1}(y/x) \), and \( \theta = \tan^{-1}[(y+H)/x] \). \( n \) is the unit outward normal on the boundaries of the fluid domain. Boundary condition (1d) at infinity is given by conservation of mass.

Note that expanding speeds \( W_1 \) and \( W_2 \) are prescribed time dependent functions. Translation velocities \( U, V_1, V_2 \) and \( V_2 \) are given time dependent functions for prescribed motions. The dynamic problem of the two cylinders, under the influences of gravity, buoyancy, and hydrodynamic loads, are governed by Newton’s second law.

\[
U_i = \frac{1}{2\pi A_i^2 \rho_i} \left( 1 - \frac{\rho_i}{\rho} \right) g \cos \gamma \sin \delta, \quad \text{for} \quad i = 1, 2, \\
V_i = \frac{1}{2\pi A_i^2 \rho_i} \left( 1 - \frac{\rho_i}{\rho} \right) g \cos \gamma \cos \delta, \quad \text{for} \quad i = 1, 2, 
\]

where \( g \) is the gravitational acceleration; \( \delta \) and \( \gamma \) are the angles between the gravity direction and \( y \) and \( z \) axes, respectively. \( \rho_i \) is the density of the fluid, and \( \rho_i' \) is the average density of cylinder \( i \); and \( (F_i, F) \) are the hydrodynamic forces on the cylinders.

To solve problem (1), we introduce a linear fractional conformal mapping between the physical plane of \( Z = x + iy \) and the mapped plane of \( \varsigma = \rho e^{i\theta} \),

\[
Z = iC \frac{\varsigma + C}{\varsigma - C} = iC \coth \alpha, \quad \text{(3)}
\]

where, as usual, “coth” denotes the hyperbolic cotangent, and

\[
C = \frac{(A_1^2 + A_2^2 - H^2)^2 - 4A_1^2 A_2^2}{2H}, \quad \alpha = \sinh(C/A_1). \quad \text{(4)}
\]

“asinh” in (4) denotes inverse hyperbolic sine. Similarly, “sinh” and “cosh,” to be used later on, denote hyperbolic sine and cosine, respectively.

The conformal mapping maps the domain outside two circles \( C_1, \ |Z| = A_1, \) and \( C_2, \ |Z - H| = A_2, \) in the plane \( Z \) to the domain between two concentric circles \( B_1, \ |\varsigma| = \rho_1, \) and \( B_2, \ |\varsigma| = \rho_2, \) in the plane \( \varsigma, \) as sketched in Fig. 1(b). \( \rho_1 \) and \( \rho_2 \) are

\[
\rho_1 = Ce^\alpha, \quad \rho_2 = Ce^{\beta}, \quad \beta = \sinh(C/A_2). \quad \text{(5)}
\]

To transform the boundary value problem of (1) in the physical plane \( Z \) to the mapped plane \( \varsigma \), the following variables are needed:

\[
\cos \theta|_{C_1} = 2e^{\alpha} \sinh \alpha \sin \Theta \Delta, \\
\sin \theta|_{C_1} = -2e^{\alpha} \frac{1 - \cosh \alpha \cos \Theta}{\Delta}, \quad \text{(6a)}
\]

\[
\cos \theta|_{C_2} = 2e^{\beta} \sin \beta \sin \Theta \Delta, \\
\sin \theta|_{C_2} = 2e^{\beta} \frac{1 - \cosh \beta \cos \Theta}{\Delta}, \quad \text{(6b)}
\]

\[
J = \left| \frac{dZ}{d\varsigma} \right|_{C_1} = \frac{2}{\Delta}, \quad J_2 = \left| \frac{dZ}{d\varsigma} \right|_{C_2} = \frac{2}{\Delta_2}, \quad \text{(6c)}
\]

where

\[
\Delta = e^{2\alpha} - 2e^\alpha \cos \Theta + 1, \quad \Delta_2 = e^{2\beta} - 2e^\beta \cos \Theta + 1. \quad \text{(7)}
\]

Point \( \varsigma = C \) in the mapped plane corresponds to the point \( Z = \infty \) in the physical plane, where it is a singular point source with the strength of \( -2\pi(A_1 W_1 + A_2 W_2) \), according to (1d). To simplify problem (1), we remove the contribution of the point source from \( \varphi \). The volume flux of the fluid across the circular \( C_2 \) in the physical plane is at the value of \( 2\pi A_2 W_2 \), which remains at the same value in the mapped plane \( \varsigma \). The point \( \varsigma = 0 \) inside \( B_2 \) in the mapped plane corresponds to the point \( Z = -iC(1 + \coth \alpha) \) inside the circle \( C_2 \) in the physical plane. We further remove the potential due to a point source...
at $s=0$ with the strength of $2\pi A_2 W_2$ from $\varphi$, which provides
the volume flux of $2\pi A_2 W_2$ across $B_2$ associated with the
deforming of the circle $C_2$,

$$
\varphi = A_2 W_2 \ln \rho - (A_1 W_1 + A_2 W_2) \ln |s - C| + \phi. \tag{8}
$$

The boundary problem of $\phi$ then becomes

$$
\phi_{\rho\rho} + \frac{1}{\rho} \phi_{\rho} = 0, \tag{9a}
$$

$$
\frac{\partial \phi}{\partial \rho} = \frac{A_1 W_1 - A_2 W_2}{C} \frac{1 - e^{\alpha \cos \Theta}}{\Delta} + 4 V_1 e^{\alpha \cos \Theta} \frac{(1 - \cosh \alpha \cos \Theta)}{\Delta^2} - 4 U_1 e^{\alpha \sin \alpha \sin \Theta} \frac{\Delta^2}{\Delta}, \text{ on } B_1, \tag{9b}
$$

$$
\frac{\partial \phi}{\partial \rho} = \frac{A_1 W_1 - A_2 W_2}{e^{-\beta \cos \Theta}} \frac{1 - e^{-\beta \cos \Theta}}{\Delta_2} + 4 V_1 e^\beta \frac{(1 - \cosh \alpha \cos \Theta)}{\Delta_2} + 4 U_1 e^{\alpha \sin \alpha \sin \Theta} \frac{\Delta_2^2}{\Delta_2}, \text{ on } B_2. \tag{9c}
$$

