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An equivariant analogue of Glauberman's *ZJ*-Theorem

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Dedicated to J.G. Thompson on his 70th birthday

1. Introduction

We will prove an analogue of Glauberman's *ZJ*-Theorem, [4] and [6, Theorem 8.2.1, p. 279], that can be used to study finite groups that admit a coprime group of automorphisms. This analogue is unusual in that no hypothesis of p-stability is required. Used in conjunction with the Bender Method, it may make it possible to prove very general results about finite groups that admit a coprime group of automorphisms. The reader is referred to [7] and [8] for a fuller discussion of the Bender Method, Glauberman's *ZJ*-Theorem and p-stability.

Before stating the main result of this paper, we introduce some notation. If R and G are groups then we say that *R acts coprimely on G* if R acts as a group of automorphisms on G, if R and G have coprime orders and if at least one of R or G is soluble. Suppose that R acts coprimely on G and that p is a prime. Define

$$O_p(G;R) = \bigcap \mathcal{M}_G^*(R,p).$$

Recall that $\mathcal{M}_G(R,p)$ is the set of R-invariant p-subgroups of G and that $\mathcal{M}_G^*(R,p)$ is the set of maximal members of $\mathcal{M}_G(R,p)$ under inclusion. Sylow's Theorems for Groups with Operators [6, Theorem 6.2.2, p. 224] asserts that $\mathcal{M}_G^*(R,p)$ consists of Sylow p-subgroups of G and that $C_G(R)$ acts transitively by conjugation on $\mathcal{M}_G^*(R,p)$. It is a consequence of this last assertion that

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$O_p(G;R)$ is characterized as being the unique maximal $RC_G(R)$-invariant p-subgroup of G.

The main result proved in this paper is the following:

Theorem A. *Suppose that the group R acts coprimely on the group $G \neq 1$, that $p > 3$ is a prime and that $F^*(G) = O_p(G)$. Set $P = O_p(G;R)$. Then*

$$K^{\infty}(P) \trianglelefteq G.$$

In particular, P contains a nontrivial characteristic subgroup that is normal in G.

Theorem A is proved by invoking Glauberman's K^{∞}-Theorem [5, Theorem A] and Theorem B below on modules. The original idea was to use $ZJ(P)$ instead of $K^{\infty}(P)$ and to mimic the proof of Glauberman's *ZJ*-Theorem, using Theorem B as a substitute for p-stability. However the proof of the *ZJ*-Theorem requires the Frattini Argument, which cannot be applied to the subgroups $O_p(G;R)$. Fortunately, there is no such impediment to applying the rather less well known K^{∞}-Theorem.

We remark that the exact definition of $K^{\infty}(P)$ is unimportant for applications, rather it is the conclusion that some nontrivial characteristic subgroup of P is normal in G. In fact, the definition of $K^{\infty}(P)$ is more formidable than the definition of $ZJ(P)$. But curiously the reverse observation is true for the proofs of the K^{∞} and *ZJ*-Theorems.

Before stating Theorem B we recall that of V is a G-module and $g \in G$ then *g acts quadratically on V* if $[V,g,g] = 0$ and $[V,g] \neq 0$. If V is a faithful G-module then we may regard G as being contained in the ring $\mathrm{End}(V)$ and we often express the condition $[V,g,g] = 0$ as $(g-1)^2 = 0$. Of course, $[V,g] \neq 0$ is just another way of saying that g acts nontrivially on V.

Theorem B. *Suppose that the group R acts coprimely on the group G, that $p > 3$ is a prime and that V is a faithful GR-module over a field of characteristic p. Then any element of $O_p(G;R)$ that acts quadratically on V is contained in $O_p(G)$.*

The proof of Theorem B requires the following result of independent interest.

Theorem C. *Suppose that G is a group, that $p > 3$ is a prime, that V is a faithful G-module over a field of characteristic p and that L is a 2-local subgroup of G. Then any element of $O_p(L)$ that acts quadratically on V is contained in $O_p(G)$.*

We remark that the spin module for $\hat{A}_n$ shows that the conclusion of Theorem C may fail if $p = 3$.

Finally, we give an example of how Theorem A can be used to study the automorphism group of a simple group. It is a well known consequence of the

Classification of Finite Simple Groups that an abelian group that acts coprimely and faithfully on a simple group must be cyclic. The proof of the following corollary to Theorem A sheds a little light on this observation.

Corollary D. *Suppose that the abelian group R acts coprimely and faithfully on the simple group G. Let $\mathcal{M}$ be the set of proper subgroups of G that are maximal subject to being R-invariant and containing $C_G(R)$.*

Suppose that $p > 3$ and that $F^*(M) = O_p(M)$ for all $M \in \mathcal{M}$. Then R is cyclic.

2. Preliminaries to the proof of Theorem C

Suppose that G is a group and that p is a prime. For any p-element $a \in G$ define:

$$\mathcal{X}_G(a) = \{X \leqslant G \mid \text{(i) } a \in X, \quad \text{(ii) } a \notin O_p(X) \text{ and} \quad \text{(iii) } a \in O_p(Y) \text{ whenever } a \in Y < X\}.$$

Notice that $\mathcal{X}_G(a) \neq \emptyset$ provided $a \notin O_p(G)$.

