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The fixed points of coprime action

By

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Abstract. Let P be an odd π-group that acts as a group of automorphisms on the soluble π'-group G. We obtain generators for the fixed points of P on $[G,P]$.

Let π be a set of primes and suppose that the π-group P acts as a group of automorphisms on the finite π'-group G. It is convenient to work in the semidirect product GP. An elementary but important consequence of the Schur-Zassenhaus Theorem is

$$G = C_G(P)[G,P].$$

Moreover, if G is abelian then $G = C_G(P) \times [G,P]$ and in particular, $C_{[G,P]}(P) = 1$. In general however, P does have fixed points on $[G,P]$. This certainly happens if P has prime order and $[G,P]$ is not nilpotent. It is the purpose of this paper to obtain generators for $C_{[G,P]}(P)$.

For each $g \in G$ define

$$[g,P] = \langle [g,\alpha] \mid \alpha \in P \rangle.$$

Then $[g,P]$ is P-invariant but not necessarily g-invariant. An elementary argument shows that

$$\langle P, P^g \rangle = [g,P]P \qquad \text{and} \qquad [g,P] = \langle P, P^g \rangle \cap G.$$

Sometimes it is helpful to think of $[g,P]$ in this way. We have

$$[G,P] = \langle [g,P] \mid g \in G \rangle.$$

We shall prove the following result.

Theorem A. *Let π be a set of odd primes and suppose that the π-group P acts as a group of automorphisms on the soluble finite π'-group G. Then*

$$C_{[G,P]}(P) = \langle C_{[g,P]}(P) \mid g \in G \rangle.$$

The restriction that π consist only of odd primes is essential. Indeed, if $|P| = 2$ then $\langle P, P^g \rangle$ is dihedral so $[g,P]$ is inverted by P. Thus $C_{[g,P]}(P) = 1$ for all $g \in G$.

Conjecture. *Theorem A holds for all finite π'-groups G.*

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In what follows we assume Theorem A to be false and that G is a minimal counterexample. Then P acts non trivially on G. Since $G = C_G(P)[G,P]$ we have $[G,P] = [[G,P],P]$ so the minimality of G forces

$$G = [G,P].$$

Let

$$D = \langle C_{[g,P]}(P) \mid g \in G \rangle$$

so that D is a proper normal subgroup of $C_G(P)$. Note also that

$$G \neq [g,P]$$

for all $g \in G$. Moreover G is nonabelian since otherwise $C_{[G,P]}(P) = 1$.

Lemma 1. *Let $V \neq 1$ be a proper P-invariant normal subgroup of G. Then the following hold.*

(a) $C_G(P) = DC_V(P)$ *and* $C_V(P) \nleqq D$.
(b) *There is a prime q such that $F(G)$ is a q-group.*
(c) *If V is nilpotent then V is elementary abelian.*
(d) $Z(G) \cap D = 1$ *and* $Z(G) \leqq C_G(P)$.

Proof. (a) Let $\overline{G} = G/V$. Then as $G = [G,P]$ we have $\overline{G} = [\overline{G},P]$ and the minimality of G implies that

$$C_{\overline{G}}(P) = \langle C_{[\overline{g},P]}(P) \mid \overline{g} \in \overline{G} \rangle.$$

If $g \in G$ then $[g,P]$ maps onto $[\overline{g},P]$ so $C_{[g,P]}(P)$ maps onto $C_{[\overline{g},P]}(P)$ by [2, Theorem 6.2.2(iv), p. 224]. Thus $C_{\overline{G}}(P) = \overline{D}$ whence $C_G(P) \leqq DV$ and then $C_G(P) = DC_V(P)$. This proves (a).

(b) Let q be a prime divisor of $|F(G)|$. Since a Sylow q-subgroup of $F(G)$ is normal in G, it follows from (a) that $C_G(P)/D$ is a nontrivial q-group. Applying this argument again, we see that q is the only prime divisor of $|F(G)|$.

(c) Suppose that $\Phi(V)$, the Frattini subgroup of V, is nontrivial. From (a) we have $C_G(P) = DC_{\Phi(V)}(P)$. Then

$$V = [V,P]C_V(P) = [V,P](D \cap V)C_{\Phi(V)}(P).$$

Recall that $[V,P] \trianglelefteq V$ so $[V,P](D \cap V)$ is a subgroup. As $V = [V,P](D \cap V)\Phi(V)$ we deduce that $V = [V,P](D \cap V)$. Thus $C_V(P) = C_{[V,P]}(P)(D \cap V)$. The minimality of G implies that $C_{[V,P]}(P) \leqq D$ whence $C_V(P) \leqq D$, a contradiction. We deduce that $\Phi(V) = 1$, so V is elementary abelian.