The right-hand sides of (9b) and (9c) can be further expanded
as the Fourier series in $\Theta$ as follows:

$$
\frac{\partial \phi}{\partial \rho} = -\frac{A_1 W_1 - A_2 W_2}{C} \sum_{n=1}^{\infty} \frac{\cos (n\Theta)}{e^{(n+1)\alpha}} - V_1 \sum_{n=1}^{\infty} \frac{\cos (n\Theta)}{e^{(n+1)\alpha}} - U_1 \sum_{n=1}^{\infty} \frac{n \sin (n\Theta)}{e^{(n+1)\alpha}}, \text{ on } B_1, \tag{10a}
$$

$$
\frac{\partial \phi}{\partial \rho} = -\frac{A_1 W_1 - A_2 W_2}{C} \sum_{n=1}^{\infty} \frac{\cos (n\Theta)}{e^{(n+1)\beta}} - V_1 \sum_{n=1}^{\infty} \frac{\cos (n\Theta)}{e^{(n+1)\beta}} + 2 U_1 \sum_{n=1}^{\infty} \frac{n \sin (n\Theta)}{e^{(n+1)\beta}}, \text{ on } B_2, \tag{10b}
$$

by using the following integration formulas:

$$
\int_0^{2\pi} \frac{\cos n\Theta d\Theta}{a^2 - 2a \cos \Theta + 1} = \frac{2\pi}{a^2(a^2 - 1)}, \quad a > 1, \tag{11a}
$$

$$
\int_0^{2\pi} \frac{\cos n\Theta d\Theta}{(a^2 - 2a \cos \Theta + 1)^2} = \frac{2\pi}{a^2(a^2 - 1)} \left[2 + (n+1)(a^2 - 1)\right], \quad a > 1. \tag{11b}
$$

Assume the solution of (9) takes the form

$$
\phi = \sum_{n=1}^{\infty} (A_n \rho^n + B_n \rho^{-n}) \cos (n\Theta) + (C_n \rho^n + D_n \rho^{-n}) \sin (n\Theta). \tag{12a}
$$

Determining its coefficients with (9b), (9c), and (10), and
substituting it into (8), we obtain

$$
\varphi = \varphi_1 + \varphi_2 + \varphi_3, \tag{12a}
$$

$$
\varphi_1 = A_2 W_2 \ln \rho - (A_1 W_1 + A_2 W_2) \ln (\rho^2 - 2\rho C \cos \Theta + C^2) - (A_1 W_1 - A_2 W_2) \sum_{n=1}^{\infty} \frac{1}{n} \frac{\cos (n\Theta)}{n \sin (n\alpha + n\beta)} \times \left[ \frac{\sinh (n\beta)}{\rho} - n \sin (n\alpha) \left( \frac{\rho_2}{\rho_1} \right)^n \right], \tag{12b}
$$

$$
\varphi_2 = -2 C \sum_{n=1}^{\infty} \frac{\cos (n\Theta)}{1 - e^{-2n(\alpha + \beta)}} \left[ e^{-n\alpha} (V_1 - V_2 e^{-2n\beta}) \left( \frac{\rho}{\rho_1} \right)^n - e^{-n\beta} (V_2 - V_1 e^{-2n\alpha}) \left( \frac{\rho_2}{\rho} \right)^n \right], \tag{12c}
$$

$$
\varphi_3 = -2 C \sum_{n=1}^{\infty} \frac{\sin (n\Theta)}{1 - e^{-2n(\alpha + \beta)}} \left[ e^{-n\alpha} (U_1 + U_2 e^{-2n\beta}) \left( \frac{\rho}{\rho_1} \right)^n + e^{-n\beta} (U_2 + U_1 e^{-2n\alpha}) \left( \frac{\rho_2}{\rho} \right)^n \right]. \tag{12d}
$$

Solutions $\varphi_1$, $\varphi_2$, and $\varphi_3$ are the potentials for two circular cylinders expanding at speeds $W_1$ and $W_2$, translating along the line of centers at velocities $V_1$ and $V_2$, and translating perpendicular to the line at velocities $U_1$ and $U_2$, respectively, in an otherwise still fluid. The three series in (12) are absolutely convergent in the whole fluid domain corresponding to $\rho_2 \leq \rho \leq \rho_1$ and $0 < \Theta \leq 2\pi$.

We further consider a special case where the two circles and their motions are identical. As $A_1 = A_2$, and $W_1 = W_2$, (12b) is simplified as follows:

$$
\varphi_1 = A_2 W_2 \ln \rho - 0.5 (A_1 W_1 + A_2 W_2) \times \ln (\rho^2 - 2\rho C \cos \Theta + C^2) = 0.5 A_2 W_2 \ln \frac{\rho^2}{\rho^2 - 2\rho C \cos \Theta + C^2} - 0.5 A_1 W_1 \ln (\rho^2 - 2\rho C \cos \Theta + C^2). \tag{13}
$$

Using the conformal mapping of (4), we have

$$
|Z + iC (\coth \alpha - 1)| = \left| iC \left( \frac{s + C}{s - C} - 1 \right) \right| = \left| iC \frac{2C}{s - C} \right| = \frac{2C^2}{\sqrt{\rho^2 - 2\rho C \cos \Theta + C^2}}. \tag{14a}
$$

$$
|Z + iC (\coth \alpha + 1)| = \left| iC \left( \frac{s + C}{s - C} + 1 \right) \right| = \left| iC \frac{2s}{s - C} \right| = \frac{2C\rho}{\sqrt{\rho^2 - 2\rho C \cos \Theta + C^2}}. \tag{14b}
$$

Substituting the above formula into (13), we have

$$
\varphi_2 = A_1 W_1 \ln |Z + iC (\coth \alpha - 1)| + A_2 W_2 \ln |Z + iC (\coth \alpha + 1)| + \text{const.}. \tag{15}
$$

Thus, the velocity potentials for the two identical circular cylinders deforming at the same speed can be expressed in
terms of two point sources located at \( Z_{1,2} = -iC(\coth \alpha \mp 1) \), and at the same strength of \( 2\pi A_W \). It is easy to verify using (12b) that, in general, (15) stands for two circles at different radii satisfying \( A_W = A_{2W} \).

As \( A_1 = A_2, U_1 = U_2, \) and \( V_1 = V_2 \), (12c) and (12d) are simplified as

\[
\varphi_2 = -CV_1 \sum_{n=1}^{\infty} \cos(n\Theta) \left( \frac{\rho}{\rho_1} \right)^n - \left( \frac{\rho_2}{\rho} \right)^n, \quad (16a)
\]

\[
\varphi_3 = -CU_1 \sum_{n=1}^{\infty} \sin(n\Theta) \left( \frac{\rho}{\rho_1} \right)^n + \left( \frac{\rho_2}{\rho} \right)^n. \quad (16b)
\]

When the two circles are far apart, \( H \to \infty \), the exact solutions of the velocity potential of (12b)–(12d), in the flow field not far away from the circle \( C_1 \), can be simplified as follows (cf. Appendix A):

\[
\lim_{H \to \infty} \varphi_1 = A_W \ln |Z|, \quad (17a)
\]

\[
\lim_{H \to \infty} \varphi_2 = -\frac{A_W^2 V_W y}{r^2}, \quad (17b)
\]

\[
\lim_{H \to \infty} \varphi_3 = -\frac{A_W^2 U_W x}{r^2}. \quad (17c)
\]

As expected, the limits of (17a)–(17c) are the velocity potentials of a single circular cylinder deforming at the speed of \( W \), and moving at the velocities of \( V_1 \) and \( U_1 \) along \( y \) and \( x \) axes, respectively, in an unbounded inviscid fluid.

