On the Fitting height of a soluble group that is generated by a conjugacy class

Paul Flavell

The School of Mathematics and Statistics
The University of Birmingham
Birmingham B15 2TT
United Kingdom
e-mail: p.j.flavell@bham.ac.uk

Let $G$ be a finite group and suppose that $P$ is a soluble $\{2, 3\}'$-subgroup of $G$. The reader will lose only a little by assuming that $P$ is a subgroup of prime order $p > 3$. Define

$\Sigma_G(P) = \{ A \leq G \mid A \text{ is soluble and } A = \langle P, P^a \rangle \text{ for some } a \in A \}.$

This set is partially ordered by inclusion and we let $\Sigma_G^*(P)$ denote the set of maximal members of $\Sigma_G(P)$.

This article grew out of the following discovery:

**Theorem A.** Let $G$ be a finite group, let $P$ be a soluble $\{2, 3\}'$-subgroup of $G$ and choose $A \in \Sigma_G^*(P)$. Then

$$F(A)V$$

is nilpotent for every nilpotent subgroup $V$ of $G$ that is normalized by $A$. 

1
The article of Bender [1] indicates the usefulness of such results in the study of finite groups. An immediate consequence is the following:

**Corollary B.** Assume the hypotheses of Theorem A and that $G$ is soluble. Then

$$\pi(F(A)) \subseteq \pi(F(G)).$$

Thus at least when $G$ is soluble, the members of $\Sigma^*_G(P)$ reflect the global properties of $G$. This is a little surprising since, in some senses, the members of $\Sigma^*_G(P)$ can be small compared to $G$. Indeed if $P$ is cyclic then every member of $\Sigma^*_G(P)$ has cyclic abelianization.

It is tempting to conjecture that the conclusion of Corollary B can be replaced by the stronger assertion that

$$F(A) \leq F(G).$$

This is false. However, by considering the notion of Fitting height, it is possible to prove a result that is just as good.

The following consequence of Theorem A is the starting point.

**Theorem C.** Let $G$ be a finite soluble group and let $C$ be a conjugacy class of $\{2, 3\}$'-subgroups of $G$. If $G$ is generated by $C$ then there exist two members of $C$ that generate a subgroup with the same Fitting height as $G$.

Moreover, the two members of $C$ may be chosen to be conjugate in the subgroup that they generate.

This leads us to define:

$$\Sigma_G^f(P)$$

to be the set consisting of those members of $\Sigma_G(P)$ with maximal Fitting height. Moreover, if $G \neq 1$ is soluble we let

$$\psi(G) = \bigcap\{ K \trianglelefteq G \mid f(G/K) < f(G) \},$$

where $f(G)$ denotes the Fitting height of $G$. Then $\psi(G)$ is the unique smallest normal subgroup of $G$ such that $f(G/\psi(G)) < f(G)$ and it is also the case that $1 \neq \psi(G) \leq F(G)$. If $G = 1$ then we let $\psi(G) = 1$. Using Theorem C we obtain:
Theorem D. Let $P$ be a $\{2,3\}'$-subgroup of the finite soluble group $G$ and choose $A \in \Sigma^f_G(P)$. Then
\[ \psi(A) \leq F(G). \]

Combining this with the Baer-Suzuki Theorem we obtain:

Corollary E. Let $C$ be a conjugacy class of $\{2,3\}'$-subgroups of the finite group $G$. Then $C$ generates a soluble subgroup if and only if every four members of $C$ generate a soluble subgroup.

Assume now the hypotheses of Theorem D. The conclusion of that theorem says that we can write down a subnormal nilpotent subgroup of $G$ just by examining the subgroups that are generated by two conjugates of $P$. If $G = \langle P^G \rangle$ then we can go a little further:

Theorem F. Let $P$ be a $\{2,3\}'$-subgroup of the finite soluble group $G$ and suppose that $G = \langle P^G \rangle$. Then
\[ \psi(G) = \left\langle \psi(A) \mid A \in \Sigma^f_G(P) \right\rangle. \]

In other words, we can write down a characteristic nilpotent subgroup of $G$ in terms of subgroups that are ‘localized’ at $P$. 

1 Preliminaries

Henceforth the word ‘group’ shall mean ‘finite group’.

**Lemma 1.1.** Let $P$ be a soluble $\{2,3\}'$-subgroup of the group $G$, let $V$ be a soluble normal subgroup of $G$, set $\overline{G} = G/V$ and suppose that $\overline{A} \in \Sigma_{\overline{P}}(\overline{P})$. Then $\overline{A}$ has an inverse image which is a member of $\Sigma_G(P)$.

**Proof.** Choose $a \in G$ with $\overline{a} \in \langle \overline{P}, \overline{P}^a \rangle = \overline{A}$ and $\langle P, Pa \rangle$ minimal. Then $\langle P, Pa \rangle \in \Sigma_G(P)$. □

For a group $G$ we let $F(G)$ be the Fitting subgroup of $G$, that is, the largest normal nilpotent subgroup of $G$. For a prime $q$ we let $O_q(G)$ be the largest normal $q$-subgroup of $G$.

If $G$ is soluble we let $f(G)$ denote the Fitting height of $G$. This is the smallest integer $n$ such that $G$ possesses a series

$$1 = F_0 \trianglelefteq F_1 \trianglelefteq \cdots \trianglelefteq F_n = G$$

with $F_{i+1}/F_i$ nilpotent for all $i$. If $G \neq 1$ then $f(G/F(G)) = f(G) - 1$.

The following two results are elementary.

**Lemma 1.2.** Let $H \leq G$ with $G$ soluble and $f(H) = f(G)$. Then

$$\psi(H) \leq \psi(G) \leq F(G).$$

**Lemma 1.3.** Let $G$ be soluble, let $N \trianglelefteq G$, set $\overline{G} = G/N$ and suppose that $\overline{G} \neq 1$. Then the following are equivalent:

(i) $\psi(\overline{G}) \neq 1$.