(d) Note that $Z(G) \cap D$ and $[Z(G),P]$ are proper P-invariant normal subgroups of G. Moreover, as $Z(G)$ is abelian we have $C_{[Z(G),P]}(P) = 1$. Now apply (a). □

Lemma 2. *G is not nilpotent.*

Proof. Assume that G is nilpotent. Let $\overline{G} = G/G'$. Now G is soluble so $\overline{G} \neq 1$. As $G = [G,P]$ we have $\overline{G} = [\overline{G},P]$ and as $\overline{G}$ is abelian we have $C_{\overline{G}}(P) = 1$ so $C_G(P) \leqq G'$. Let $g \in G$. The subgroup $G'[g,P]$ is P-invariant and normal in G. Since $G \neq [g,P]$ and

$G' \leqq \Phi(G)$ we have $G'[g,P] \neq G$. Lemma 1(c) implies that $G'[g,P]$ is abelian. Now $G = [G,P]$ so we deduce that $G' \leqq Z(G)$. Using Lemma 1(d) it follows that

$$G' = Z(G) = C_G(P).$$

Let $g \in G$ and $\alpha \in P$. Working in the semidirect product GP we have $\langle P, P^g \rangle = [g,P]P$. By the Schur-Zassenhaus Theorem there exists $h \in [g,P]$ such that $P^g = P^h$. Let $c = gh^{-1}$ so that $g = ch$. We have $[P,c] \leqq P \cap G = 1$ so $c \in C_G(P) = Z(G)$. The previous paragraph implies that $[g,P]$ is abelian so since it is P-invariant, we see that h commutes with h^α. Since $c \in C_G(P) = Z(G)$ it now follows that

$$[g, g^\alpha] = 1.$$

Since $G' = Z(G) = C_G(P)$ the maps $x \mapsto [g,x]$ and $x \mapsto [x,g]$ are endomorphisms of G and $[x^\alpha, y^\alpha] = [x,y]$ for all $x, y \in G$. Choose $x, y \in G$. Then

$$\begin{aligned} 1 &= [xy, (xy)^\alpha] \\ &= [xy, x^\alpha][xy, y^\alpha] \\ &= [x, x^\alpha][y, x^\alpha][x, y^\alpha][y, y^\alpha] \\ &= [y, x^\alpha][x, y^\alpha] \\ &= [x^\alpha, y]^{-1}[x, y^\alpha] \\ &= [x^\alpha, y^{-1}][x^\alpha, y^{\alpha^2}] \\ &= [x^\alpha, y^{-1}y^{\alpha^2}]. \end{aligned}$$

It follows that $[y, \alpha^2] \in Z(G)$ and then that α^2 acts trivially on $\overline{G}$. Since α^2 acts trivially on G' and since α^2 has order coprime to G it follows that α^2 acts trivially on G. Since α is an arbitrary element of P and since P has odd order we deduce that P acts trivially on G, a contradiction. Thus G is not nilpotent. □

We have shown that $G \neq F(G)$ so Lemma 1 implies that $F(G)$ is an elementary abelian q-group. Let K be the inverse image in G of a minimal P-invariant normal subgroup of $G/F(G)$. Then K is P-invariant and normal in G. Since $G/F(G)$ is soluble, $K/F(G)$ is an elementary abelian r-group for some prime r. Since $F(G)$ is a q-group we have $r \neq q$. By [2, Theorem 2.2.6(i), p. 224], K possesses a P-invariant Sylow r-subgroup R. Then $K = RF(G)$ and R is elementary abelian. The Frattini Argument yields

$$G = N_G(R)F(G).$$

Now K is not nilpotent since $F(G) < K$ whence $[F(G), R] \neq 1$. As $G = N_G(R)F(G)$ it follows that $[F(G), R]$ is a P-invariant normal subgroup of G. Let V be a minimal P-invariant normal subgroup of G contained in $[F(G), R]$. Since $F(G)$ is abelian we have $C_{[F(G),R]}(R) = 1$ whence

$$C_V(R) = 1.$$

Lemma 3. $G = RV$.

Proof. We consider V as a PR-module over $GF(q)$. Since $[RV, P] \trianglelefteq PRV$ we see that $V \cap [RV, P]$ is a PR-submodule. As PR has order coprime to q it follows from Maschke's

Theorem that V contains a PR-submodule W such that

$$(*) \qquad V = W \times (V \cap [RV, P]).$$

Now $[W, P] \leqq W \cap (V \cap [RV, P]) = 1$ so P acts trivially on W. Then so also does $[R, P]$. We have

$$R = [R, P] C_R(P)$$

so as $C_V(R) = 1$ we see that $C_W(C_R(P)) = 1$. Consequently

$$W = [W, C_R(P)].$$

Recall that $D \trianglelefteq C_G(P)$ so Lemma 1 (a) implies that $C_G(P)/D$ is a q-group. Since $r \neq q$ it follows that $C_R(P) \leqq D$. Since P acts trivially on W we have $W \leqq C_G(P)$ whence $W = [W, C_R(P)] \leqq D$.