### III. FORCE FORMULA

The pressure field can be obtained from the Bernoulli equation

\[
\frac{p}{\rho_f} = -\varphi - \frac{1}{2}(\varphi_x^2 + \varphi_y^2). \quad (18)
\]

Only the force acting on body 1 is considered, as the force on body 2 can be obtained simply by rotating the parameters between them. The force components \( F_x \) and \( F_y \) on body 1 are

\[
\frac{F_x}{\rho_f} = -\oint_{c_1} p \cos \theta \, dl = \left( \frac{d}{dt} - \frac{W_1}{A_1} \right) \oint_{c_1} \varphi \cos \theta \, dl + \frac{1}{2} \oint_{c_1} (\varphi_x^2 + \varphi_y^2) \cos \theta \, dl, \quad (19a)
\]

\[
\frac{F_y}{\rho_f} = -\oint_{c_1} p \sin \theta \, dl = \left( \frac{d}{dt} - \frac{W_1}{A_1} \right) \oint_{c_1} \varphi \sin \theta \, dl + \frac{1}{2} \oint_{c_1} (\varphi_x^2 + \varphi_y^2) \sin \theta \, dl. \quad (19b)
\]

Notice that the time dependency of \( C_1 \) has been considered in the first terms on the right-hand side of (19a) and (19b). Taking the first term in (19a) as an example, we have

\[
\oint_{c_1} \varphi_x \cos \theta \, dl = \int_0^{2\pi} \varphi_x \cos \theta \, d\theta = \int_0^{2\pi} [(A_1 \varphi_x - A_1 \varphi) \cos \theta] \, d\theta = \left( \frac{d}{dt} - \frac{A_1}{A_1} \right) \oint_{c_1} A_1 \varphi \cos \theta \, d\theta = \left( \frac{d}{dt} - \frac{W_1}{A_1} \right) \oint_{c_1} \varphi \cos \theta \, dl. \quad (20)
\]

Denoting the four integrals in (19a) and (19b) as \( I_1(t), I_2(t), J_1(t), \) and \( J_2(t) \),

\[
I_1(t) = \oint_{c_1} \varphi \cos \theta \, dl, \quad I_2(t) = \frac{1}{2} \oint_{c_1} (\varphi_x^2 + \varphi_y^2) \cos \theta \, dl, \quad (21a)
\]

\[
\frac{F_x}{\rho_f} = I_1 - \frac{W_1}{A_1} I_1 + I_2. \quad (22a)
\]

\[
\frac{F_y}{\rho_f} = J_1 - \frac{W_1}{A_1} J_1 + J_2. \quad (22b)
\]

\( I_1(t), I_2(t), J_1(t), \) and \( J_2(t) \) can be integrated analytically with the following results (cf. Appendix B):

\[
I_1 = 4\pi C \sum_{n=1}^{\infty} nb_n e^{-2na}, \quad (23a)
\]

\[
I_2 = \frac{\pi}{C} \sinh \alpha \left( 2c_0 d_1 e^{-\alpha} + \sum_{n=1}^{\infty} f_n e^{-2na} \right), \quad (23b)
\]

\[
J_1 = 4\pi C \sum_{n=1}^{\infty} na_n e^{-2na}, \quad (23c)
\]

\[
J_2 = \frac{\pi}{C} \left( c_0^2 + 2c_0 c_1 e^{-\alpha} \cosh \alpha + 2 \sum_{n=1}^{\infty} g_n e^{-2na} \right). \quad (23d)
\]
\[ a_n = \frac{A_1 W_1 + A_2 W_2 - A_1 W_1 - A_2 W_2}{2n} - \frac{2e^{-2n\beta} + e^{-2n(\alpha + \beta)}}{1 - e^{-2n(\alpha + \beta)}} \]
\[ b_n = -C \frac{U_1 + 2U_2 e^{-2n\beta} + U_1 e^{-2n(\alpha - 2\beta)}}{1 - e^{-2n(\alpha + \beta)}} \]
\[ c_n = -a_n b_n + a_n b_{n+1} + (c_n d_{n+1} - c_n d_n) \]

The six series in (23a)-(23d) and (25a) and (25b) are absolutely convergent.

As \( U_1 = U_2 = V_1 = V_2 = 0 \), and \( A_1 W_1 = A_2 W_2 \), the force acting on circle 1 can be simplified as follows from (22)-(24):

\[ \frac{F_x}{\rho_f} = 2\pi \frac{d}{dt} \left( \frac{A_1^2 W_1 e^{-\alpha t}}{A_1 W_1} \right) - \frac{\pi}{C} (A_1 W_1)^2. \]

As two moving circles are far apart, the asymptotic behaviors of the force acting on the circle \( C_1 \) have been derived in Appendix C as follows:

\[ \lim_{H \to \infty} \frac{F_x}{\rho_f} = \left( \frac{d}{dt} - \frac{W_1}{A_1} \right) \lim_{H \to \infty} I_1 + \lim_{H \to \infty} I_2 = -\pi A_1^2 U_1 + 2\pi \left( \frac{d}{dt} \left( \frac{A_1^2 W_1}{A_1 W_1} \right) - \frac{W_1}{A_1} \frac{A_1^2 W_1}{A_1 W_1} \right) \frac{1}{H} \]

\[ + \left( \frac{2\pi}{d} \left( \frac{A_1^2 W_1}{A_1 W_1} \right) - \frac{W_1}{A_1} \frac{A_1^2 W_1}{A_1 W_1} \right) \frac{1}{H^2} + O(H^{-3}). \]

A few conclusions can be drawn from the above formula. First, as the two circles are far apart, the limits of the forces reduce to that acting on a single circular cylinder accelerating in an unbounded inviscid fluid:

\[ F_x = -\pi \rho \dot{A}_1^2 U_1, \quad F_y = -\pi \rho \dot{A}_1^2 V_1. \]

Second, the dynamic influences of the circle \( C_2 \) far away to the circle \( C_1 \) are the first order of magnitude \( O(H^{-3}) \), in the direction along the line of centers, and are the second order of magnitude \( O(H^{-2}) \), in the direction perpendicular to the
line of centers. The first-order dynamic forces are due to the
deformations of the two circles, and the second-order forces
are due to the nonlinear coupling of their deformations and
translations. As two circular cylinders far apart are only in
translations, the forces between them are the third-order
small quantities of \( O(H^{-3}) \).