(ii) $f(\overline{G}) = f(G)$.

(iii) $\psi(\overline{G}) = \psi(G)$. 

4
2 Modules for soluble groups

Lemma 2.1. Let \( G \) be a group, let \( V \neq 1 \) be an irreducible \( G \)-module over an algebraically closed field and let \( K \trianglelefteq G \) be such that \( G/K \) is cyclic. If \( V \) is homogeneous as a \( K \)-module then it is irreducible as a \( K \)-module.

Proof. This follows from [2, Theorems 11.20, p.278 and 11.46, p.303]. \( \square \)

Lemma 2.2. Let \( Q \) be an extraspecial \( q \)-group of exponent \( q \) and order \( q^{1+2t} \). Suppose that \( A \) is a noncyclic abelian normal subgroup of \( Q \) with order \( q^{1+k} \). Then the following hold:

(a) \( Z(Q) \leq A \) and \( A \) possesses exactly \( q^k \) hyperplanes \( A_1, \ldots, A_{q^k} \) that do not contain \( Z(Q) \). These hyperplanes are permuted transitively by \( Q \).

(b) Let \( V \) be a faithful homogeneous \( Q \)-module over a field of characteristic prime to \( q \). For each \( i \) set \( V_i = C_V(A_i) \) and set \( \Omega = \{V_1, \ldots, V_{q^k}\} \). Then \( \Omega \) is permuted transitively by \( Q \),

\[ V = V_1 \oplus \cdots \oplus V_{q^k} \quad \text{and} \quad C_A(V_i) = A_i \text{ for all } i. \]

(c) Let the \( q \)-group \( P \) act as a group of automorphisms of \( Q \). Then \( Q \) possesses a \( P \)-invariant abelian normal subgroup with order \( q^{1+t} \).

Proof. Let \( \overline{Q} = Q/Z(Q) \). Since \( Q \) is extraspecial we have \( Z(Q) = Q' = \Phi(Q) \cong \mathbb{Z}_q \) so \( \overline{Q} \) may be regarded as a \( GF(q) \)-vectorspace and the map \((,): \overline{Q} \times \overline{Q} \longrightarrow Z(Q) \) defined by

\[ (Z(Q)x, Z(Q)y) = [x, y] \]

is a nondegenerate symplectic form on \( \overline{Q} \).

(a). The first assertion is true since \( Z(Q) = Q' \cong \mathbb{Z}_q \) and \( 1 \neq A \trianglelefteq Q \). The second assertion follows from a counting argument and the fact that \( A \) is elementary abelian. Let \( B = A_1 \). The group \( \overline{Q} \) acts by conjugation on the set of hyperplanes of \( A \). Since \( Z(Q) \cap B = 1 \) it follows that

\[ N_{\overline{Q}}(B) = \overline{B}^\perp, \]
so $B$ has $|Q : B^\perp|$ conjugates. Now $\text{codim } B^\perp = \dim B$ so $|Q : B^\perp| = |B|$. As $Z(Q) \cap B = 1$ we have $|B| = q^k$. Consequently, $B$ has $q^k$ conjugates. This proves (a).

(b). The first assertion follows from (a). We may suppose that $Q$ acts irreducibly on $V$. In particular, $C_V(Z(Q)) = 0$. Let $U \leq V$ be an irreducible $A$-module. Since $A$ is elementary abelian it follows that $C_A(U)$ is a hyperplane of $A$. As $C_V(Z(Q)) = 0$ we have $C_A(U) = A_i$ for some $i$. Then $\langle \Omega \rangle \neq 0$ and then the irreducibility of $Q$ on $V$ forces $V = \langle \Omega \rangle$.

If $i \neq j$ then $V_i \cap V_j \leq C_V(A) = 0$. Let $i < q^k$ be such that $V_1 + \cdots + V_i = V_i \oplus \cdots \oplus V_i$. Now $A$ is abelian so it normalizes each $V_j$ and as $V_{i+1} = C_V(A_{i+1})$ we have

$$V_{i+1} \cap (V_1 \oplus \cdots \oplus V_i) = (V_{i+1} \cap V_i) \oplus \cdots \oplus (V_{i+1} \cap V_i) = 0.$$ We deduce that $V = V_1 \oplus \cdots \oplus V_{q^k}$. The final assertion in (b) follows from $C_V(Z(Q)) = 0$.

(c). Since $Z(Q) \cong \mathbb{Z}_q$ we have $[Z(Q), P] = 1$ so $P$ acts as a group of isometries on $Q$. Let $U$ be a maximal $P$-invariant isotropic subspace of $Q$. Suppose that $U < U^\perp$. Then $P$ acts on the nontrivial $GF(q)$-vectorspace $U^\perp / U$. Since $P$ is a $q$-group, it fixes a nonzero vector $v + U \in U^\perp / U$. Then $U \oplus \langle v \rangle$ is a $P$-invariant isotropic subspace, contrary to the maximal choice of $U$. Thus $U = U^\perp$ and then $\dim U = t$. The inverse image of $U$ in $Q$ has the desired properties.

\[ \square \]

**Lemma 2.3.** Let $G$ be a soluble non-nilpotent primitive linear group over an algebraically closed field. Then there exists $Q \trianglelefteq G$ such that $Q$ is an extraspecial $q$-group and $G$ acts nontrivially and irreducibly on $Q/\Phi(Q)$. Moreover, if $q \neq 2$ then $Q$ has exponent $q$.