Lemma 2.1. *Suppose that $X \in \mathcal{X}_G(a)$. Then*:

(i) *$X = \langle a, a^x \rangle$ for some $x \in X$.*
(ii) *a is contained in a unique maximal subgroup of X.*

Proof. (i) is a consequence of the Baer–Suzuki Theorem [1, Theorem 1.1, p. 4] and the fact that $a \notin O_p(X)$. (ii) follows from Wielandt's First Maximizer Lemma [9, Lemma 7.3.1, p. 222]. Alternatively, J.H. Walter's proof of the Baer–Suzuki Theorem as given in [1] can be trivially adapted to prove (ii). □

The following result is essentially due to Glauberman [2, Theorem 3.2].

Theorem 2.2. *Assume the following*:

(i) *G is a group, p is an odd prime and a is a p-element of G.*
(ii) *V is a faithful G-module over an algebraically closed field of characteristic p.*
(iii) *a acts quadratically on V.*
(iv) *$X \in \mathcal{X}_G(a)$.*

Then the following hold:

(a) *X contains a unique involution.*

(b) *Let i be the unique involution in X. Set*

$$X_0 = C_X\big(C_V(i)\big).$$

Then X_0 is a normal subgroup of X with index 1 or p, $X = X_0\langle a\rangle$ and there exists $x \in X_0$ such that $X = \langle a, a^x\rangle$.

(c) *If $C_V(X) = 0$ then $X \cong SL_2(p)$ and V is a direct sum of natural $SL_2(p)$-modules.*

Remark. A natural $SL_2(p)$-module over an arbitrary field is an $SL_2(p)$-module that has a basis with respect to which any matrix $\binom{a\ b}{c\ d}$ is represented by itself.

3. The proof of Theorem C

Assume Theorem C to be false and consider a counterexample with $|G|$ and then $\dim V$ minimized. Then

$$a \notin O_p(G). \tag{1}$$

We may suppose that the field of definition of V is algebraically closed. Since L is a 2-local subgroup of G there is a 2-subgroup $S \neq 1$ such that $L = N_G(S)$. Now p is odd and $a \in O_p(L) \leqslant C_L(S)$ whence $[a, S] = 1$. Choose S_1 with $S \leqslant S_1 \in \mathrm{Syl}_2(C_G(a))$. Then $a \in C_G(S_1) \leqslant C_G(S)$ so $a \in O_p(C_G(S_1)) \leqslant O_p(N_G(S_1))$. Hence we may replace S by S_1 to suppose that

$$S \in \mathrm{Syl}_2\big(C_G(a)\big). \tag{2}$$

Moreover, the minimality of $|G|$ implies that

$$G = \big\langle a^G\big\rangle S. \tag{3}$$

Step 1. *G acts irreducibly on V.*

Proof. Assume false. Let U be a G-composition factor of V and set $\overline{G} = G/C_G(U)$. Note that $O_p(\overline{G}) = 1$ since the field of definition for U has characteristic p.

We claim that $[\langle \bar{a}^{\overline{G}}\rangle, \overline{S}] = 1$. Let H be the inverse image of $N_{\overline{G}}(\overline{S})$ in G, so that $L \leqslant H$. If $H = G$ then $\overline{S} \trianglelefteq \overline{G}$ and as $[\bar{a}, \overline{S}] = 1$, the claim follows. If $H \neq G$ then the minimality of $|G|$ and $\dim V$ forces $a \in O_p(H)$ and then $\bar{a} \in O_p(\overline{G}) = 1$. Thus the claim follows in this case also.

What we have just done implies that

$$\big[\big\langle a^G\big\rangle, S\big] \leqslant \bigcap C_G(U),$$

where the intersection is over all G-composition factors U of V. This intersection is well known to equal $O_p(G)$. Since $G = \langle a^G \rangle S$ it follows that $SO_p(G) \trianglelefteq G$. Now S is a Sylow 2-subgroup of $SO_p(G)$ so the Frattini Argument yields

$$G = N_G(S)O_p(G).$$

But $a \in O_p(N_G(S))$ whence $a \in O_p(G)$. This contradiction completes the proof of Step 1. $\square$

Choose $X \in \mathcal{X}_G(a)$. Theorem 2.2 implies that X has a unique involution, which we shall denote by i. Now $i \in C_G(a)$ and $S \in \mathrm{Syl}_2(C_G(a))$ so conjugating X by a suitable element of $C_G(a)$, we may suppose that

$$i \in S. \tag{4}$$

The following step shows that there are two cases to be considered. The first case is a rather dull wreathed configuration that is easy to eliminate. Towards the end of the second, more interesting case, we use an idea of Stark [10].

Step 2. *One of the following holds*:

(i) i *has two conjugates in* G, $|S : C_S(i)| = 2$ *and* $[\langle a^G \rangle, C_S(i)] = 1$.
(ii) $i \in Z(G)$.