As the centers of the two circles are at fixed locations,
the force on circle 1 due to the deforming of circle 2 is
simplified as

\[
F_y = 2\pi\rho \left( \frac{d}{dt} \left( A_1^2 W_2 - A_1 A_2 W_1 W_2 \right) \right) + O(H^{-3}).
\]

When the two circles undergo small-amplitude radial har-
monic pulsations at the same frequency, i.e.,

\[
A_i(t) = A_{ik} \left[ 1 + \delta_i \cos(\omega t + \psi_i) \right], \quad i = 1, 2,
\]

with \( \delta < 1 \), where \( \omega \) is the pulsation frequency, and \( \psi_1 \) and
\( \psi_2 \) are their initial phases. Substituting (32) into (31), and
averaging over one cycle, one can obtain the averaging force as follows:

\[
\langle F_y \rangle = -\frac{2\pi\rho\omega A_{10} A_{20}}{H} \delta_1 \delta_2 \cos(\psi_1 - \psi_2) + O(H^{-3}).
\]

It can be seen from (33) that the two circles attract each other
when their pulsations are in phase, and vice versa the two
circles repel each other when their pulsations are out of
phase. This is a further affirmation of the presence of a
Bjerknes-type effect.

Blake, Taib, and Doherty\textsuperscript{12} developed the Kelvin im-
pulse criterion for bubbles using the concept of the Bjerknes
force, which has been widely used in predicting the direc-
tions of the motion of bubbles and the liquid jets formed in
bubbles collapsing (cf. Refs. 13–17). The Bjerknes effect has
also been developed and applied widely in predicting the
coupled pulsation and translation of two gas bubbles in an
acoustic wave.\textsuperscript{18–22} Those studies are based on the approxi-
mate analytical solution of the velocity potential for two os-
illating spheres far apart. The exact solutions of the velocity
potential for two moving circles obtained here can be used to
provide a more accurate Kelvin impulse criterion for
cylindrical-type bubbles, as well as an accurate Bjerknes
force for predicting the coupled pulsation and translation of
two cylindrical-type bubbles in an acoustic wave.

IV. DYNAMIC ANALYSES

The forces on the circular cylinders given in Sec. III are
calculated using the MATLAB. The series in (23) and (25)
converge rapidly, since their terms decay exponentially. The
calculations have been performed at a very high accuracy
with the series truncated when the terms are at \( O(10^{-8}) \), since
the CPU time needed is minimal. The computational results
are given in terms of the scaled distance of centers \( H^* \) and
scaled force \( F_x^* \) as follows:

\[
H^* = \frac{H}{A}, \quad F_x^* = \frac{F_x}{\rho_j V_j^2 A^2}, \quad V = \sqrt{U_j^2 + V_j^2 + W_j^2}.
\]

We first consider the case where only one of the two
circles is moving. Figure 2 shows the scaled force on circle 1
versus the distance of centers as one of the two circles is (a)
expanding (contracting), (b) translating along the line of cen-
ters, and (c) translating perpendicular to the line. As circle 1
is stationary, the referenced velocity \( V \) in (23) is chosen as
\( V = \sqrt{U_2^2 + V_2^2 + W_2^2} \). The ratio of their radii is \( A_2/A_1 = 0.5, 1.0, 2.0 \). Note that circle 1 at \( A_2/A_1 = 0.5, 1.0, 2.0 \) corresponds to
circle 2 at \( A_2/A_1 = 2.0, 1.0, 0.5 \), respectively. Circle 1 ex-
periences repulsion as it alone deforms [Fig. 2(a)]. As circle 2
alone deforms, circle 1 experiences attraction as the clear-
ance between them is not small; however, it experiences repel-
ance as the clearance is small. They are repelled from
each other as one of them translates along the line of centers
[Fig. 2(b)], whereas they are attracted to each other as one of
them translates perpendicular to the line [Fig. 2(c)]. The
forces on the two circles increase with their proximity. The
force on the stationary circle appears larger than that on the
moving one, but the difference becomes small when the clear-
ance between them is small. This is consistent with the
attraction (repellence) phenomenon for a vessel passing by
(approaching to) a stationary vehicle observed in experi-
ments by Dand.\textsuperscript{23} He also noticed that the force on the sta-
nary body is larger than that on the moving one.

We next consider the case where both of the two circles
are moving. Figure 3 shows the scaled force on circle 1
versus the distance of centers as the two circles deform at
\( W_2/W_1 = -2, -1, 0, 1, 2 \). They are attracted to each other
as one expands and the other contracts (dashed line), and
conversely they are repelled from each other as both of them
expand or contract (solid line). The force on circle 1 in-
creases with the deforming speed of circle 2 and their prox-
imity. The mechanism underlying the phenomenon can be
explained as follows. Being an axisymmetric case, the flow
velocity along the axis of symmetry is along the axis. As
both of them expand (contract), the flow velocity along the
axis of symmetry changes direction from one circle to the
other. A stagnation point thus occurs on the axis of symmetry
between the two circles, and a high-pressure region is thus
formed over there, which repels both of the two circles.
When one circle expands and the other contracts, the fluid
flow between the two circles is pushed by the expanding
circle and attracted by the contracting one, and is conse-
quently faster than the flow beside them, which is mainly
produced by one of them only.

Figure 4 shows the force on circle 1 versus the distance
of centers as they both either expand or contract at \( W_2/W_1
= 1 \) and \( A_2/A_1 = 0.5, 1.0, 2.0 \) (dashed line), and as one ex-
pands and the other contracts at \( W_2/W_1 = -1 \) and \( A_2/A_1
= 0.5, 1.0, 2.0 \) (solid line). As expected, the magnitude of the
force on body 1 increases with the size of body 2.