**Proof.** Let $\overline{G} = G/F(G)$ and choose a prime $r$ such that $O_r(\overline{G}) \neq 1$, let $K$ be the inverse image of $O_r(\overline{G})$ in $G$ and choose $R \in \text{Syl}_r(K)$. Then $K = R O_r(F(G))$ and since $R \not\subseteq O_r(G)$ we have $[O_r(F(G)), R] \neq 1$. Choose a prime $q \neq r$ such that $[O_q(G), R] \neq 1$ and set $S = [O_q(G), R]$. Note that $S = [S, R]$. The Frattini Argument implies that

$$G = N_G(R) O_r(F(G))$$

and it follows that $S \trianglelefteq G$. 

6
Let $Q$ be a subgroup of $S$ that is minimal subject to $Q \trianglelefteq G$ and $[Q, R] \neq 1$. The hypotheses imply that every abelian normal subgroup of $G$ is cyclic and contained in $Z(G)$. In particular, $Q$ is nonabelian. Suppose that $T < Q$ with $T \trianglelefteq G$. The minimality of $Q$ implies that $R$ acts trivially on $T$ and as $S = [S, R]$ it follows that $S$ acts trivially on $T$, whence $T \leq Z(S)$. We deduce that

$$1 \neq Q' \leq \Phi(Q) \leq Z(Q) \leq Z(S) \leq Z(G).$$

(1)

Since $S$ is nilpotent we have $[Q, S] \leq Q$ whence

$$[Q, S] \leq Z(S) \leq Z(G)$$

also.

We claim that $Q' = [Q, S] \cong \mathbb{Z}_q$. Let $x \in Q$ and $y \in S$. Since $[Q, S] \leq Z(S)$ we have $[x, y]^g = [x^g, y]$. But $x^g \in \Phi(Q) \leq Z(S)$ whence $[x, y]^g = 1$. Since $Z(S)$ is cyclic and since $Q' \neq 1$ the claim follows.

Now $[C_Q(R), S] \leq [Q, S] \leq C_Q(R)$ so $C_Q(R) \trianglelefteq S$. As $S = [S, R]$ it follows that $C_Q(R) \leq Z(S)$. Since $Q = C_Q(R)[Q, R]$ we have $Q = Z(Q)[Q, R]$ and then $Q' = [Q, R]'$. By the previous paragraph we have $Q' = [Q, S]'$ and in particular $[Q, R] \leq S$. Recall that $G = N_G(R)O_r(F(G))$ and that $Q \leq S = [O_q(G), R]$. It follows that $[Q, R] \leq G$ and as $1 \neq [Q, R] = [Q, R, R]'$, the minimal choice of $Q$ yields

$$Q = [Q, R].$$

Let $Q^* = Q/Q'$ and consider the action of $G$ on $Q^*$. Now $Q^* = [Q^*, R]$ and since $Q^*$ is abelian we have $C_{Q^*}(R) = 1$ by [3, Theorem 5.2.3, p.177]. Suppose that $T^*$ is a proper $G$-invariant subgroup of $Q^*$. Let $T$ be the inverse image of $T^*$ in $G$. The minimal choice of $Q$ implies that $[T, R] = 1$ whence $T^* \leq C_{Q^*}(R) = 1$. This implies that $Q^*$ is elementary abelian and then that the action of $G$ on $Q^*$ is irreducible. This action is nontrivial since $Q^* = [Q^*, R]$. From (1) we have $1 \neq Q' \leq \Phi(Q) \leq Z(Q) < Q$ whence $Q' = \Phi(Q) = Z(Q)$. We have seen that $Q' \cong \mathbb{Z}_q$ so $Q$ is extraspecial. If $q \neq 2$ then by [3, Theorem 5.3.10, p.184] and the minimal choice of $Q$ we have $Q = \Omega_1(Q)$ and then the fact that $Q' \leq Z(Q)$ implies that $Q$ has exponent $q$. □
**Theorem 2.4.** Let $G$ be a soluble group, suppose that $P$ is a $\{2, 3\}'$-subgroup of $G$ such that $G = \langle P^G \rangle$, suppose that $V \neq 0$ is a $G$-module that does not involve the trivial $G$-module and let $p = \min \pi(P)$. Then

$$\dim C_V(P) \leq \frac{2}{p} \dim V.$$ 

**Proof.** Assume false and consider a counterexample in which $|G| + |P|$ is minimized and then $\dim V$ is minimized. We may suppose that $F$, the underlying field for $V$, is algebraically closed and that $G$ acts irreducibly on $V$. We have

$$\dim C_V(P) > \frac{2}{p} \dim V.$$ 

**Step 1** Let $\Omega = \{V_1, \ldots, V_n\}$ be a $P$-invariant collection of subspaces of $V$, all of the same dimension, such that

$$V = V_1 \oplus \cdots \oplus V_n.$$ 

Let $m = |\text{Fix}_\Omega(P)|$. Then

$$n < pm.$$ 

**Proof.** Let $\Omega_1, \ldots, \Omega_k$ be the orbits of $P$ on $\Omega$. Then

$$\dim C_V(P) = \sum_{i=1}^{k} \dim C_{\langle \Omega_i \rangle}(P) \leq \sum_{i=1}^{k} \frac{1}{|\Omega_i|} \dim \langle \Omega_i \rangle.$$ 

We may suppose that $\Omega_1, \ldots, \Omega_m$ are the orbits of size 1, so the other orbits all have size at least $p$. Then

$$\dim C_V(P) \leq m \dim V_1 + \frac{1}{p} \sum_{i=m+1}^{k} \dim \langle \Omega_i \rangle \leq m \dim V_1 + \frac{1}{p} (\dim V - m \dim V_1).$$ 

Now $\dim C_V(P) > \frac{2}{p} \dim V$ whence

$$\frac{1}{p} \dim V \leq m \left( 1 - \frac{1}{p} \right) \dim V_1.$$ 

Since $\dim V = n \dim V_1$, the result follows. □
**Step 2** $G$ is primitive on $V$. In particular, every normal subgroup of $G$ is homogeneous on $V$ and $Z(G)$ is the unique maximal abelian normal subgroup of $G$.