From $(*)$ we have

$$C_V(P) = W \times (V \cap C_{[RV,P]}(P)).$$

Lemma 1 (a) implies that $C_V(P) \nleqq D$ whence $C_{[RV,P]}(P) \nleqq D$. The minimality of G forces $G = RV$. $\square$

We are now in a position to complete the proof of Theorem A. We will regard V as a GP-module over $GF(q)$. Let $\overline{G} = G/V$. Since $G = [G, P]$ we have $\overline{G} = [\overline{G}, P]$. The previous lemma implies that $\overline{G}$ is abelian whence $C_{\overline{G}}(P) = 1$ and then $C_G(P) \leqq V$. In particular, $C_R(P) = 1$. Choose $g \in R^\sharp$. Now $[g, P] \neq 1$ is a P-invariant subgroup of R, R is abelian and $G = RV$. The minimal choice of R implies that $R = [g, P]$.

Let $v \in V$ and consider the P-invariant subgroup $[gv, P]$. We have $\overline{[gv, P]} = [\overline{gv}, P] = [\overline{g}, P] = \overline{R} = \overline{G}$ whence

$$G = [gv, P] V.$$

Now $G \neq [gv, P]$ and V is a minimal P-invariant normal subgroup of G. Then $[gv, P] \cap V = 1$ and it follows that $[gv, P]$ is a P-invariant Sylow r-subgroup of G. Since R is also a P-invariant Sylow r-subgroup of G, there exists $u \in C_G(P)$ such that $[gv, P]^u = R$ by [2, Theorem 6.2.2(ii), p. 224].

Working in the semidirect product GP, we have

$$\langle P, P^g \rangle = [g, P] P = RP \quad \text{and} \quad \langle P, P^{gv} \rangle = [gv, P] P.$$

Then $P^{gvu} \leqq ([gv, P] P)^u = RP$ and $P^g \leqq RP$. Now $C_G(P) \leqq V$ so $u \in C_V(P)$ and also $vu \in V$. Then

$$[vu, P^g] \leqq \langle P^g, P^{gvu} \rangle \cap V \leqq PR \cap V = 1$$

so $vu \in C_V(P^g)$. We deduce that $V = C_V(P) C_V(P^g)$ whence

$$\dim(V) \leqq 2 \dim(C_V(P)).$$

Recall that $C_V(R) = 1$, that $C_R(P) = 1$ and that R is abelian. The following lemma provides a contradiction and completes the proof of Theorem A.

Lemma 4. *Suppose that V is an X-module over a field of characteristic q where $X = RP$, R is an abelian normal subgroup of X, $(|P|, |R|) = 1$, $R = [R, P]$, $C_V(R) = 0$ and X is a q'-group. If t is the smallest prime dividing $|P|$ then*

$$\dim(V) \geqq t \dim(C_V(P)).$$

Proof. We may assume that the field is algebraically closed and that X acts irreducibly and faithfully on V. Let $V_1, \ldots, V_n$ be the homogeneous components of R, so that $V = V_1 \oplus \ldots \oplus V_n$ and P acts transitively on $\{V_1, \ldots, V_n\}$. If $n = 1$ then since R is abelian, it is cyclic and acts as scaler transformations on V. But then $[R, P] = 1$, a contradiction. Thus $n > 1$.

For each $i > 1$, choose $\alpha_i \in P$ such that $V_1\alpha_i = V_i$ and define $\beta_i : V_1 \longrightarrow [V, P]$ by $v_1\beta_i = v_1\alpha_i - v_1$. Clearly $\dim \mathrm{Im}(\beta_i) = \dim(V_1)$ and it is readily verified that $\mathrm{Im}(\beta_i) \cap (\mathrm{Im}(\beta_2) + \cdots + \mathrm{Im}(\beta_{i-1})) = 0$ for all $i \geqq 3$. Consequently,

$$\dim([V, P]) \geqq (n - 1) \dim(V_1).$$

Now $\dim(V) = \dim([V, P]) + \dim(C_V(P))$ and $\dim(V) = n \dim(V_1)$ whence

$$\dim(V) \geqq n \dim(C_V(P)).$$

Since P acts transitively on $\{V_1, \ldots, V_n\}$ we have that n divides $|P|$ and as $n > 1$ we have $n \geqq t$. Hence result □

Remark. Theorem A can be used to construct signalizer functors when studying variants of the problem considered in [1].

References

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