Figure 5(a) shows the scaled force on circle 1 versus the
distance of centers as the two circles translate along the line
of centers at \( V_2/V_1 = -2, -1, 0, 1, 2 \). The force is always
repulsion, no matter if they approach to \( V_2/V_1 < 0 \) or de-
part from \( V_2/V_1 > 0 \) each other. The repulsion increases
with the relative velocities between them and their proximity.
This phenomenon reduces head-on collisions between float-
ing bodies. To interpret this phenomenon, we choose an

\[
\text{Downloaded 19 Nov 2004 from 129.11.77.160. Redistribution subject to AIP license or copyright, see http://pof.aip.org/pof/copyright.jsp}
\]
ertial coordinate system moving at the speed of \((V_1 + V_2)/2\), assuming the two circles are moving at constant speeds. In this system, the two circles move in contrary direction, at the velocities of \((V_1 - V_2)/2\) and \(-(V_1 - V_2)/2\), respectively, and the flow velocity along the axis of symmetry changes direction from one circle to the other. A stagnation point is thus formed on the axis of symmetry between the two circles. Since the Bernoulli equation stands in any inertial coordinate system, a high-pressure region occurs near the stagnation point, which repels both the circles.

Figure 5(b) shows the scaled force on circle 1 versus the distance of centers as the two circles translate perpendicular to the line of centers at \(U_2/U_1 = -2, -1, 0, 1,\) and 2. The force is attraction, which increases with their proximity. This is because the flow passing through the two circles is restricted by both of them and thus moves faster than the flow beside

![Figure 2](image-url)
them restricted only by one of them. The attraction is larger as the two circles translate in the same direction than in opposite directions.

At last, we consider a circular cylinder falling to a wall, under gravity, buoyancy, and hydrodynamic loads, which can be analyzed with the exact solutions obtained here using the image method. Falling velocity $V_1$ is obtained by integrating $\Delta t = \gamma = 0$. Figures 6(a) and 6(b) show the scaled velocity and center height versus time for the fall at $\rho_f/\rho_i = 1.05$, and started with its center at a height of $2A_1$ from the wall at zero velocity. The length scale is chosen as $A_1$ and the time scale as $\sqrt{A_1/g}$. We also depict the corresponding results for the fall without the hydrodynamic load for comparison.

The falling cylinder is slowed down apparently by the repel-lence of the wall at the later stage, and the moment of the cylinder at impact to the wall is reduced by about one-third by the hydrodynamic load. Unlike a body moving in an un-bounded inviscid fluid with the added mass being a constant, the added mass in this problem increases with time. Similar phenomena were observed for a sphere falling to a wall by Milne-Thomson, and a two-dimensional flat plate falling to a wall by Yih.

V. CONCLUSIONS

The unsteady problem of two parallel circular cylinders moving and deforming in an inviscid fluid is analyzed analytically, by using a linear fractional conformal mapping that transforms two arbitrary circles to concentric circles. Exact solutions are obtained for the flow induced and the hydrodynamic forces acting on them, as the two circular cylinders, with any radii and at any locations, expand (contract) and translate arbitrarily at time dependent speeds.

As the two bodies are far apart, the center distance $H \to \infty$, the exact solutions of the flow and forces obtained reduce to those for a single circular cylinder in an unbounded inviscid fluid. The dynamic forces on the two bodies far apart are the first order of magnitude $O(H^{-1})$ in the direction along the line of centers, and are the second order of magni-
The first-order dynamic forces are due to the deformations of the two circles, and the second-order forces are due to the nonlinear coupling of their deformations and translations. As two circular cylinders far apart are in translations, the forces between them are the third-order small quantities of \( O(H^{-3}) \).

As the two circular cylinders at fixed positions oscillate at the same frequency, the averaging force cause them to either attract or repel each other depending upon whether their pulsations are in phase or out of phase, respectively. This is analogous to the Bjerknes-type force between the two pulsating spherical bubbles noticed by Bjerknes\(^9\) and Bjerknes.\(^{10}\) The force between the two oscillating spherical bubbles is inversely proportional to the square of the distance between them; whereas the force between the two oscillating circular cylinders is inversely proportional to the distance between them.

Numerical analyses are performed for the two circular cylinders deforming and translating at constant speeds and in various ways, as well as a circular cylinder falling to a wall. It has been noticed that the two circular cylinders are attracted to each other, as one of them expands and the other contracts, or as they translate perpendicular to the line of centers; whereas they are repelled from each other, as both of them expand, contract, or as they translate along the line of centers. The repulsion is larger as they translate in opposite directions along the line of centers than that in the same direction. Conversely, the attraction is larger as they translate in the same direction perpendicular to the line of centers than in opposite direction. The force on one of them increases with the size and speed of the other, as well as their proximity. It has also been noticed that a circular cylinder falling to a wall is slowed down significantly by the presence of the wall.

The exact solutions obtained here can be applied to analyze two nearly parallel slender bodies of revolution in close interaction using the method of matched asymptotic expansions, which is similar to the two-dimensional cross-flow solutions deployed in the classical slender body theory. The solutions can be used to provide an accurate Kelvin impulse criterion for cylindrical-type bubbles in predicting the directions of the motion of bubbles and the liquid jets formed in bubbles collapsing, refering Blake, Taib, and Doherty.\(^{12}\) The solutions can also be used to provide an accurate Bjerknes force for predicting the coupled pulsation and translation of two cylindrical-type bubbles in an acoustic refering, Harkin, Kaper, and Nadin.\(^{21}\) etc.

**ACKNOWLEDGMENTS**

The author wishes to express his sincere thanks to the two referees of this paper for their valuable comments and suggestions.

**APPENDIX A: THE LIMIT OF THE VELOCITY POTENTIAL SOLUTIONS AS THE TWO CIRCLES ARE FAR APART**

In this appendix, we calculate the limits of the velocity potential solutions of (12) as the two circles are far apart. It can be obtained from the conformal mapping (4) and (5) that, as \( H \to \infty \), we have

\[
C = 0.5H + O(H^{-1}), \quad \alpha = \ln \frac{H}{A_1} + O(H^{-2}), \quad (A1)
\]

\[
\beta = \ln \frac{H}{A_2} + O(H^{-2}), \quad (A2)
\]

As \( H \to \infty \), and \( |Z| = O(A_1) \), i.e., for the flow field not far away from the circle \( C_1 \), the conformal mapping (3) becomes

\[
W \to 0.5i \frac{H^2}{Z} \quad \text{or} \quad \rho(\cos \Theta + i \sin \Theta) \to 0.5 \frac{H^2}{r} (\sin \theta + i \cos \theta). \quad (A3)
\]

Therefore,
\[ \rho \cos \Theta \to 0.5 \frac{H^2}{r} \sin \theta, \quad \rho \sin \Theta \to 0.5 \frac{H^2}{r} \cos \theta. \]  
(A4)

\[ O(\rho) = \frac{H^2}{r}. \]  
(A5)