*Proof.* Assume false. Then there exists a collection $\Omega = \{V_1, \ldots, V_n\}$ of subspaces of $V$ that are permuted transitively by $G$ such that

$$V = V_1 \oplus \cdots \oplus V_n \text{ and } n \geq 2.$$ 

Choose such a collection with $n$ minimal. Let

$$K = \ker (G \to S_\Omega) \text{ and } \overline{G} = G/K.$$ 

Then $\overline{G}$ is a faithful primitive permutation group on $\Omega$. Let $\overline{L}$ be a minimal normal subgroup of $\overline{G}$. Since $\overline{G}$ is soluble and primitive on $\Omega$ it follows that $\overline{L}$ is an elementary abelian $l$-group for some prime $l$, that $\overline{L}$ is regular on $\Omega$ and that

$$\overline{G} = \text{Stab}_{\overline{G}}(U) \overline{L} \text{ and } \text{Stab}_{\overline{G}}(U) \cap \overline{L} = 1 \text{ for all } U \in \Omega.$$ 

Step 1 implies that $\text{Fix}_\Omega(P) \neq \emptyset$. We claim that $C_{\overline{L}}(\overline{P})$ acts regularly on $\text{Fix}_\Omega(P)$. Let $U, W \in \text{Fix}_\Omega(P)$. There exists $g \in \overline{L}$ such that $Wg = U$. Then $\overline{P}, \overline{P}^g \leq \text{Stab}_{\overline{G}}(U)$ so

$$[g, \overline{P}] \leq \text{Stab}_{\overline{G}}(U) \cap \overline{L} = 1$$

and hence $g \in C_{\overline{L}}(\overline{P})$. Since $\overline{L}$ is regular on $\Omega$ we have proved the claim.

Now $|\Omega| = |L|$ and $|\text{Fix}_\Omega(P)| = |C_{\overline{L}}(\overline{P})|$ so Step 1 implies that

$$|L : C_{\overline{L}}(\overline{P})| < p.$$ 

Using the fact that every nonidentity element of $\overline{P}$ has order at least $p$, it follows that $\overline{L} = C_{\overline{L}}(\overline{P})$ and then that $\overline{P}$ acts trivially on $\Omega$. Since $\overline{G} = \langle \overline{P}^g \rangle$ and since $\overline{G}$ acts transitively on $\Omega$, we have obtained a contradiction. Thus every normal subgroup of $G$ acts homogeneously on $V$.

The final two assertions follow from the first. □

Now $G$ is irreducible on $V$ so $C_V(G) = 0$ whence $P < G$. Since $G = \langle P^G \rangle$, it follows that $G$ is not nilpotent. Step 2 and Lemma 2.3 imply that there exists a prime $q$ and $Q \leq G$ such that $Q$ is an extraspecial $q$-group and that $G$ acts irreducibly and nontrivially on $Q/\Phi(Q)$. Also, $C_V(Q) = 0$ so $q \neq \text{char}(F)$.
Step 3 $P/C_P(Q)$ is a $q$-group.

Proof. Assume false. Then there exists a prime $p_0 \neq q$ and a cyclic $p_0$-group $P_0 \leq P$ such that $[Q, P_0] \neq 1$. Set $G_0 = P_0[Q, P_0]$. Since $p_0 \neq q$ we have $[Q, P_0] = [Q, P_0, P_0]$ whence $G_0 = \langle P_0^{G_0} \rangle$. Now $Q$ acts homogeneously and faithfully on $V$ and as $[Q, P_0] \leq Q$ it follows from Maschke’s Theorem that $V$ does not involve the trivial $[Q, P_0]$-module. In particular, $V$ does not involve the trivial $G_0$-module and then the minimality of $|G| + |P|$ forces $G = G_0, P = P_0, p = p_0$ and $Q = [Q, P]$. Then $G = PQ$ and $C_P(Q) \leq G$.

Since $C_V(P) \neq 0$ and since $G$ is irreducible on $V$ we have $C_P(Q) = 1$. Also, Lemma 2.1 implies that $Q$ acts irreducibly on $V$.

By [3, Theorem 5.5.5, p.208] there exists an integer $t$ such that

$$|Q| = q^{1+2t} \quad \text{and} \quad \dim V = q^t.$$ 

We have $|P| = p^n$ for some integer $n \geq 1$. Since $P$ is a $\{2,3\}'$-group we have $p > 3$ so the first paragraph of the proof of [3, Lemma 11.2.5, p.368] implies that

$$p^n \text{ divides } q^t + 1.$$ 

The argument now splits into two cases depending on whether $F$ has characteristic $p$ or not.

Case $p = \text{char}(F)$. The Hall-Higman Theorem [3, Theorem 11.2.1, p.364] implies that the Jordan canonical form for a generator of $P$ consists of $(q^t + 1)/p^n$ Jordan blocks. Recalling that $\dim V = q^t$ we have

$$\dim C_V(P) = \frac{\dim V + 1}{p^n} \leq \frac{2}{p} \dim V,$$

a contradiction.

Case $p \neq \text{char}(F)$. Let $\chi$ be the character of $G$ afforded by $V$. Using the Coprime Hall-Higman Theorem [4, Satz V.17.13, p.574] together with the fact that $p^n | q^t + 1$ we have

$$\chi_P = \frac{q^t + 1}{p^n} \rho - \mu$$

10
where \( \rho \) is the regular character of \( P \) and \( \mu \) is a linear character of \( P \). Then

\[
\dim C_V(P) \leq \frac{\dim V + 1}{p^n} \leq \frac{2}{p} \dim V.
\]

This contradiction completes the proof of Step 3.