Proof. Since $i \in Z(X)$ we have $a \notin O_p(C_G(i))$. On the other hand, $a \in O_p(C_G(S))$. Hence we may choose T maximal subject to

$$i \in T < S \quad \text{and} \quad a \notin O_p\bigl(C_G(T)\bigr).$$

Choose S_0 with $T \trianglelefteq S_0 \leqslant S$ and $|S_0 : T| = 2$. Since $a \notin O_p(C_G(T))$, the minimality of $|G|$ yields

$$G = C_G(T)S_0. \tag{5}$$

In particular, $T \trianglelefteq G$. Then $[\langle a^G \rangle, T] = 1$ since $[a, T] = 1$. Now $X = \langle a^X \rangle$ whence $X \leqslant C_G(T)$. It follows that

$$i \in Z(T). \tag{6}$$

If $[i, S_0] = 1$ then (5) and (6) imply that (ii) holds. Hence we shall assume that $[i, S_0] \neq 1$ and prove that (i) holds.

Now $|S_0 : T| = 2$ so we see that i has two conjugates in G. Recall that $a \notin O_p(C_G(i))$. Then as $C_G(i) < G$ the minimality of G implies that $a \notin O_p(C_G(C_S(i)))$. From (6) we have $T \leqslant C_S(i)$ so the maximal choice of T forces $T = C_S(i)$. From (5) we obtain

$$S = C_S(T)S_0 \leqslant C_S(i)S_0 = TS_0 = S_0,$$

whence $S = S_0$. The final two assertions in (i) now follow from $|S_0 : T| = 2$ and $[\langle a^G \rangle, T] = 1$. $\square$

Step 3. *The first possibility of Step* 2 *does not hold.*

Proof. Assume that it does. Choose $s \in S - C_S(i)$. Then s interchanges i and i^s by conjugation. Now i is an involution and $i \notin Z(G)$. Consequently $C_V(i) \neq 0$. Set $U = C_V(i)$. Since $i^G = \{i, i^s\}$ it follows from the irreducibility of G on V that

$$V = U \oplus Us.$$

Let $K = \langle a^G \rangle \trianglelefteq G$, so that $X \leqslant K$. Since $[i, K] = 1$ it follows that K normalizes U and Us. Let $X_0 = C_X(U)$. Then $X_0^s = C_{X^s}(Us)$ and

$$\left[X_0, X_0^s\right] \leqslant C_K(U) \cap C_K(Us) = 1.$$

It follows that

$$\left\langle X_0, X_0^s \right\rangle = X_0 \times X_0^s. \tag{7}$$

By Theorem 2.2 there exists $x \in X_0$ such that $X = \langle a, a^x \rangle$. Now

$$xx^s \in C_G(S)$$

because s interchanges X_0 and X_0^s, because $|S : C_S(i)| = 2$ and because $X_0 \times X_0^s \leqslant K \leqslant C_G(C_S(i))$. By hypothesis $a \in O_p(C_G(S))$, whence

$$\left\langle a, a^{xx^s} \right\rangle$$

is a p-group.

Let $H = \langle X_0, X_0^s, a \rangle$. Since a normalizes both X_0 and X_0^s we see from (7) that $X_0^s \trianglelefteq H$. Set $\overline{H} = H/X_0^s$. Then $\overline{x^s} = 1$ so $\langle \bar{a}, \bar{a}^{\bar{x}} \rangle$ is also a p-group. Recall that $X = \langle a, a^x \rangle$, so $\overline{X_0}$ is a p-group. Now $X_0 \cap X_0^s = 1$ so X_0 is a p-group. But $X = X_0\langle a \rangle$ by Theorem 2.2 so X is a p-group. This contradicts the fact that $a \notin O_p(X)$ and completes the proof of this step. □

Step 4. *The second possibility of Step* 2 *does not hold.*

Proof. Assume that it does. Then $i \in Z(G)$ so the irreducibility of G on V yields $C_V(i) = 0$. In particular, $C_V(X) = 0$ so by Theorem 2.2 we have that $X \cong SL_2(p)$ and that V is a direct sum of natural $SL_2(p)$-modules.

Let $P = \langle a \rangle$. We may suppose that P corresponds to the subgroup $\left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)$. Choose $Q \leqslant X$ such that Q corresponds to the subgroup $\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)$. Let $T = N_X(P) \cap N_X(Q)$ so that $T = \langle t \rangle$ where t corresponds to the matrix $\left(\begin{smallmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{smallmatrix}\right)$ for some generator τ of $GF(p)^{\times}$.

Since V is a direct sum of natural $SL_2(p)$-modules we have that

$$V = [V, P] \oplus [V, Q];$$

that t acts on $[V, P]$ as scalar multiplication by τ and on $[V, Q]$ as scalar multiplication by τ^{-1}; and that $V/[V, P] \cong_T [V, Q]$.