Using (A1), (A2), and (A5), we have the following estimation:

\[ \phi_0 = \lim_{H \to \infty} \sum_{n=1}^{\infty} \frac{1}{n} \frac{\cos(n\Theta)}{n!} \sinh(n\beta) \left( 1 - \sinh(n\alpha) \right) \left( \frac{p}{\rho_1} \right)^n \\
= \lim_{H \to \infty} \sum_{n=1}^{\infty} \frac{1}{n} \frac{\cos(n\Theta)}{n!} \sinh(n\beta) \left( \frac{p}{\rho_1} \right)^n. \]  
(A6)

Introduce

\[ b = e^{-a} \frac{\rho}{\rho_1}. \]  
(A7)

It is easy to check that \( 0 \leq b < 1, \) as \( H \to \infty \) and \(|Z| = O(A_1)\). Using the following formulas:

\[ \sum_{n=1}^{\infty} \frac{1}{n} b^n \cos(n\Theta) = -0.5 \ln(b^2 - 2b \cos \Theta + 1), \quad 0 \leq b < 1, \]  
(A8)

(A6) becomes

\[ \phi_0 = \lim_{H \to \infty} \sum_{n=1}^{\infty} \frac{1}{n} \frac{\cos(n\Theta)e^{-n\alpha}}{n!} \left( \frac{p}{\rho_1} \right)^n \\
= \lim_{H \to \infty} \sum_{n=1}^{\infty} \frac{1}{n} b^n \cos(n\Theta) \\
= -0.5 \lim_{H \to \infty} \ln(b^2 - 2b \cos \Theta + 1) = 0, \]  
(A9)

since \( b \to 0 \), as \( H \to \infty \) and \(|Z| = O(A_1)\).

Substituting (A9) into (12b), one has

\[ \lim \varphi_1 = \lim_{H \to \infty} \frac{A_1 W_1 \ln \rho - 0.5(A_1 W_1 + A_2 W_2) \ln(\rho^2 - 2\rho C \cos \Theta + C^2)}{H^2} \]  
\[ = \lim_{H \to \infty} \frac{0.5A_2 W_2 \ln \rho}{\rho^2 - 2\rho C \cos \Theta + C^2} \]  
\[ - 0.5A_1 W_1 \ln(\rho^2 - 2\rho C \cos \Theta + C^2). \]  
(A10)

Using (A1), (A5), and (A10) becomes

\[ \lim \varphi_1 = -0.5A_1 W_1 \lim_{H \to \infty} \frac{\ln(\rho^2 - 2\rho C \cos \Theta + C^2)}{H^2} \]  
\[ = - A_1 W_1 \lim_{H \to \infty} \frac{\ln \rho - A_1 W_1 \ln|\rho|}{H^2}. \]  
(A11)

(12c) can be estimated as follows using (A1), (A2), and (A5):

\[ \lim_{H \to \infty} \varphi_2 = -2CV_1 \lim_{H \to \infty} \sum_{n=1}^{\infty} b^n \cos(n\Theta). \]  
(A12)

By using the following formulas:

\[ \frac{1 - b^2}{b^2 - 2b \cos \Theta + 1} = 1 + 2 \sum_{n=1}^{\infty} b^n \cos(n\Theta), \quad 0 \leq b < 1, \]  
(A13)

one can obtain

\[ \lim_{H \to \infty} \varphi_2 = 2 \lim_{H \to \infty} \frac{b^2 - b \cos \Theta}{b^2 - 2b \cos \Theta + 1} \]  
\[ = -2V_1 \lim_{H \to \infty} Cb \cos \Theta. \]  
(A14)

Substituting (A2), (A4), and (A7) into (A14), we have

\[ \lim \varphi_2 = -2V_1 \lim_{H \to \infty} b C \cos \Theta \]  
\[ = -2V_1 \lim_{H \to \infty} e^{-a} \frac{\rho}{\rho_1} \cos \Theta \]  
\[ = -V_1 \lim_{H \to \infty} e^{-a} \rho \cos \Theta \]  
\[ = -2V_1 \lim_{H \to \infty} \left( \frac{A}{H} \right)^2 \frac{0.5H^2}{r} \sin \theta \\
= - \frac{A_2^2 V_1}{r} \sin \theta = - \frac{A_2^2 V_1 y}{r^2}. \]  
(A15)

Similarly, using (A1), (A2), and (A5), the limit of \( \varphi_3 \) of (12c) can be expressed as

\[ \lim \varphi_3 = -2CU_1 \lim_{H \to \infty} \sum_{n=1}^{\infty} b^n \sin(n\Theta). \]  
(A16)

By using the following formulas:

\[ \frac{\sin \Theta}{b^2 - 2b \cos \Theta + 1} = \sum_{n=1}^{\infty} b^n \sin(n\Theta), \quad 0 \leq b < 1, \]  
(A17)

one can obtain

\[ \lim \varphi_3 = -2U_1 \lim_{H \to \infty} b C \sin \Theta = - \frac{A_2^2 U_1}{r} \sin \Theta = - \frac{A_2^2 U_1 x}{r^2}. \]  
(A18)

As expected, the limits of the velocity potentials of (12a)–(12c) given in (A11), (A15), and (A18) are the velocity potentials of a single circular cylinder deforming at the speed of \( W_1 \), and moving at the velocities \( V_1 \) and \( U_1 \) along \( y \) and \( x \) axes, respectively, in an unbounded inviscid fluid.
APPENDIX B: THE DERIVATION OF (23)

We first transform the four integrals in (21) along the circle \( C_1 \) in the physical plane \( Z \) to that along the circle \( B_1 \), \(|s|=\rho_1\), in the transformed plane \( \zeta \):

\[
I_1 = \oint_{C_1} \varphi \cos \theta \, dl = \oint_{B_1} \varphi \cos \theta J \, dl, \tag{B1a}
\]

\[
I_2 = \frac{1}{2} \oint_{C_1} (\varphi_x^2 + \varphi_y^2) \cos \theta \, dl \\
= \frac{1}{2} \oint_{B_1} \left( \varphi_\rho^2 + \frac{1}{\rho^2} \varphi_\theta^2 \right) \cos \theta J^{-1} \, dl, \tag{B1b}
\]

\[
J_1 = \oint_{C_1} \varphi \sin \theta \, dl = \oint_{B_1} \varphi \sin \theta J \, dl, \tag{B1c}
\]

\[
J_2 = \frac{1}{2} \oint_{C_1} (\varphi_x^2 + \varphi_y^2) \sin \theta \, dl \\
= \frac{1}{2} \oint_{B_1} \left( \varphi_\rho^2 + \frac{1}{\rho^2} \varphi_\theta^2 \right) \sin \theta J^{-1} \, dl. \tag{B1d}
\]

We next expand \( \varphi \), \( \varphi_\rho \), and \( \rho \varphi_\theta \) on \( B_1 \) in terms of the Fourier series in \( \Theta \) using (12),

\[
\varphi|_{B_1} = -A_1 W_1 \ln \rho_1 + 2 \sum_{n=1}^{\infty} \left[ a_n \cos(n \Theta) \right] e^{-n a}, \tag{B2a}
\]

\[
\varphi_\rho|_{B_1} = 2 \sum_{n=1}^{\infty} \left[ -n a_n \sin(n \Theta) + n b_n \cos(n \Theta) \right] e^{-n a}, \tag{B2b}
\]

\[
\rho \varphi_\theta|_{B_1} = c_0 + 2 \sum_{n=1}^{\infty} \left[ c_n \cos(n \Theta) + d_n \sin(n \Theta) \right] e^{-n a}, \tag{B2c}
\]

where \( a_n \), \( b_n \), \( c_n \), and \( d_n \) are given in (24a)–(24c).