Recall that \( G \) acts nontrivially and irreducibly on \( Q/\Phi(Q) \) and that \( p = \min \pi(P) \). Since \( G = \langle P^G \rangle \) and \( P \) is a \( \{2, 3\}' \)-group it follows from the previous step that

\[ q > 3 \quad \text{and that} \quad p \leq q. \]

Then Lemma 2.3 implies that \( Q \) has exponent \( q \).

**Step 4** Let \( A \) be an abelian normal subgroup of \( Q \) that is normalized by \( P \). Then \( A \) is centralized by \( P \).

**Proof.** Let \( |A| = q^{1+k} \) and note that \( A \) is elementary abelian. Since \( Z(Q) \leq Z(G) \) we have \([Z(Q), P] = 1\). If \( k = 0 \) then \( A = Z(Q) \), hence we may suppose that \( k \geq 1 \). We assume the notation of Lemma 2.2. Now \( P \) normalizes \( A \) so it permutes \( \Omega \). Let \( m = |\text{Fix}_\Omega(P)| \). Step 1 together with \( p \leq q \) implies that

\[ q^k < qm. \]

Hence \( m > 1 \). Lemma 2.2(b) implies that if \( V_i \in \text{Fix}_\Omega(P) \) then \( P \) normalizes \( A_i \). Note that \( |A_i| = q^k \) for all \( i \).

Since \( m > 1 \) we may suppose that \( V_1, V_2 \in \text{Fix}_\Omega(P) \). If \( k = 1 \) then \( |A_1| = |A_2| = q \) and then Step 3 implies that \( P \) centralizes \( A_1 \) and \( A_2 \). But \( A = \langle A_1, A_2 \rangle \) so \( P \) centralizes \( A \). Hence we may assume that \( k \geq 2 \). Now \( q^k < qm \) so \( q < m \).

Now \( |A : A_1 \cap A_2| = q^2 \) and \( Z(Q) \cap A_1 \cap A_2 = 1 \) so it follows that \( A \) contains exactly \( q \) hyperplanes which contain \( A_1 \cap A_2 \) but not \( Z(Q) \). Since \( q < m \) we may suppose that \( V_3 \in \text{Fix}_\Omega(P) \) and that \( A_1 \cap A_2 \nsubseteq A_3 \). Then \( A_1 \cap A_2 \) and \( A_1 \cap A_3 \) are distinct hyperplanes of \( A_1 \) whence \( A = Z(Q)(A_1 \cap A_2)(A_1 \cap A_3) \).

Recall that \( P \) normalizes \( A_1 \) and \( A_2 \) so \( P \) normalizes \( Z(Q)(A_1 \cap A_2) \). This subgroup has index \( q \) in \( A \) and it is normal in \( Q \) since \( Q' = Z(Q) \). By induction, \( P \) centralizes \( Z(Q)(A_1 \cap A_2) \). Similarly \( P \) centralizes \( Z(Q)(A_1 \cap A_3) \) and we deduce that \( P \) centralizes \( A \).
We are now in a position to obtain a final contradiction. Let \( \overline{Q} = Q/Z(Q) \) and regard \( \overline{Q} \) as a \( GF(q)G \)-module. Since \( G \) acts irreducibly and nontrivially on \( \overline{Q} \) we have that \( \overline{Q} \) does not involve the trivial \( G \)-module. Since \( Z(Q) \) is in the kernel of the action of \( G \) on \( Q \), we may invoke the minimality of \( G \) to obtain

\[
\dim C_{\overline{Q}}(P) \leq \frac{2}{p} \dim \overline{Q}.
\]

Choose \( t \) such that \( |Q| = q^{1+2t} \), so that \( \dim \overline{Q} = 2t \). Using Step 3, Lemma 2.2(c) and Step 4 we see that

\[
\dim C_{\overline{Q}}(P) \geq \frac{1}{2} \dim \overline{Q}.
\]

But \( p \geq 5 \) so this contradicts the previous inequality and completes the proof of this theorem. \( \square \)

**Remark**  By modifying the conclusion, it ought to be possible to remove the hypothesis that \( P \) is a \( \{2,3\}' \)-group.

**Corollary 2.5.** Assume the hypotheses of Theorem 2.3. Then

\[
\dim C_{\overline{Q}}(P) < \frac{1}{2} \dim V.
\]

**Remark**  It is in fact Corollary 2.5 that we shall use rather than the stronger Theorem 2.4. If it is desired to prove only Corollary 2.5 then a simpler proof is possible. In particular, the appeal to Hall-Higman theory in Step 3 may be replaced by a more elementary argument. Indeed, in Step 3 we have

\[
G = PQ
\]

where \( Q \) is an extraspecial \( q \)-group, \( P \) is a cyclic \( p \)-group that acts faithfully and irreducibly on \( Q/\Phi(Q) \) and \( V \) is a faithful \( G \)-module on which \( Q \) acts irreducibly.