From $[P, S] = 1$ it is clear that S normalizes $[V, P]$. The next objective is to show that some suitable conjugate of S normalizes both $[V, P]$ and $[V, Q]$. Consider the chain

$$V > [V, P] > 0. \tag{8}$$

Let H be the stabilizer of this chain, so that $H = N_G([V, P])$. Let K be the subgroup of H consisting of those elements which act trivially on each factor of (8). Then K is an elementary abelian p-group and $K \trianglelefteq H$. A generator for P acts quadratically on V so

$$P \leqslant K \leqslant C_G(P).$$

Now T and S normalize P so $T, S \leqslant H$. Since t acts on $[V, P]$ as scalar multiplication by τ and on $V/[V, P]$ as scalar multiplication by τ^{-1} we deduce that $[T, S] \leqslant K$. In particular, TSK is a soluble subgroup of G. Now S is a 2-group and $T \cong \mathbb{Z}_{p-1}$ so K is a normal Sylow p-subgroup of TSK.

Hall's Theorem implies that there exists $k \in K$ such that $\langle T, S^k\rangle$ is a p-complement in TSK. Then as $[T, S] \leqslant K$ we have $[T, S^k] = 1$. Recall that $k \in K \leqslant C_G(P)$, that $P = \langle a\rangle$ and that $a \in O_p(C_G(S))$. Then $a \in O_p(C_G(S^k))$. In particular, we may replace S by S^k to suppose that

$$[T, S] = 1.$$

Recall that τ is a generator for $GF(p)^\times$. Then $\tau \neq \tau^{-1}$ since $p > 3$. Consequently t has exactly two eigenspaces: the τ-eigenspace $[V, P]$ and the τ^{-1}-eigenspace $[V, Q]$. By the previous paragraph $[t, S] = 1$, so these eigenspaces are both S-invariant.

Let $s \in S$. Then $[V, Q] = [V, Q^s]$ so as Q is quadratic on V it follows that $\langle Q, Q^s\rangle$ acts trivially on each factor of the chain $V > [V, Q] > 0$. This implies that $\langle Q, Q^s\rangle$ is an elementary abelian p-group.

Choose $e \in P$ such that e corresponds to $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $f \in Q$ such that f corresponds to $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$. Then

$$(ef)^2 = i \in Z(G).$$

Let $g = f^s$. Then as $e \in C_G(S)$ we have

$$(ef)^2 = i = i^s = (eg)^2.$$

Thus $fef = geg$ and then $(g^{-1}f)^e = gf^{-1}$. Now g and f commute since $\langle Q, Q^s\rangle$ is abelian. Consequently

$$\left(g^{-1}f\right)^e = \left(g^{-1}f\right)^{-1}.$$

But e has odd prime order p so this forces $g^{-1}f = 1$. Thus $g = f$ and so s commutes with f. Now $X = \langle e, f\rangle$ so we deduce that s commutes with X. Since s was an arbitrary member of S, we have shown that

$$X \leqslant C_G(S).$$

But $a \in O_p(C_G(S))$ and $a \notin O_p(X)$. This contradiction completes the proof of Step 4. □

Step 2 is contradicted by Steps 3 and 4. The proof of Theorem C is complete.

4. Preliminaries to the proof of Theorem B

Lemma 4.1. *Suppose that the group R acts coprimely on the group G and that p is a prime. Then*:

(i) $\mathcal{N}^*_G(R, p) \subseteq \mathrm{Syl}_p(G)$.
(ii) *$C_G(R)$ acts transitively by conjugation on $\mathcal{N}^*_G(R, p)$.*
(iii) *If H is an R-invariant subgroup of G then*

$$O_p(G; R) \cap H \leqslant O_p(H; R).$$

(iv) *Suppose that K is a normal subgroup of GR and set $\overline{GR} = GR/K$. Then*

$$C_{\overline{G}}(\overline{R}) = \overline{C_G(R)}$$

and

$$\overline{O_p(G; R)} \leqslant O_p(\overline{G}; \overline{R}).$$

Proof. (i) and (ii) are Sylow's Theorems for Groups with Operators. A proof of these and the first assertion in (iv) may be found in [6, Theorem 6.2.2, p. 224]. The remaining assertions follow from the fact that $O_p(G; R)$ is the largest $RC_G(R)$-invariant p-subgroup of G.

In (iv) we remark that $\overline{R}$ acts coprimely on $\overline{G}$ and that the containment may be strict. □

Lemma 4.2. *Suppose that the group R acts coprimely on the group G. Let Σ be a G-conjugacy class of subgroups of G and suppose that Σ is R-invariant. Then*:

(i) *Σ contains at least one member that is R-invariant.*
(ii) *$C_G(R)$ acts transitively by conjugation on the set of R-invariant members of Σ.*

Proof. A Frattini Argument followed by an application of the Schur–Zassenhaus Theorem. □

Lemma 4.3. *Suppose that the group R acts coprimely on the group G. Suppose that $K_1, \ldots, K_n$ are distinct subgroups of G that are permuted amongst* *themselves by R and that $[K_i, K_j] = 1$ for all $i \neq j$. Let p be a prime and set $R_1 = N_R(K_1)$. Then*

$$O_p(G; R) \cap K_1 \leqslant O_p(K_1; R_1).$$

Proof. We may suppose that R acts transitively on $\{K_1, \ldots, K_n\}$. For each i choose $t_i \in R$ such that $K_i = K_1^{t_i}$. Then $\{t_1, \ldots, t_n\}$ is a right transversal to R_1 in R.