To calculate \( I_1 \) and \( J_1 \), we express \( J \cos \theta \) and \( J \sin \theta \) on \( B_1 \) as

\[
\left( J \cos \theta \right)|_{B_1} = \frac{2 e^{a2} \sinh \alpha \sin \theta}{(e^{a2} - e^{2a} \cos \Theta + 1)^2} = \sum_{n=1}^{\infty} \frac{n \sin(n \Theta)}{e^{n a}}, \tag{B3a}
\]

\[
\left( J \sin \theta \right)|_{B_1} = \frac{2 e^{a2} - (e^{2a} + 1) \cos \Theta}{(e^{a2} - e^{2a} \cos \Theta + 1)^2} = \frac{1}{e^{a2}} \sum_{n=1}^{\infty} \frac{n \cos(n \Theta)}{e^{n a}}, \tag{B3b}
\]

by using (6), (7), and (11b). Substituting (B2a) and (B3a) into (B1a), we have

\[
I_1 = \oint_{B_1} \varphi J \cos \theta \, dl \\
= 2 e^{-a \rho_1} \int_{0}^{2\pi} \left( \sum_{n=1}^{\infty} n e^{-n a} \sin(n \Theta) \right) \\
\times \left[ -A_1 W_1 \ln \rho_1 + 2 \sum_{n=1}^{\infty} \left[ a_n e^{-n a} \cos(n \Theta) \right] + b_n e^{-n a} \sin(n \Theta) \right] \, d\Theta \\
= 4 \pi C \sum_{n=1}^{\infty} n b_n e^{-2na} \sin^2(n \Theta) \, d\Theta \\
= 4 \pi C \sum_{n=1}^{\infty} n b_n e^{-2na}. \tag{B4}
\]

In the above equation, we have used the following integral formula:

\[
\int_{0}^{2\pi} \sin(n \Theta) \cos(m \Theta) \, d\Theta = 0, \tag{B5}
\]

\[
\int_{0}^{2\pi} \sin(n \Theta) \sin(m \Theta) \, d\Theta = \pi \delta_{nm}, \quad \text{for } n,m \geq 1,
\]

where \( \delta_{nm} \) is the Kronecker delta, i.e., \( \delta_{nm}=0 \) as \( n \neq m \), and \( \delta_{nn}=1 \) as \( n=m \).

Similarly, substituting (B2a) and (B3b) into (B1a), one can obtain \( J_1 \) as follows:

\[
J_1 = \oint_{B_1} \varphi J \sin \theta \, dl \\
= 2 C \int_{0}^{2\pi} \left( \sum_{n=1}^{\infty} n e^{-n a} \cos(n \Theta) \right) \left[ -A_1 W_1 \ln \rho_1 \\
+ 2 \sum_{n=1}^{\infty} \left[ a_n e^{-n a} \cos(n \Theta) + b_n e^{-n a} \sin(n \Theta) \right] \right] \, d\Theta \\
= 4 \pi C \sum_{n=1}^{\infty} n a_n e^{-2na}. \tag{B6}
\]

To calculate \( I_2 \) and \( J_2 \), we need to calculate their integrants. Using (6) and (7), we have

\[
\left( J^{-1} \cos \theta \right)|_{B_1} = e^a \sinh \alpha \sin \theta, \tag{B7a}
\]

\[
\left( J^{-1} \sin \theta \right)|_{B_1} = e^a (1 - \cosh \alpha \cos \Theta). \tag{B7b}
\]

Using (B2b) and (B2c), we have
\[
(\rho_1^2 \varphi_p^2 + \varphi_\theta^2)|_{B_1} = c_0^2 + 4c_0d_1e^{-\alpha} \cos \Theta + 4c_0d_1e^{-\alpha} \sin \Theta + 2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} e^{-(n+m)\alpha} \left(\rho_1^2 \varphi_p^2 + \varphi_\theta^2 \right) (-nma_n b_m + c_m d_m)
\times \left[ \sin((n + m)\Theta) + \sin((n - m)\Theta) \right] + (nmb_n b_m + c_m d_m) \left[ \cos((n + m)\Theta) + \cos((n - m)\Theta) \right] + (nma_n b_m + c_m d_m)
\times \left[ \cos((n + m)\Theta) - \cos((n + m)\Theta) \right].
\]

(B8)

Substituting (B7) and (B8) into (B1b) and (B1c), we obtain

\[
I_2 = \frac{1}{2} \int_{B_1} \left( \varphi_p^2 + \frac{1}{\rho^2} + \varphi_\theta^2 \right) \cos \Theta J^{-1} d\theta
\]

\[
= \frac{1}{2 \rho_1} \int_0^{2\pi} \left( \rho_1^2 \varphi_p^2 + \varphi_\theta^2 \cos \Theta J^{-1} d\theta \right)
\]

\[
= \frac{1}{2 \rho_1} \int_0^{2\pi} e^{\alpha} \sin \alpha \left( 4c_0d_1e^{-\alpha} \sin \Theta + 2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} e^{-(n+m)\alpha} \sin((n - m)\Theta) \right) d\Theta
\]

\[
= \frac{1}{2 \rho_1} \int_0^{2\pi} e^{\alpha} \sin \alpha \left( 2c_0d_1 + \sum_{n=1}^{\infty} f_n e^{-2n\alpha} \right) d\Theta
\]

\[
= \frac{\pi}{\rho_1} \sin \alpha \left( 2c_0d_1 + \sum_{n=1}^{\infty} f_n e^{-2n\alpha} \right),
\]