Let \( E \) be the enveloping algebra of \( Q \) on \( V \). By Weddurburn's Theorem [3, Theorem 3.6.3, p.86] we have \( E = \text{End}(V) \), so then \( \dim E = (\dim V)^2 \). Choose \( x \in P \) with prime order \( p \).
The linear transformations $y : V \rightarrow V$ with $[V, x] \leq \ker y$ and $\text{Im } y \leq C_V(x)$ constitute a subspace of $C_E(x)$ with dimension $(\dim C_V(x))^2$. Considering the scalar transformations, it follows that

$$\dim C_E(x) \geq (\dim C_V(x))^2 + 1.$$  

Either by considering the action of $\langle x \rangle Q/\Phi(Q)$ on $E$ or by the argument of [3, Lemma 11.2.4, p.367] we have

$$\dim C_E(x) = \frac{\dim E - 1}{p} + 1.$$  

But $\dim E = (\dim V)^2$ and $p \geq 5$ so these inequalities yield

$$\dim C_V(x) < \frac{1}{2} \dim V.$$  

Also, another proof of Corollary 2.5 is possible by using a result of Robinson [5, Corollary 1.2].
The proofs of Theorems A–E

The proof of Theorem A. Assume false and consider a counterexample with $|G| + |V|$ minimal. Then $G = AV$ and there exist distinct primes $r$ and $q$ such that $V$ is an $r$-group and $O_q(A)$ does not centralize $V$. Since $[V, O_q(A)]$ is normalized by $A$, the minimality of $V$ forces $V = [V, O_q(A)]$. Note that $G \notin \Sigma_G(P)$.

Let $\overline{G} = G/\Phi(V)$. Then $\overline{G} = \overline{A} V$, $V$ is elementary abelian, $V = [V, O_q(A)] \neq 1$ and then $C_T(O_q(A)) = 1$. Let $\overline{U} \leq \overline{V}$ be a minimal normal subgroup of $\overline{G}$ and suppose that $\overline{U} < \overline{V}$. Let $U$ be the inverse image of $\overline{U}$ in $G$. The minimality of $|V|$ implies that $[O_q(A), U] = 1$ whence $\overline{U} \leq C_T(O_q(A)) = 1$, a contradiction. Thus $V$ is a minimal normal subgroup of $G$. Since $V$ is abelian, this implies that $A$ is a maximal subgroup of $G$ and that $A \cap V = 1$.

Clearly $A \in \Sigma_G(P)$. Suppose that $A \notin \Sigma^*_G(P)$. Then since $A$ is a maximal subgroup of $G$ we have $\overline{G} = \langle \overline{P}, \overline{P}^g \rangle$ for some $g \in G$. Then $G = \Phi(V)\langle P, P^g \rangle$ whence $V = \Phi(V)(V \cap \langle P, P^g \rangle)$ so $V \leq \langle P, P^g \rangle$ and then $G = \langle P, P^g \rangle \in \Sigma_G(P)$, a contradiction. We deduce that $A \in \Sigma^*_G(P)$ and then the minimality of $|G|$ forces $\Phi(V) = 1$. In particular, $A$ is a complement to $V$.

Set $N = O_q(A)V \leq G$ and note that $O_q(A) \in \text{Syl}_q(N)$. Since $C_V(O_q(A)) = 1$ we have $V \cap N_G(O_q(A)) = 1$ and it follows that the complements to $V$ in $G$ are the normalizers of the Sylow $q$-subgroups of $N$. In particular, $V$ acts transitively by conjugation on its set of complements.

Choose $a \in A$ such that $A = \langle P, P^a \rangle$. Let $v \in V$ and set $B = \langle P, P^{av} \rangle$. Since $G = AV$ we have $G = BV$. Now $G \notin \Sigma_G(P)$ and $V$ is a minimal normal subgroup of $G$ so it follows that $B$ is a complement to $V$. By the previous paragraph there exists $u \in V$ such that $B^u = A$. Then

$$\langle P^u, P^{av} \rangle = A = \langle P, P^a \rangle.$$ 

Thus $u \in C_V(P)$ and $vu \in C_V(P^a)$. Since $v$ was arbitrary, we deduce that $V = C_V(P)C_V(P^a)$. 

14
Regarding $V$ as a $GF(r)A$-module, this implies that
\[ \dim C_V(P) \geq \frac{1}{2} \dim V. \]

But $A$ acts irreducibly and nontrivially on $V$ and $A = \langle P^A \rangle$, so Corollary 2.5 supplies a contradiction.

\[ \square \]

The proof of Corollary B. This follows from Theorem A and the fact that $C_G(F(G)) \leq F(G)$.

\[ \square \]

The proof of Theorem C. Choose $P \in \mathcal{C}$. It suffices to show that there exists $A \in \Sigma_G(P)$ with $f(A) = f(G)$. Assume this to be false and let $G$ be a minimal counterexample. Choose $q \in \pi(F(G))$ and set
\[ \overline{G} = G/O_q(G). \]

Using Lemma 1.1 we see that $f(\overline{G}) = f(G) - 1$. Then $F(G) = O_q(G)$ since otherwise $G$ would embed into a direct product of two groups, both with Fitting height $f(G) - 1$.

The minimality of $G$ implies that there exists $\overline{A} \in \Sigma_{\overline{G}}(\overline{P})$ such that $f(\overline{A}) = f(\overline{G})$. By Lemma 1.1 there exists $A \in \Sigma_G(P)$ such that $A$ maps onto $\overline{A}$. Choose $A^*$ such that
\[ A \leq A^* \in \Sigma^*_G(P). \]

Now $f(G) - 1 = f(\overline{A}) \leq f(A) \leq f(A^*)$ so as $G$ is a counterexample, we deduce that $f(\overline{A}) = f(A) = f(A^*)$. By Lemma 1.2 we have $\psi(A) \leq F(A^*)$ so Theorem A implies that $\psi(A)O_q(G)$ is nilpotent. Now $F(G) = O_q(G)$ and $G$ is soluble so $C_G(O_q(G)) \leq O_q(G)$. We deduce that $\psi(A)$ is a $q$-group.

Since $f(\overline{A}) = f(A)$ it follows from Lemma 1.3 that $\psi(\overline{A})$ is a $q$-group.