Choose $P_1 \in \mathcal{U}^*_{K_1}(R_1, p)$ and set $P = \langle P_1^{t_1}, \ldots, P_1^{t_n} \rangle$. Now $[K_i, K_j] = 1$ for all $i \neq j$ and $P_1^{t_i} \leqslant K_i$ so it follows that P is a p-group. If $g \in R$ then $g = ht_i$ for some $h \in R_1$ and some i. Then $P_1^g = P_1^{ht_i} = P_1^{t_i} \leqslant P$. It follows that P is R-invariant. Choose P^* with $P \leqslant P^* \in \mathcal{U}^*_G(R, p)$. We have

$$P_1 \leqslant P \cap K_1 \leqslant P^* \cap K_1 \in \mathcal{U}_{K_1}(R_1, p)$$

so as $P_1 \in \mathcal{U}^*_{K_1}(R_1, p)$ we deduce that

$$P_1 = P^* \cap K_1.$$

Since $P^* \in \mathcal{U}^*_G(R, p)$ the definition of $O_p(G; R)$ yields $O_p(G; R) \leqslant P^*$. Consequently

$$O_p(G; R) \cap K_1 \leqslant P_1.$$

Now P_1 was an arbitrary member of $\mathcal{U}^*_{K_1}(R_1, p)$ so

$$O_p(G; R) \cap K_1 \leqslant O_p(K_1; R_1)$$

as desired. □

Lemma 4.4. *Suppose that G is a group, that V is a faithful G-module over a field of characteristic $p \neq 2$, and that the Sylow 2-subgroups of G are abelian. Then any element of G that acts quadratically on V is contained in $O_p(G)$.*

Proof. See [6, Theorem 3.8.3, p. 108]. □

5. Quadratic modules

Throughout this section we assume the following:

Hypothesis 5.1.

(i) *G is a group and $p > 3$ is a prime.*
(ii) *V is a faithful G-module over an algebraically closed field of characteristic p.*
(iii) $O_p(G) = 1$.

(iv) *G contains elements that act quadratically on V.*

Following Thompson [11], set

$$\begin{aligned} \mathcal{Q} &= \{g \in G \mid g \text{ acts quadratically on } V\}, \\ d &= \min_{g \in \mathcal{Q}} \dim V(g-1) \quad \text{and} \\ \mathcal{Q}_d &= \{g \in \mathcal{Q} \mid \dim V(g-1) = d\}. \end{aligned}$$

Define an equivalence relation $\sim$ on $\mathcal{Q}_d$ by

$$a \sim b \quad \text{iff} \quad V(a-1) = V(b-1) \text{ and } C_V(a) = C_V(b).$$

Let

$$\Sigma = \{A \subseteq G \mid 1 \in A \text{ and } A^\# \text{ is an equivalence class of } \sim\}.$$

The members of Σ are elementary abelian p-groups. To see this, given $A \in \Sigma$ choose $a \in A^\#$ and observe that A acts trivially on every factor of the chain

$$V > V(a-1) > 0.$$

Moreover, distinct members of Σ have trivial intersection.

The following result of Thompson [11] is fundamental. Timmesfeld gives a proof in [12, 20.9, p. 120].

***SL_2*-Lemma.** *Suppose that $X = \langle a, b\rangle$ for some $a, b \in \mathcal{Q}_d$ and suppose that X is not a p-group. Then $X \cong SL_2(p^n)$ for some $n \in \mathbb{N}$, V is completely reducible as an X-module and every nontrivial X-composition factor of V is two dimensional.*

Lemma 5.2. *Continue with the notation of the SL_2-Lemma. Let i be the unique involution in X and let A be the member of Σ that contains a. Then $[A, i] = 1$.*

Proof. We have $V = C_V(i) \oplus [V, i]$. Now $C_V(i) \leqslant C_V(a)$ so the definition of $\sim$ yields that A acts trivially on $C_V(i)$. Also, $[V, A] = [V, a] \leqslant [V, i]$ so A normalizes $[V, i]$. But i is an involution so i acts as scalar multiplication by -1 on $[V, i]$. Consequently $[A, i] = 1$. □

Theorem 5.3. *Suppose that $A, B \in \Sigma$. Then either $\langle A^G\rangle$ and $\langle B^G\rangle$ commute or A and B are G-conjugate. Moreover, if $G = \langle A^G\rangle$ then Σ is a single G-conjugacy class of subgroups.*

Proof. The first assertion follows from Timmesfeld [12, 3.16, p. 23] or a slight modification of an argument of Stark [10, Theorem I]. Now $B \nleqslant Z(G)$ since $O_p(G) = 1$. Thus the second assertion follows from the first. □

Theorem 5.4. *Let $\widetilde{P} \in \mathrm{Syl}_p(G)$ and set $P = \langle \widetilde{P} \cap \mathcal{Q} \rangle$. Then some member of Σ is normal in P.*

Proof. Suppose that $T \trianglelefteq P$ with $T = \langle T \cap \mathcal{Q}_d \rangle$. We claim that either $[T, P] \cap \mathcal{Q}_d \neq \emptyset$ or $[T, P] = 1$. Suppose that $[T, P] \neq 1$. Then there exist $t \in T \cap \mathcal{Q}_d$ and $s \in P \cap \mathcal{Q}$ such that $[t, s] \neq 1$. Let $H = \langle t, s \rangle$. By a result of Glauberman [3, Theorem 3.3] there exists $h \in H' \cap \mathcal{Q}_d$. But $H' = [\langle t \rangle, \langle s \rangle] \leqslant [T, P]$ so the claim is proved.