(B9a)

\[
J_2 = \frac{1}{2} \int_{B_1} \left( \varphi_p^2 + \frac{1}{\rho^2} + \varphi_\theta^2 \right) \sin \Theta J^{-1} d\theta
\]

\[
= \frac{1}{2 \rho_1} \int_0^{2\pi} \left( \rho_1^2 \varphi_p^2 + \varphi_\theta^2 \sin \Theta J^{-1} d\theta \right)
\]

\[
= \frac{1}{2 \rho_1} \int_0^{2\pi} e^{\alpha} (-1 + \cosh \alpha \cos \Theta)
\times \left( c_0^2 + 4c_0d_1e^{-\alpha} \cos \Theta + 2 \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} (nma_n a_m + nmb_n b_m + c_m d_m)
\times \left[ \cos((n + m)\Theta) + \cos((n - m)\Theta) \right] \right) d\Theta
\]

\[
= \frac{\pi}{C} \left[ -c_0^2 + 2c_0d_1e^{-\alpha} \cos \alpha + 2 \sum_{n=1}^{\infty} g_n e^{-2n\alpha} \right],
\]

(B9b)

where \(f_n\) and \(g_n\) are given in (24e) and (24f). The following integral formula have been used in the above equations:

\[
\int_0^{2\pi} \sin \Theta \cos((n \pm m)\Theta) d\Theta = 0,
\]

(B10a)

\[
\int_0^{2\pi} \sin \Theta \sin((n \pm m)\Theta) d\Theta = 0,
\]

(B10b)

\[
\int_0^{2\pi} \cos \Theta \cos((n \pm m)\Theta) d\Theta = 0,
\]

(B10c)

\[
\int_0^{2\pi} \cos \Theta \sin((n \pm m)\Theta) d\Theta = \begin{cases} 0, & \text{as } |n - m| \neq 1 \\
\pi, & \text{as } n = m + 1 \\
-\pi, & \text{as } n = m - 1 \end{cases}
\]

(B10d)

for \(n, m \gg 1\).

**APPENDIX C: THE ASYMPTOTIC BEHAVIORS OF THE FORCES AS THE TWO MOVING CIRCLES FAR APART**

In this section, we consider the asymptotic behaviors of the forces between two moving circles far apart. As \(H \to \infty\), using (A1) and (A2), \(a_n, b_n, c_n, d_n\) in (24) are reduced to

\[
\lim_{H \to \infty} a_n = \frac{A_1}{n} - CV_1 + V_2 A_2^2 + O(H^{-2}),
\]

(C1)

\[
\lim_{H \to \infty} b_n = -CU_1 - U_2 A_2^2 + O(H^{-2}),
\]

(C2)

\[
c_n = -(A_1 W_1 + nCV_1), \quad d_n = -nCU_1.
\]

(C3)

Substituting (C1)–(C3) into (24e) and (24f), one can obtain

\[
\lim_{H \to \infty} f_n = CU_1 (A_1 W_1 - A_2 W_2) + O(1),
\]

(C4)

\[
\lim_{H \to \infty} g_n = 0.5CU_1 (-A_1 W_1 + A_2 W_2) + O(1).
\]

(C5)

Substituting (C1)–(C5) and (A1) into (23a)–(23d), one has
\[ \lim I_1 = 4\pi \lim_{H \to \infty} \sum_{n=1}^{\infty} nb_n e^{-2n\alpha} \]

\[ = 4\pi \lim_{H \to \infty} \left( -CU_1 - U_2 \frac{A_1^2}{H} + O(H^{-3}) \right) e^{-2\alpha} \]

\[ = -4\pi \lim_{H \to \infty} C^2 e^{-2\alpha} U_1 - 2\pi U_2 \frac{A_1 A_2^2}{H^2} + O(H^{-3}) \]

\[ = -\pi A_1^2 U_1 - 2\pi U_2 \frac{A_1 A_2^2}{H^2} + O(H^{-3}), \quad \text{(C6)} \]

\[ \lim I_2 = \lim_{H \to \infty} \frac{\pi}{2} \left( 2A_1 W_1 \right) C U_1 \]

\[ + C U_1 (A_1 W_1 - A_2 W_2) \sum_{n=1}^{\infty} e^{-2n\alpha} + O(H^{-3}) \]

\[ = \lim_{H \to \infty} \frac{\pi}{2} \left( 2A_1 W_1 C U_1 + C U_1 (A_1 W_1 - A_2 W_2) \frac{A_2^2}{H^2} \right) + O(H^{-3}) \]

\[ = \pi A_1 W_1 U_1 + \frac{\pi}{2} U_1 (A_1 W_1 - A_2 W_2) \frac{A_2^2}{H^2} + O(H^{-3}), \quad \text{(C7)} \]

\[ \lim J_1 = 4\pi \lim_{H \to \infty} \sum_{n=1}^{\infty} n a_n e^{-2n\alpha} \]

\[ = 4\pi \lim_{H \to \infty} \left( \frac{A_1 W_2}{n} - CV_1 + \frac{V_2 A_1^2}{H} + O(H^{-3}) \right) e^{-2\alpha} \]

\[ = 4\pi \lim_{H \to \infty} \left( A_2 W_2 - CV_1 + \frac{V_2 A_2^2}{H} e^{-2\alpha} + O(H^{-3}) \right) \]

\[ = 4\pi A_2 W_2 \lim_{H \to \infty} C e^{-2\alpha} - 4\pi V_1 \lim_{H \to \infty} C^2 e^{-2\alpha} + 2\pi V_2 \frac{A_1 A_2^2}{H^2} \]

\[ + O(H^{-3}) \]

\[ = -\pi A_1 V_1 + 2\pi A_2 W_2 \frac{A_1^2}{H} + 2\pi V_2 \frac{A_1 A_2^2}{H^2} + O(H^{-3}). \quad \text{(C8)} \]

\[ \lim J_2 = \lim_{H \to \infty} \frac{\pi}{C} \left( -c_1^2 + 2c_1 c_2 e^{-\alpha} \cos \alpha + 2 \sum_{n=1}^{\infty} g_n e^{-2n\alpha} \right) \]

\[ = \lim_{H \to \infty} \frac{\pi}{C} \left( -(A_1 W_1)^2 + A_1 W_1 (A_1 W_1 + CV_1) \right) \]

\[ + C U_1 (A_1 W_1 - A_2 W_2) \frac{A_2^2}{H^2} + O(H^{-3}) \]

\[ = \pi A_1 W_1 V_1 + \pi U_1 (A_1 W_1 - A_2 W_2) \frac{A_1^2}{H^2} + O(H^{-3}). \quad \text{(C9)} \]

Substituting (C6)–(C9) into the force formula of (22), one can derive the asymptotic behaviors of the forces acting on the circle \( C_1 \) up to the second-order magnitude \( O(H^2) \):