Recall that $f(\overline{A}) = f(\overline{G})$ so Lemma 1.2 implies that $\psi(\overline{A}) \leq F(\overline{G})$. However, $\overline{G} = G/O_q(G)$ so $F(\overline{G})$ is a $q$'-group and then $\psi(\overline{A}) = 1$. This implies that $\overline{G} = 1$ and then that $G = O_q(G)$. Since $G = \langle P^{G} \rangle$, this forces $G = P$ and then $P$ is a member of $\Sigma_G(P)$ with Fitting height $f(G)$. This contradiction completes the proof. 

\[ \square \]
The proof of Theorem D. Set $H = \langle P^G \rangle$ and note that $A \leq H$. Then $A \in \Sigma^f_H(P)$. If $H < G$ then by induction we have $\psi(A) \leq F(H)$. But $H \leq G$ so $F(H) \leq F(G)$. Hence we may suppose that $H = G$. Then by Theorem C we have $f(A) = f(G)$ and then Lemma 1.2 forces $\psi(A) \leq F(G)$. \qed

The proof of Corollary E. Choose $P \in \mathcal{C}$ and $A \in \Sigma^f_G(P)$. Let $g \in G$ and set $H = \langle A, A^g \rangle$. Then $H$ is soluble since it is generated by four members of $\mathcal{C}$. By Theorem D we have $\langle \psi(A), \psi(A)^g \rangle \leq F(H)$. In particular, $\langle \psi(A), \psi(A)^g \rangle$ is nilpotent for all $g \in G$ so the Baer-Suzuki Theorem forces $\psi(A) \leq F(G)$. Now apply induction to $G/F(G)$. \qed
4 Generators for $\psi(G)$

**Lemma 4.1.** Let $G$ be a soluble group. Suppose that $f(G) \geq 2$ and that $\psi(G)$ is a $q$-group. Set $\overline{G} = G/\psi(G)$ and let $K$ be the inverse image of $O_q'(\psi(\overline{G}))$ in $G$. Then

$$\psi(G) = [\psi(G), K].$$

**Proof.** Let $L$ be the inverse image of $\psi(\overline{G})$ in $G$ and choose $Q \in \text{Syl}_q(L)$. Since $\psi(G)$ is a $q$-group and since $\psi(\overline{G})$ is nilpotent we have

$$L = KQ, \ K \trianglelefteq L \quad \text{and} \quad Q \trianglelefteq L. \quad (2)$$

Set

$$G^* = G/[\psi(G), K].$$

Now $K^*/\psi(G)^* \cong K/\psi(G)$, which is nilpotent. Since $\psi(G)^* \leq Z(K^*)$ we deduce that $K^*$ is nilpotent. Then using (2) we see that $L^*$ is nilpotent. We have

$$G^*/L^* \cong G/L \cong \overline{G}/\psi(\overline{G}).$$

Since $f(G) \geq 2$ we have $f(\overline{G}/\psi(\overline{G})) = f(G) - 2$. Now $L^*$ is nilpotent so $f(G^*/L^*) \geq f(G^*) - 1$ whence $f(G) - 2 \geq f(G^*) - 1$ so $f(G) > f(G^*)$. But then $\psi(G) \leq [\psi(G), K]$. \qed

17
The proof of Theorem F. Assume false and let $G$ be a minimal counterexample. Set

$$T = \left\langle \psi(A) \mid A \in \Sigma_f^G(P) \right\rangle.$$  

Using Theorem D we have

$$T \leq \psi(G) \text{ but } T \neq \psi(G).$$

**Step 1** Suppose that $V \neq 1$ is a normal subgroup of $G$ such that $f(G/V) = f(G)$. Then

$$\psi(G) = T(\psi(G) \cap V) \text{ and } \psi(G) \cap V \nleq T.$$  

**Proof.** Set $\overline{G} = G/V$. Since $f(\overline{G}) = f(G)$ we have $\psi(\overline{G}) = \overline{\psi(G)}$ by Lemma 1.3. The minimality of $G$ implies that $\psi(\overline{G}) = \langle \psi(\overline{A}) \mid \overline{A} \in \Sigma^f_G(P) \rangle$. Let $\overline{A} \in \Sigma^f_G(P)$. Theorem C implies that $f(\overline{A}) = f(\overline{G})$ and Lemma 1.1 implies that $\overline{A}$ has an inverse image $A \in \Sigma_G(P)$. Since $f(G) = f(\overline{G})$ it follows that $A \in \Sigma^f_G(P)$ and then Lemma 1.3 yields $\psi(\overline{A}) = \overline{\psi(A)}$. Consequently $\psi(G) \leq \langle \psi(A) \mid A \in \Sigma^f_G(P) \rangle V = TV$. Since $T \leq \psi(G)$ we have $\psi(G) = T(\psi(G) \cap V)$ and since $T \neq \psi(G)$ we have $\psi(G) \cap V \nleq T$.  

**Step 2** $\psi(G)$ is an elementary abelian $q$-group for some prime $q$.

**Proof.** Suppose that $q$ and $r$ are distinct prime divisors of $|\psi(G)|$. Then $\psi(G) \nleq O_q(\psi(G))$ so $f(G/O_q(\psi(G))) = f(G)$ and then Step 1 implies that $|\psi(G) : T|$ is a power of $q$. Similarly, $|\psi(G) : T|$ is a power of $r$ whence $\psi(G) = T$, a contradiction. Thus $\psi(G)$ is a $q$-group for some prime $q$. Suppose that $\Phi(\psi(G)) \neq 1$. Since $\Phi(\psi(G)) \neq \psi(G)$ we may apply Step 1 to conclude that $\psi(G) = T\Phi(\psi(G))$. But then $\psi(G) = T$, a contradiction. We deduce that $\Phi(\psi(G)) = 1$ and then that $\psi(G)$ is elementary abelian.  