Note that $P \cap \mathcal{Q}_d \neq \emptyset$ since $\mathcal{Q}_d$ is a union of conjugacy classes. Hence we may choose T minimal subject to $1 \neq T \trianglelefteq P$ and $T = \langle T \cap \mathcal{Q}_d \rangle$. Since P is a p-group we have $[T, P] < T$ so the claim and the minimal choice of T yield $[T, P] = 1$. Choose $a \in T \cap \mathcal{Q}_d$ and let A be a member of Σ that contains a. Now $[a, P] = 1$ and distinct members of Σ have trivial intersection. Thus P normalizes A.

The definition of P implies that P is maximal subject to being a p-group and generated by members of $\mathcal{Q}$. Now $A^{\#} \subseteq \mathcal{Q}$ whence $A \leqslant P$, which completes the proof. □

6. The proof of Theorem B

Assume Theorem B to be false and consider a counterexample with $|G|$ and then $\dim V$ minimized. We may suppose that the field of definition of V is algebraically closed. Let

$$\mathcal{Q} = \{ g \in G \mid g \text{ acts quadratically on } V \}.$$

Then

$$\mathcal{Q} \cap O_p(G; R) \nsubseteq O_p(G). \tag{9}$$

Step 1. *GR acts irreducibly on V. In particular,*

$$O_p(G) = 1.$$

Proof. Assume false. Let U be a GR-composition factor of V and set

$$\overline{G} = GR/C_{GR}(U).$$

Lemma 4.1 implies that

$$\overline{O_p(G; R)} \leqslant O_p(\overline{G}; \overline{R}).$$

Now $\overline{GR}$ acts irreducibly on U so $O_p(\overline{G}) \leqslant O_p(\overline{GR}) = 1$. Then the minimality of $\dim V$ implies that no element of $O_p(\overline{G}; \overline{R})$ acts quadratically on U. Consequently

$$\mathcal{Q} \cap O_p(G; R) \subseteq \bigcap C_G(U),$$

where the intersection is over all GR-composition factors U of V. But this intersection is $O_p(G)$ so (9) is contradicted. We deduce that GR acts irreducibly on V. $\square$

Step 2. *Suppose that* $Q \neq 1$ *is an* R*-invariant subgroup of* $O_p(G;R)$ *with* $Q = \langle Q \cap \mathcal{Q} \rangle$. *Let* $C = C_G(Q)$. *Then*:

(i) *The Sylow* 2*-subgroups of* C *are contained in* $Z(G)$.
(ii) $\mathcal{Q} \cap C \subseteq O_p(C)$.

Proof. Note that C is R-invariant. Choose $S \in \text{И}^*_C(R,2)$ and let $L = C_G(S)$. Using Lemma 4.1(iii) we have

$$Q \leqslant O_p(G;R) \cap L \leqslant O_p(L;R).$$

If $L \neq G$ then the minimality of $|G|$ forces $Q \leqslant O_p(L)$ and then Theorem C yields $Q \leqslant O_p(G) = 1$, a contradiction. Thus $L = C$ and (i) is proved. In particular, the Sylow 2-subgroups of C are abelian so Lemma 4.4 implies (ii). $\square$

Hypothesis 5.1 is satisfied so we assume the notation defined there. Our first objective is to find an R-invariant member of Σ that has nontrivial intersection with $O_p(G;R)$. An argument similar to one used near the end of the proof of Theorem C then yields a contradiction.

Step 3. *Let* $\widetilde{P} \in \text{И}^*_G(R,p)$ *and set* $P = \langle \widetilde{P} \cap \mathcal{Q} \rangle$. *Then there exists* $A \in \Sigma$ *such that* $A \trianglelefteq P$. *Moreover, for any such* A *we have*

$$A \cap O_p(G;R) \neq 1.$$

Proof. The existence of A follows from Theorem 5.4. To prove the second assertion let $Q = \langle O_p(G;R) \cap \mathcal{Q} \rangle$ and let $Z = A \cap Z(P) \neq 1$. Recall that $A^{\#} \subseteq \mathcal{Q}$ so Step 2 yields

$$Z \leqslant O_p\big(C_G(Q)\big).$$

Now Q is $RC_G(R)$-invariant whence $O_p(C_G(Q)) \leqslant O_p(G;R)$ and then $1 \neq Z \leqslant A \cap O_p(G;R)$. $\square$