18
Let \( \overline{G} = G/\psi(G) \) and \( \overline{K} = O_{\psi}(\psi(\overline{G})) \).

Let \( K \) be the inverse image of \( \overline{K} \) in \( G \). The minimality of \( G \) implies that

\[ \overline{K} = \langle O_{\psi}(\psi(A)) \mid A \in \Sigma_{G}^{f}(P) \rangle. \]

By Lemma 1.1, each member of \( \Sigma_{G}^{f}(P) \) has an inverse image in \( \Sigma_{G}(P) \) so we let

\[ \Sigma = \{ A \in \Sigma_{G}(P) \mid \overline{A} \in \Sigma_{G}(P) \} \]

and for each \( A \in \Sigma \) we let

\[ \Pi(A) \]

denote the inverse image of \( O_{\psi}(\psi(\overline{A})) \) in \( A \). Then

\[ K = \psi(G)\langle \Pi(A) \mid A \in \Sigma \rangle. \]

**Step 3** \( \psi(G) = \langle [\psi(G), \Pi(A)] \mid A \in \Sigma \rangle. \)

**Proof.** We will apply Lemma 4.1. If \( f(G) < 2 \) then \( G \) is nilpotent so as \( G = \langle P^{G} \rangle \) we have \( G = P \) and then \( G \in \Sigma_{G}(P) \), a contradiction. Thus \( f(G) \geq 2 \) and Lemma 4.1 implies that

\[ \psi(G) = [\psi(G), K]. \]

Now \( K = \psi(G)\langle \Pi(A) \mid A \in \Sigma \rangle \) and \( \psi(G) \) is abelian. Then

\[ \psi(G) = \langle [\psi(G), \Pi(A)] \mid A \in \Sigma \rangle \]

because \( K \) centralizes the quotient of the left hand side by the right hand side. \( \square \)
In what follows, we fix $A \in \Sigma$ such that

$$[\psi(G), \Pi(A)] \not\leq T.$$ 

Such an $A$ exists by Step 3 and the fact that $\psi(G) \not\leq T$. Set

$$H = A[\psi(G), \Pi(A)].$$

Choose $B$ such that

$$A \leq B \in \Sigma^*_H(P).$$

**Step 4** $[\psi(G), \Pi(A)] = [\psi(G), \Pi(A), \Pi(A)]$.

*Proof.* This is because $\Pi(A)/\psi(G) \cap \Pi(A)$ is a $q'$-group and $\psi(G)$ is abelian. \qed

**Step 5** $f(\overline{A}) = f(G) - 1$, $B \in \Sigma^*_G(P)$ and $f(H) = f(G)$.

*Proof.* Since $\overline{A} \in \Sigma^*_G(P)$ and $\overline{G} = G/\psi(G) = \langle \overline{P} \overline{G} \rangle$, Theorem C implies that $f(\overline{A}) = f(G) - 1$. We claim that $f(B) = f(G)$. Assume false. Then

$$f(G) - 1 \geq f(B) \geq f(A) \geq f(\overline{A}) = f(G) - 1$$

whence

$$f(B) = f(A) = f(\overline{A}).$$

Lemma 1.3 implies that $\Pi(A) \leq \psi(A)(A \cap \psi(G))$ and then using Lemma 1.2 we have $\Pi(A) \leq F(B)$. Now $B \in \Sigma^*_H(P)$ so Theorem A implies that $\Pi(A)F(H)$ is nilpotent. But $[\psi(G), \Pi(A)] \leq F(H)$ so it follows from Step 4 that $[\psi(G), \Pi(A)] = 1$, contrary to the choice of $A$. We deduce that $f(B) = f(G)$ so $B \in \Sigma^*_G(P)$ and then also $f(H) = f(G)$. \qed
Step 6 $[\psi(G), \Pi(A)] \leq \psi(H)$.

Proof. Set $H^* = H/\psi(H)$. By Step 5 we have $f(H) = f(G)$ so $\psi(H) \leq \psi(G)$. In particular, $\overline{A}$ is a homomorphic image of $A^*$. Then

$$f(G) - 1 = f(H^*) \geq f(A^*) \geq f(\overline{A}) = f(G) - 1$$

so $f(A^*) = f(\overline{A}) = f(H^*)$. Lemma 1.3 yields $\Pi(A^*) \leq \psi(A^*)(A \cap \psi(G))^*$ and then Lemma 1.2 forces $\Pi(A^*) \leq F(H^*)$. From Step 4 we have

$$[\psi(G), \Pi(A)]^* = [[\psi(G), \Pi(A)]^*, \Pi(A)^*].$$

Now $[\psi(G), \Pi(A)]^* \leq F(H^*)$ so as $\Pi(A)^* \leq F(H^*)$ and $F(H^*)$ is nilpotent it follows that $[\psi(G), \Pi(A)]^* = 1$. Hence $[\psi(G), \Pi(A)] \leq \psi(H)$.

We are now in a position to obtain a final contradiction. Since $A = \langle P^A \rangle$ and $H = A[\psi(G), \Pi(A)]$, it follows from Step 4 that $H = \langle P^H \rangle$. Also, $\Sigma^f_H(P) \subseteq \Sigma^f_G(P)$ since $f(H) = f(G)$. Now $[\psi(G), \Pi(A)] \not\subseteq T$ so Step 6 and the minimality of $G$ force $G = H$. Since $f(B) = f(G)$ we have $\psi(B) \leq \psi(G)$. Moreover, $A \leq B$, $\psi(G)$ is elementary abelian and $G = A[\psi(G), \Pi(A)]$ so $1 \neq \psi(B) \leq G$. By Step 5 and the definition of $T$ we have $\psi(B) \leq T$ so applying Step 1 with $V = \psi(B)$ it follows that $\psi(G) = \psi(B)$. This is a contradiction since $B \in \Sigma^f_G(P)$. 