Step 4. Σ *is a single* G*-conjugacy class of subgroups.*

Proof. Continue with the notation in the statement of Step 3. Let $\mathcal{K}_1$ be the G-conjugacy class of Σ that contains A. Observe that R acts by conjugation on the G-conjugacy classes of Σ. Let $\{\mathcal{K}_1, \ldots, \mathcal{K}_n\}$ be the orbit that contains $\mathcal{K}_1$. For each i let $K_i = \langle \mathcal{K}_i \rangle$. Theorem 5.3 implies that $[K_i, K_j] = 1$ for all $i \neq j$. Let $R_1 = N_R(K_1)$. By Lemma 4.3 we have

$$O_p(G;R) \cap K_1 \leqslant O_p(K_1;R_1).$$

Now $A \in \Sigma$ so $A^{\#} \subseteq \mathcal{Q}$. By Step 3 we have $A \cap O_p(G; R) \neq 1$. Consequently

$$\mathcal{Q} \cap O_p(K_1; R_1) \neq \emptyset.$$

Since $K_1 \trianglelefteq G$ and $O_p(G) = 1$ it follows that $O_p(K_1) = 1$. Then the minimality of $|G|$ yields $K_1 = G$ and then Theorem 5.3 implies that Σ is a single G-conjugacy class. □

Step 5. *There exists $A \in \Sigma$ such that A is R-invariant and $A \cap O_p(G; R) \neq 1$.*

Proof. Let

$$\Delta = \{P \leqslant G \mid P \text{ is a } p\text{-group and } P = \langle P \cap \mathcal{Q}\rangle\}$$

and let Δ^* be the set of maximal members of Δ. If $P \in \Delta^*$ then $P = \langle \widetilde{P} \cap \mathcal{Q}\rangle$ for some $\widetilde{P} \in \mathrm{Syl}_p(G)$. Consequently Δ^* is a single G-conjugacy class of subgroups.

Step 4 and Lemma 4.2 imply that we may choose $A \in \Sigma$ such that A is R-invariant. Using Theorem 5.4 and Step 4 we see that $N_G(A)$ contains a member of Δ^*. Note that $N_G(A)$ is R-invariant. By Sylow's Theorem, the members of Δ^* contained in $N_G(A)$ form a single $N_G(A)$-conjugacy class. Using Lemma 4.2 there exists $P \leqslant N_G(A)$ such that $P \in \Delta^*$ and P is R-invariant.

Choose $\widetilde{P} \in \mathcal{V}_G^*(R, p)$ with $P \leqslant \widetilde{P}$. Since $P \in \Delta^*$ we have $P = \langle P \cap \mathcal{Q}\rangle = \langle \widetilde{P} \cap \mathcal{Q}\rangle$. Also $A \leqslant P$ since $A^{\#} \subseteq \mathcal{Q}$, whence $A \trianglelefteq P$. Step 3 implies that $A \cap O_p(G; R) \neq 1$, completing the proof of this step. □

Choose $A \in \Sigma$ in accordance with Step 5 and choose

$$a \in A^{\#} \cap O_p(G; R).$$

The definition of Σ and $\sim$ imply that $[V, A] = V(a-1)$. Then as A is R-invariant it follows that

$$V(a-1) \text{ is } R\text{-invariant.} \tag{10}$$

Recall the definition of $\mathcal{X}_G(a)$ given in Section 2. Choose $X \in \mathcal{X}_G(a)$.

Step 6. *$X \cong SL_2(p)$ and V is a direct sum of natural $SL_2(p)$-modules.*

Proof. The SL_2-Lemma and Theorem 2.2 imply that $X \cong SL_2(p)$, that $V = C_V(X) \oplus [V, X]$ and that $[V, X]$ is a direct sum of natural $SL_2(p)$-modules. Thus it suffices to show that $C_V(i) = 0$ where i is the unique involution in X.

We claim that $i \in Z(GR)$. Lemma 5.2 implies that $[i, A] = 1$. Now $A \cap O_p(G; R)$ is R-invariant and $A^{\#} \subseteq \mathcal{Q}$ so Step 3 implies that $i \in Z(G)$. Since $[V, X]$ is a direct sum of natural $SL_2(p)$-modules it follows that i acts as scalar multiplication by -1 on $V(a-1)$. Then using (10) we obtain

$$0 \neq V(a-1) \leqslant C_V([i, R]).$$

But $[i, R]$ is R-invariant and also G-invariant because $i \in Z(G)$. The irreducibility of GR on V forces $C_V([i, R]) = V$, whence $[i, R] = 1$. This proves the claim.

The irreducibility of GR on V and the claim prove that $C_V(i) = 0$, completing the proof of this step. □

Let $P = \langle a \rangle$. By Step 6 and a suitable choice of basis we may suppose that P corresponds to the subgroup $\{\begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix}\}$. Choose $Q \leqslant X$ such that Q corresponds to the subgroup $\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\}$. Let $T = N_X(P) \cap N_X(Q)$ so that $T = \langle t \rangle$ where t corresponds to the matrix $\begin{pmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{pmatrix}$ for some generator τ of $GF(p)^\times$. Since V is a direct sum of natural $SL_2(p)$-modules it follows that

$$V = [V, P] \oplus [V, Q];$$

that t acts on $[V, P]$ as scalar multiplication by τ and on $[V, Q]$ and $V/[V, P]$ as scalar multiplication by τ^{-1}. By (10) we know that $[V, P]$ is R-invariant.

Step 7. *There is a choice of X such that $[V, Q]$ is R-invariant.*

Proof. Let H be the G-stabilizer of the chain

$$V > [V, P] > 0, \tag{11}$$

and let K be the subgroup consisting of those elements of H that act trivially on every factor of (11). Then $K \trianglelefteq H$ and K is an elementary abelian p-group. Since $P = \langle a \rangle$ and $a \in \mathcal{Q}$ we have

$$a \in K \leqslant C_G(a).$$

Now R stabilizes (11) by (10), hence H is R-invariant. Also, $T \leqslant H$ since $T \leqslant N_X(P)$. Since t acts on $[V, P]$ as scalar multiplication by τ and on $V/[V, P]$ as scalar multiplication by τ^{-1} it follows that

$$[T, R] \leqslant K.$$

Consequently, TK is R-invariant.

Recall that K is a p-group and that $T \cong \mathbb{Z}_{p-1}$. Thus T is a p-complement in TK and then by Lemma 4.2 there is an R-invariant p-complement. Hence there exists $k \in K$ such that $R \leqslant N_{GR}(T^k)$. In particular, $[T^k, R] \leqslant T^k \cap K = 1$. (Because R centralizes TK/K and K is a p-group.)

Now $k \in K \leqslant C_G(a)$ so $X^k \in \mathcal{X}_G(a)$. Thus, replacing X by X^k we may suppose that $[T, R] = 1$. Since $p > 3$ and τ is a generator for $GF(p)^\times$ we have $\tau \neq \tau^{-1}$. In particular, t has precisely two eigenspaces: the τ-eigenspace $[V, P]$ and the τ^{-1}-eigenspace $[V, Q]$. As $[t, R] = 1$, the eigenspaces for t are R-invariant. This completes the proof of Step 7. □

We are now in a position to complete the proof of Theorem B. Choose X in accordance with Step 7. Let Q^* be the subgroup of G consisting of those elements that act trivially on every factor of the chain

$$V > [V, Q] > 0. \tag{12}$$

Then Q^* is an elementary abelian p-group. Recall that V is a direct sum of natural $SL_2(p)$-modules, so the elements of $Q^\#$ act quadratically on V. In particular,

$$Q \leqslant Q^*.$$

Now $[V, Q]$ is R-invariant so $Q^* \in \text{И}^*_G(R, p)$. The definition of $O_p(G; R)$ implies that $\langle O_p(G; R), Q^*\rangle$ is a p-group. Now $P = \langle a\rangle$ and $a \in O_p(G; R)$ so $\langle P, Q\rangle$ is a p-group. But $\langle P, Q\rangle = X \cong SL_2(p)$. This contradiction completes the proof of Theorem B.

7. The proof of Theorem A

First we restate part of [5, Theorem A].

Theorem (Glauberman). *Suppose that G is a group, that p is a prime, that $F^*(G) = O_p(G)$ and that P is a p-subgroup of G with $O_p(G) \leqslant P$. If no element of P acts quadratically on any chief factor X/Y of G with $X \leqslant O_p(G)$ then $K^\infty(P) \trianglelefteq G$.*

Assume the hypotheses of Theorem A. We may replace R by $R/C_R(G)$ to suppose that R acts faithfully on G. Then as $F^*(G) = O_p(G)$ and as R and G have coprime orders it follows that $F^*(GR) = O_p(G) = O_p(GR)$.

Suppose that X/Y is a chief factor of GR with $X \leqslant O_p(G)$. Set $V = X/Y$ and regard V as a GR-module over $GF(p)$. Let $\overline{GR} = GR/C_{GR}(V)$. Now GR acts irreducibly on V since V is a chief factor of GR. Consequently $O_p(\overline{G}) = 1$. Lemma 4.1(iv) implies that $\overline{P} \leqslant O_p(\overline{G}; \overline{R})$. Invoking Theorem B, we see that no element of P acts quadratically on V. Glauberman's Theorem, with GR in place of G, implies that $K^\infty(P) \trianglelefteq G$. The proof of Theorem A is complete.

8. The proof of Corollary D

Assume the hypotheses of Corollary D. Let

$$P = O_p(G; R).$$

Choose $M \in \mathcal{M}$ and let $P_0 = O_p(M; R)$. Since $C_G(R) \leqslant M$ it follows that

$$P_0 = P \cap M.$$

By hypothesis, $F^*(M) = O_p(M)$ so Theorem A implies that $K^\infty(P_0) \trianglelefteq M$. Now G is simple, whence $M = N_G(K^\infty(P_0))$. Then $N_P(P_0) \leqslant N_P(K^\infty(P_0)) \leqslant M \cap P = P_0$. We deduce that $P_0 = P$, that $M = N_G(K^\infty(P))$ and that $\mathcal{M} = \{N_G(K^\infty(P))\}$. In particular,

$$G \neq \langle \mathcal{M} \rangle.$$

Assume now that R is non cyclic. Then $G = \langle C_G(a) \mid a \in R^\# \rangle$. Since R is abelian, for each $a \in R^\#$ we know that $C_G(a)$ is R-invariant. Also, $C_G(R) \leqslant C_G(a)$. It follows that

$$G = \langle \mathcal{M} \rangle.$$

This contradicts the previous paragraph. We conclude that R is cyclic.

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